



## ON THE CAYLEY TRANSFORM OF MATRIX CLASSES\*

TIKESH VERMA<sup>†</sup>, DEBASISHA MISHRA<sup>†</sup>, AND MICHAEL TSATSOMEROS<sup>‡</sup>

**Abstract.** The Cayley transform of a square matrix  $A$ , defined as  $F = (I + A)^{-1}(I - A)$ , is tantamount to the factorization  $A = (I + F)^{-1}(I - F)$ . In this context, Fallat *et al.* [Electron. J. Linear Algebra, 9:190-196, 2002] and Mondal *et al.* [Linear Algebra Appl., 681:1-20, 2024] studied the Cayley transform of matrix positivity classes, namely,  $P$ -matrices, positive definite matrices, as well as  $H$ -matrices,  $M$ -matrices, and their inverse classes. The Cayley transform of  $J$ -symplectic, Toeplitz, and dual matrices has also been considered in the literature. In this paper, the discussion is extended by examining the Cayley transform of positive semidefinite matrices,  $Q^*$ -matrices, EP-matrices, weighted-EP matrices, GP matrices, idempotent matrices,  $T$ -Hermitian matrices,  $T$ -EP matrices,  $S$ -skew symmetric matrices,  $S$ -normal matrices, centrosymmetric matrices, tridiagonal matrices, block triangular matrices, and semiconvergent matrices. The results complement the existing literature and are illustrated with examples. Connections among the matrix classes considered are also discussed, and a summary of all results on the matrix Cayley transform known to date is compiled in the form of a table.

**Key words.** Cayley transform,  $P$ -matrix, EP matrix,  $T$ -EP matrix, Centrosymmetric matrix, Group inverse.

**AMS subject classifications.** 15A09, 15A23, 15A24, 15B57.

**1. Introduction.** The Cayley transform was first introduced by Arthur Cayley [5] in 1846. The Cayley transform for a matrix  $A \in M_n(\mathbb{C})$  is defined as

$$C(A) = (I + A)^{-1}(I - A) = (I - A)(I + A)^{-1},$$

when  $-1 \notin \sigma(A)$  (where  $\sigma(A)$  denotes the set of all the eigenvalues of  $A$ ).  $C(A)$  is an extension of the conformal mapping  $f(z) = \frac{1-z}{1+z}$  in the complex plane to the vector space of square matrices. Stein [33] and Taussky [34] established a link between stable and convergent matrices using the Cayley transform indirectly. Haynes [15] proved that a matrix is convergent if and only if it is the Cayley transform of a stable matrix.

When it is well-defined, the Cayley transform of  $A$ ,  $F = C(A)$ , is an involution, that is,  $A = C(F) = (I + F)^{-1}(I - F)$ . Thus, the Cayley transform of a matrix  $A$  results into a factorization of  $A$ . In this regard, Fallat and Tsatsomeros [11] investigated the Cayley transform and resulting factorizations for different positivity classes of matrices, namely,  $P$ -matrices, positive definite matrices,  $H$ -matrices,  $M$ -matrices, and their inverse classes. In 2024, Mondal *et al.* [30] provided a deeper analysis for the classes of  $M$ -matrices,  $H$ -matrices, and inverse  $H$ -matrices. In the same year, Gigola *et al.* [13] discussed the Cayley transform of  $J$ -symplectic matrices. More recently, the authors investigated the Cayley transform of Toeplitz and dual matrices in [37].

The matrix Cayley transform serves as the iteration matrix in an iterative method to solve the Linear Complementarity Problem [16]. It is also applied in areas related to machine learning, signal processing, and statistics [17, 21, 28]. In this regard, we are motivated to investigate the Cayley transform for several additional classes of matrices, such as the Hermitian  $P_0$ -matrices,  $Q^*$ -matrices, EP-matrices, weighted-EP matrices, GP matrices, idempotent matrices,  $T$ -Hermitian matrices,  $T$ -EP matrices,  $S$ -skew symmetric

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<sup>†</sup>Department of Mathematics, National Institute of Technology Raipur, India (rprikesh@gmail.com, dmishra@nitrr.ac.in).

<sup>‡</sup>Department of Mathematics and Statistics, Washington State University, Washington, USA (tsat@wsu.edu).

matrices,  $S$ -normal matrices, centrosymmetric matrices, tridiagonal matrices, block upper (lower) triangular matrices, and convergent matrices. We undertake this task in Section 3 after setting the background in Section 2.

Looking at the existing literature summarized above and at the results herein, we believe a comprehensive study of the Cayley transform has been achieved. As a consequence, we provide in Section 4 a table reviewing the results on matrix Cayley transforms.

**2. Background results.** Let  $M_{m \times n}(\mathbb{R})$  ( $M_{m \times n}(\mathbb{C})$ ) denote the space of all real (complex)  $m \times n$  matrices. We write  $M_n(\mathbb{R})$  ( $M_n(\mathbb{C})$ ) when  $m = n$ . We denote the transpose of  $A$  by  $A^t$ , the conjugate transpose of  $A$  by  $A^*$ , the range space of  $A$  by  $R(A)$ , and the null space of  $A$  by  $N(A)$ . The rank of  $A$  is denoted by  $r(A)$ .

Let  $A[\alpha, \beta]$  be the submatrix of  $A$  whose rows and columns are indexed by  $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ , where the elements of  $\alpha$  and  $\beta$  are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered empty; if it is square, it is conventionally given a determinant of 1. We abbreviate  $A[\alpha, \alpha]$  by  $A[\alpha]$  and refer to it as a *principal submatrix* of  $A$ .

The *Moore–Penrose inverse* [3] of a matrix  $A \in M_{m \times n}(\mathbb{C})$  is denoted by  $A^\dagger$  and defined as the unique matrix  $X \in M_{n \times m}(\mathbb{C})$  that satisfies the following four matrix equations:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad (XA)^* = XA.$$

Given  $A \in M_n(\mathbb{C})$ , the unique matrix  $X \in M_n(\mathbb{C})$  satisfying

$$AXA = A, \quad XAX = X, \quad \text{and} \quad AX = XA,$$

is called the *group inverse* [3] of  $A$ . It is denoted by  $A^\#$ . It is important to note that the group inverse does not exist for all square matrices. The class of group invertible matrices is known as the class of GP matrices [14]. A necessary and sufficient condition for the existence of  $A^\#$  is that  $r(A) = r(A^2)$ . Equivalently, a matrix  $A \in M_n(\mathbb{C})$  is GP if and only if  $R(A) = R(A^2)$ .

Matrix  $A \in M_n(\mathbb{C})$  is *Hermitian*, *normal*, or *idempotent* if  $A = A^*$ ,  $AA^* = A^*A$ , and  $A^2 = A$ , respectively. Let  $\lambda_i(A)$  or  $\lambda_i$ , where  $i = 1, 2, \dots, n$ , be the eigenvalues of a matrix  $A \in M_n(\mathbb{C})$ . We assume that the eigenvalues of a Hermitian matrix are ordered as  $\lambda_{\min}(A) = \lambda_n(A) \leq \lambda_{n-1}(A) \leq \dots \leq \lambda_1(A) = \lambda_{\max}(A)$ .

Matrix  $A \in M_n(\mathbb{R})$  is called *nonnegative* if all its entries are nonnegative and is denoted by  $A \geq 0$ . Matrix  $A \in M_n(\mathbb{R})$  is called a *Z-matrix* if its off-diagonal entries are nonpositive. A Z-matrix can be expressed as  $A = sI - B$ , where  $s$  is a real number and  $B \geq 0$ . In particular, a matrix  $A \in M_n(\mathbb{R})$  is called a *nonsingular M-matrix* if  $A = sI - B$  with  $B \geq 0$  and  $s > \rho(B)$ , where  $\rho(B)$  denotes the spectral radius of  $B$ . More than fifty equivalent for a Z-matrix to be a nonsingular M-matrix are given in [2].

We next recall Lemma 2.1 and Lemma 2.2 of [11] in combined form.

LEMMA 2.1. ([11, Lemma 2.1 & 2.2]) *Let  $A \in M_n(\mathbb{C})$  be such that  $F = C(A)$  is well-defined. Then,  $I + F$  is invertible, i.e.,  $C(F)$  is well-defined and*

$$A = (I + F)^{-1}(I - F) = C(F),$$

$$I + F = 2(I + A)^{-1}.$$

If, in addition,  $A$  is invertible, then

$$I - F = 2(I + A^{-1})^{-1}.$$

Two (out of the four) equivalent conditions in Theorem 3.1, [4] are recalled next.

**THEOREM 2.2.** ([4, Theorem 3.1]) *Let  $A, B \in M_n(\mathbb{C})$  be such that  $A^\#$  and  $B^\#$  exist. Then, the following are equivalent:*

1.  $(AB)^\#$  exists and  $(AB)^\# = B^\#A^\#$ ;
2.  $R(AB) = R(BA)$  and  $N(AB) = N(BA)$ .

Next, we recall a result that establishes a relationship between the eigenvalues of a Hermitian matrix and the eigenvalues of a principal submatrix.

**THEOREM 2.3.** ([38, Theorem 8.10]) *Let  $A \in M_n(\mathbb{C})$  be a Hermitian matrix partitioned as*

$$A = \begin{bmatrix} Q & C \\ C^* & B \end{bmatrix},$$

where  $Q$  is an  $m \times m$  principal submatrix of  $A$ ,  $1 \leq m \leq n$ . Then,

$$\lambda_{k+n-m}(A) \leq \lambda_k(Q) \leq \lambda_k(A), \quad k = 1, 2, \dots, m.$$

In particular, when  $m = n - 1$ ,

$$\lambda_n(A) \leq \lambda_{n-1}(Q) \leq \lambda_{n-1}(A) \leq \dots \leq \lambda_2(A) \leq \lambda_1(Q) \leq \lambda_1(A).$$

The following remark is a consequence of Theorem 2.3.

**REMARK 2.4.** *Let  $A \in M_n(\mathbb{C})$  be a Hermitian matrix. If  $B$  is any principal submatrix of  $A$ , then  $\lambda_{\min}(A) \leq \lambda_{\min}(B) \leq \lambda_{\max}(B) \leq \lambda_{\max}(A)$ .*

Next, we recall the definition of the  $k^{\text{th}}$  symmetric function of  $n$  complex numbers.

**DEFINITION 2.5.** *Let  $\lambda_1, \dots, \lambda_n$  be  $n$  complex numbers. The  $k^{\text{th}}$  symmetric function of  $\lambda_1, \dots, \lambda_n$  is defined as*

$$s_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

**REMARK 2.6.** ([18, Proposition 2]) *Let  $\lambda_1, \dots, \lambda_n$  be  $n$  complex numbers, all of whose symmetric functions are nonnegative, and let  $\alpha$  be a positive number. Then,  $\lambda_1 + \alpha, \dots, \lambda_n + \alpha$  have positive symmetric functions.*

To end this section, we recall a result from [30].

**THEOREM 2.7.** ([30, Theorem 3.1]) *Let  $A \in M_n(\mathbb{C})$  be such that  $C(A)$  is well-defined. Then,  $A$  is Hermitian if and only if  $C(A)$  is Hermitian.*

**3. Cayley transform results.** This section deals with the Cayley transform of several matrix classes that play prominent roles in applications. The results are organized into three subsections.

**3.1.  $P_0$ ,  $Q^*$ , and semiconvergent matrices.** Recall that a complex square matrix is called a  $P$ -matrix ( $P_0$ -matrix) if all its principal minors are positive (nonnegative). These matrices have applications in Linear Complementarity Problems and iterative solutions of linear systems of equations. The following results recall a few characterizations of a  $P_0$ -matrix.

LEMMA 3.1. ([22, Lemma 4.8.1]) *Let  $A \in M_n(\mathbb{C})$ . Then,  $A$  is a  $P_0$ -matrix if and only if  $A + \alpha I$  is a  $P$ -matrix for all  $\alpha > 0$ .*

THEOREM 3.2. ([22, Theorem 4.8.2]) *Let  $A \in M_n(\mathbb{C})$  and assume all of its principal submatrices have real characteristic polynomials. Then,  $A$  is a  $P_0$ -matrix if and only if every real eigenvalue of every principal submatrix of  $A$  is nonnegative.*

In 2002, Fallat and Tsatsomeris [11] showed that a  $P$ -matrix  $A$  can be factorized as a product of two  $P$ -matrices in terms of  $C(A)$  (see Theorem 3.1, [11]). Recently, Tsatsomeris *et al.* [35] considered  $C(A)$  for a complex  $P$ -matrix. These results motivate us to examine the Cayley transform of Hermitian  $P_0$  matrices, namely, complex positive semidefinite matrices.

THEOREM 3.3.  *$A \in M_n(\mathbb{C})$  is a Hermitian  $P_0$ -matrix if and only if  $F = C(A)$  is well-defined, and  $(I + F)^{-1}$  and  $I - F$  are Hermitian  $P_0$ -matrices.*

*Proof.* Suppose that  $A$  is a Hermitian  $P_0$ -matrix. Then, all of its eigenvalues are nonnegative, resulting in a well-defined  $F = C(A)$ . By Lemma 2.1, we have  $(I + F)^{-1} = \frac{1}{2}(I + A)$ . If  $\lambda$  is an eigenvalue of  $A$ , then  $\frac{1}{1+\lambda}$  is an eigenvalue of  $(I + A)^{-1}$ . Again,  $\lambda_{\max}((I + A)^{-1}) \leq 1$ . And, by Lemma 3.1,  $I + A$  is a  $P$ -matrix. So,  $\frac{1}{2}(I + A)$  is a  $P$ -matrix. Since  $A$  is Hermitian and every  $P$ -matrix is a  $P_0$ -matrix,  $\frac{1}{2}(I + A) = (I + F)^{-1}$  is a Hermitian  $P_0$ -matrix. We have

$$\begin{aligned} I - F &= I - (I + A)^{-1}(I - A) \\ &= (I + A)^{-1}(I + A) - (I + A)^{-1}(I - A) \\ &= 2A(I + A)^{-1} \quad (\because A \text{ and } (I + A)^{-1} \text{ commute}) \\ &= 2(I + A - I)(I + A)^{-1} \\ &= 2(I - (I + A)^{-1}). \end{aligned}$$

So,  $I - F$  is Hermitian, and thus has real eigenvalues. Let  $(I - F)[\alpha]$  be a principal submatrix of  $(I - F)$ . Then,  $(I - F)[\alpha] = 2(I[\alpha] - (I + A)^{-1}[\alpha])$ . Since  $A$  is a Hermitian matrix, all eigenvalues of a principal submatrix are real. Let  $\mu$  be an eigenvalue of  $(I + A)^{-1}[\alpha]$ . By Remark 2.4, we have  $\mu \leq \lambda_{\max}((I + A)^{-1}[\alpha]) \leq \lambda_{\max}((I + A)^{-1}) \leq 1$ . Hence,  $1 - \mu \geq 0$ . So, all the eigenvalues of any principal submatrix of  $I - F$  are nonnegative. Therefore, by Theorem 3.2,  $I - F$  is a Hermitian  $P_0$ -matrix.

Conversely, if  $(I + F)^{-1} = \frac{1}{2}(I + A)$  and  $I - F = 2(I - (I + A)^{-1})$  are positive semidefinite matrices, then  $A$  is Hermitian and thus has real eigenvalues. Also, if  $\lambda \in \sigma(A)$ , then  $\frac{1}{2}(1 + \lambda) \geq 0$  and  $\frac{2\lambda}{1 + \lambda} \geq 0$ , which implies  $\lambda \geq 0$ . Therefore,  $A$  is a positive semidefinite matrix.  $\square$

Now, we recall a generalization of the notion of a  $P$ -matrix.

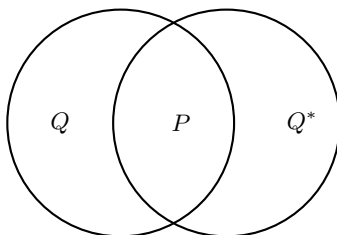
DEFINITION 3.4. Matrix  $A \in M_n(\mathbb{C})$  is called a  $Q^*$ -matrix if for each  $k \in \{1, 2, \dots, n\}$ , the sum of all  $k \times k$  principal minors of  $A$  is positive.

REMARK 3.5. A  $Q^*$ -matrix is also referred to as a  $Q$ -matrix in the linear-algebraic literature (see, e.g., [22, Definition 4.4.3]). We introduced above the term  $Q^*$ -matrix in order to distinguish this class of matrices from the notion of a  $Q$ -matrix  $A$  as used in the Linear Complementarity Problem literature, where it signifies

that  $LCP(A, q)$  has a solution for all  $q \in M_{n \times 1}(\mathbb{R})$  (see, e.g., [9]). The two notions are indeed distinct as shown in the next example.

EXAMPLE 3.6. Consider the matrices  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}$ . Then,  $A$  is a  $Q$ -matrix on account of being a nonnegative matrix with positive diagonal entries (see Theorem 3.10, [2]). However,  $A$  is not a  $Q^*$ -matrix according to Definition 3.4. In contrast,  $B$  is a  $Q^*$ -matrix according to Definition 3.4, but it is not a  $Q$ -matrix since  $LCP(B, q)$  has no solution for  $q = [-1 \ 1]^t \in M_{2 \times 1}(\mathbb{R})$ .

It is also worth noting that  $A \in M_n(\mathbb{R})$  is a  $P$ -matrix if and only if  $LCP(A, q)$  has a unique solution for all  $q \in M_{n \times 1}(\mathbb{R})$ . The below diagram illustrates the relationship among the real  $P$ -matrix,  $Q$ -matrix, and  $Q^*$ -matrix classes.



Recently, Mondal *et al.* [31] discussed the fact that  $Q^*$ -matrices are not generally similar to  $P$ -matrices. Considering the characteristic polynomial of a matrix  $A$ , the sum of principal minors of  $A$  of order  $k$  is equal to  $k^{th}$  symmetric function of the eigenvalues of  $A$  (see pp. 494 – 495, [29]). As a consequence, it is straightforward to show the following.

LEMMA 3.7. If  $A \in M_n(\mathbb{C})$  is a  $Q^*$ -matrix, then  $\alpha A$  is a  $Q^*$ -matrix for every  $\alpha > 0$ .

LEMMA 3.8. If  $A \in M_n(\mathbb{C})$  is a  $Q^*$ -matrix, then  $A + \alpha I$  is a  $Q^*$ -matrix for every  $\alpha > 0$ .

The inverse of a  $Q^*$ -matrix is again a  $Q^*$ -matrix and is shown in the next lemma.

LEMMA 3.9. If  $A \in M_n(\mathbb{C})$  is a  $Q^*$ -matrix, then  $A^{-1}$  is a  $Q^*$ -matrix.

*Proof.* Suppose that  $A$  is a  $Q^*$ -matrix. Then, for each  $k \in \{1, 2, \dots, n\}$ , we have

$$s_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} > 0,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $A$ . If  $d_k$  is the  $k^{th}$  symmetric function of  $A^{-1}$ , where  $k \in \{1, 2, \dots, n\}$ , then

$$\begin{aligned} d_1 &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n} \\ &= \frac{\lambda_2 \lambda_3 \cdots \lambda_n + \lambda_1 \lambda_3 \cdots \lambda_n + \dots + \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\lambda_1 \lambda_2 \cdots \lambda_n} \\ &= \frac{s_{n-1}}{s_n} > 0, \\ d_2 &= \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} + \frac{1}{\lambda_2} \cdot \frac{1}{\lambda_3} + \dots + \frac{1}{\lambda_n} \cdot \frac{1}{\lambda_1} \\ &= \frac{\lambda_3 \lambda_4 \cdots \lambda_n + \lambda_1 \lambda_4 \cdots \lambda_n + \dots + \lambda_2 \lambda_3 \cdots \lambda_{n-1}}{\lambda_1 \lambda_2 \cdots \lambda_n} \end{aligned}$$

$$\begin{aligned}
 &= \frac{s_{n-2}}{s_n} > 0, \\
 &\vdots \\
 d_k &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{1}{\lambda_{i_1}} \cdots \frac{1}{\lambda_{i_k}} \\
 &= \frac{s_{n-k}}{s_n} > 0, \quad \text{where } s_0 = 1.
 \end{aligned}$$

Thus,  $d_k$  is positive for each  $k \in \{1, 2, \dots, n\}$ . Hence,  $A^{-1}$  is a  $Q^*$ -matrix. □

We next see that the Cayley transform of a  $Q^*$ -matrix need not be a  $Q^*$ -matrix.

EXAMPLE 3.10. Consider the  $Q^*$ -matrix  $A = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$ . Then, its Cayley transform

$$C(A) = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & -1 \end{bmatrix} \text{ is not a } Q^*\text{-matrix.}$$

Motivated by Theorem 3.1, [11], we next show that a  $Q^*$ -matrix can be factorized as a product of two  $Q^*$ -matrices through the Cayley transform.

THEOREM 3.11. Let  $A \in M_n(\mathbb{C})$  be a  $Q^*$ -matrix such that  $F = C(A)$  is well-defined. Then,  $(I + F)^{-1}$  and  $I - F$  are  $Q^*$ -matrices.

*Proof.* Suppose that  $A$  is a  $Q^*$ -matrix. Then,  $I + A^{-1}$  is a  $Q^*$ -matrix by Lemma 3.8. So,  $2(I + A^{-1})^{-1} = I - F$  is a  $Q^*$ -matrix. Again, by Lemma 3.8,  $\frac{1}{2}(I + A) = (I + F)^{-1}$  is a  $Q^*$ -matrix. □

The next example illustrates that the Cayley transform of a  $Q$ -matrix is not necessarily a  $Q$ -matrix.

EXAMPLE 3.12. Consider the  $Q$ -matrix  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . Then, its Cayley transform  $C(A) = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$  is not a  $Q$ -matrix since  $LCP(C(A), q)$  has no solution when  $q = [0 \quad -1]^t \in M_{2 \times 1}(\mathbb{R})$ .

In the following example, we demonstrate that if a matrix  $A \in M_n(\mathbb{R})$  is a  $Q$ -matrix, then it is not necessary that the factors  $(I + F)^{-1}$  and  $I - F$  are  $Q$ -matrices.

EXAMPLE 3.13. Let  $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ . This is a  $Q$ -matrix (see page 78, [32]). Then,  $I - F = 2 \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{bmatrix}$  is a  $Q$ -matrix since it is a  $P$ -matrix, but  $(I + F)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is not a  $Q$ -matrix, because any nonnegative matrix with one of its diagonal entries equal to 0 is not a  $Q$ -matrix; see the proof of Theorem 3.10, page 279, [2].

Haynes [15] showed that a matrix is convergent (i.e.,  $\lim_{k \rightarrow \infty} A^k = 0$ ) if it is the Cayley transform of a stable matrix. We will relate this result to semiconvergent matrices.

DEFINITION 3.14.  $A \in M_n(\mathbb{C})$  is called a semiconvergent matrix if  $\lim_{k \rightarrow \infty} A^k$  exists.

THEOREM 3.15. Let  $A \in M_{m \times n}(\mathbb{C})$ . Then,  $C(A^*A)$  is a well-defined semiconvergent matrix.

*Proof.* We know that  $A^*A$  is positive semidefinite matrix. So, all the eigenvalues of  $A^*A$  are nonnegative and thus  $C(A^*A)$  is well-defined. From Theorem 2.7,  $C(A^*A)$  is Hermitian. Thus,  $C(A^*A) = UDU^*$ , where

$U$  is a unitary matrix and  $D$  is a diagonal matrix containing the eigenvalues of  $C(A^*A)$ . Every eigenvalue of  $C(A^*A)$  is of the form  $\frac{1-\lambda}{1+\lambda}$ , where  $\lambda$  is the eigenvalue of  $A^*A$ . Since  $\frac{1-\lambda}{1+\lambda}$  is a decreasing function of  $\lambda$  for  $\lambda \geq 0$ , and since  $\lim_{\lambda \rightarrow \infty} \frac{1-\lambda}{1+\lambda} = -1$ , it follows that  $-1 < \frac{1-\lambda}{1+\lambda} \leq 1$ . Thus,  $\lim_{k \rightarrow \infty} (C(A^*A))^k = U \left( \lim_{k \rightarrow \infty} D^k \right) U^*$  exists. Therefore,  $C(A^*A)$  is a semiconvergent matrix.  $\square$

**3.2. Generalizations of EP matrices.** In this subsection, we will look into the Cayley transforms of EP matrices and related classes.

DEFINITION 3.16. Matrix  $A \in M_n(\mathbb{C})$  is an *EP matrix* if  $R(A) = R(A^*)$ .

EP matrices  $A$  are also known as a “range Hermitian” matrices and their Moore–Penrose inverses satisfy  $AA^\dagger = A^\dagger A$ . It is believed that the acronym EP reflects the implied “Equal Projection” property. EP matrices include the classes of Hermitian, skew-Hermitian, unitary, and more generally the normal matrices. For the class of EP matrices,  $A^\#$  always exists.

In 2024, Gigola *et al.* [13] argued that the Cayley transform of a normal matrix is also a normal matrix. Thus, we have the following result.

THEOREM 3.17. Let  $A \in M_n(\mathbb{C})$  be a normal matrix such that  $C(A)$  is well-defined. Then,  $C(A)$  is an EP matrix.

In the next example, we show that the converse of the above result is not true.

EXAMPLE 3.18. Let  $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ . Then,  $C(A) = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$  is an EP matrix, but  $A$  is not a normal matrix.

One generalization EP matrices is the class of weighted EP matrices.

DEFINITION 3.19. ([36, Definition 3.2]) Matrix  $A \in M_n(\mathbb{C})$  is said to be a *weighted-EP matrix* with respect to two given Hermitian positive definite matrices  $M$  and  $N$  of order  $n$  if both  $MA$  and  $AN^{-1}$  are EP matrices, i.e.,  $R(MA) = R((MA)^*)$  and  $R(AN^{-1}) = R((AN^{-1})^*)$ . The matrices  $M$  and  $N$  are usually implied by the context of the discussion.

The weighted EP matrices have connections to the weighted Moore–Penrose inverse [3]. There are 42 equivalent conditions for a weighted EP matrix stated in [36, Theorem 3.5]. We recall two of them.

THEOREM 3.20. ([36, Theorem 3.5]) Let  $A \in M_n(\mathbb{C})$  and let  $M, N \in M_n(\mathbb{C})$  be two Hermitian positive definite matrices. Then, the following are equivalent:

1.  $A$  is a weighted-EP matrix, i.e.,  $MA$  and  $AN^{-1}$  are EP matrices;
2.  $r(A) = r(A^2)$  and both  $MAA^\#$  and  $NAA^\#$  are Hermitian matrices.

Next, we relate a weighted EP matrix to its Cayley transform.

THEOREM 3.21. Let  $A \in M_n(\mathbb{C})$  be such that  $F = C(A)$  is well-defined. Then,  $A$  is a weighted-EP matrix if and only if  $(I + F)^{-1}$  and  $I - F$  are weighted-EP matrices.

*Proof.* Let  $A$  be a weighted-EP matrix with respect to two Hermitian positive definite matrices  $M_1$  and  $N_1$ . By Theorem 3.20, we have  $r(A) = r(A^2)$ , and  $M_1AA^\#$  and  $N_1AA^\#$  are Hermitian. Then, by Lemma 2.1, we have  $(I + F)^{-1} = \frac{1}{2}(I + A)$ . Since an invertible matrix and its conjugate transpose have

the same range,  $M_1(I + F)^{-1}$  and its conjugate transpose must have the same range. Similarly, we have  $R((I + F)^{-1}N_1^{-1}) = R(((I + F)N_1^{-1})^*)$ . Thus,  $(I + F)^{-1}$  is a weighted-EP matrix. From the proof of Theorem 3.3, we have

$$I - F = 2A(I + A)^{-1}.$$

Also,  $A$  and  $(I + A)^{-1}$  commute and so  $R(A(I + A)^{-1}) = R((I + A)^{-1}A)$  and  $N(A(I + A)^{-1}) = N((I + A)^{-1}A)$ , and

$$\begin{aligned} r((I - F)^2) &= r(4A^2(I + A)^{-2}) \\ &= r(A^2) \\ &= r(A) \\ &= r(I - F). \end{aligned}$$

Hence,  $(I - F)^\#$  exists. Thus, by Theorem 2.2, we get  $(I - F)^\# = (2A(I + A)^{-1})^\# = \frac{1}{2}(I + A)A^\#$ . So,  $M_1(I - F)(I - F)^\# = M_1AA^\#$  and  $N_1(I - F)(I - F)^\# = N_1AA^\#$  are Hermitian matrices. Therefore, by Theorem 3.20,  $I - F$  is also a weighted-EP matrix.

Conversely, suppose that  $I - F$  is a weighted-EP matrix with respect to two Hermitian positive definite matrices  $M_2$  and  $N_2$ . Again by Theorem 3.20, we have  $r(I - F) = r((I - F)^2)$ , and  $M_2(I - F)(I - F)^\#$  and  $N_2(I - F)(I - F)^\#$  are Hermitian matrices. Then,

$$\begin{aligned} r(A) &= r(I - F) \\ &= r((I - F)^2) \\ &= r(4A^2(I + A)^{-2}) \\ &= r(A^2). \end{aligned}$$

Further,  $M_2AA^\# = M_2(I - F)(I - F)^\#$  and  $N_2AA^\# = N_2(I - F)(I - F)^\# = N_2AA^\#$  are Hermitian matrices. Therefore,  $A$  is a weighted-EP matrix.  $\square$

Every EP matrix  $A$  can be written as  $A = U \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ , where  $U$  is unitary matrix and  $A_1$  is a nonsingular matrix (see Theorem 2.1, [36]). The next result provides another factorization of an EP matrix into two commuting EP matrices, which is a special case of Theorem 3.21.

**COROLLARY 3.22.** *Let  $A \in M_n(\mathbb{C})$  be such that  $F = C(A)$  is well-defined. Then,  $A$  is an EP matrix if and only if  $(I + F)^{-1}$  and  $I - F$  are EP matrices.*

Next, we study the Cayley transform of GP matrices  $A$ , i.e.,  $R(A) = R(A^2)$ . Note that the class of GP matrices contains the class of EP matrices. First, we illustrate that the Cayley transform of a GP matrix is not necessarily a GP matrix.

**EXAMPLE 3.23.** *Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Then,  $R(A) = R(A^2)$ , but  $C(A) = \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix}$  is not a GP matrix as  $R(C(A)) \neq R(C(A)^2)$ .*

Similar to Theorem 3.3, a GP matrix can also be factored into the product of two GP matrices in terms of the Cayley transform, as proved hereunder.

**THEOREM 3.24.** *Let  $A \in M_n(\mathbb{C})$  be such that  $F = C(A)$  is well-defined. Then,  $A$  is a GP matrix if and only if  $(I + F)^{-1}$  and  $I - F$  are GP matrices.*

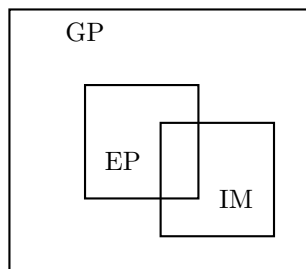
*Proof.* Suppose that  $A$  is a GP matrix, i.e.,  $R(A) = R(A^2)$ . As  $I + F$  is nonsingular,  $R((I + F)^{-1}) = R((I + F)^{-2})$ , i.e.,  $(I + F)^{-1}$  is a GP matrix. Now, from the proof of Theorem 3.3, we have  $I - F = 2A(I + A)^{-1}$ . So,

$$\begin{aligned} R((I - F)^2) &= R(4A^2(I + A)^{-2}) \\ &= R(A^2) \\ &= R(A) \\ &= R(2A(I + A)^{-1}) \\ &= R(I - F). \end{aligned}$$

Hence,  $I - F$  is a GP matrix.

Conversely, let  $I - F$  be GP, i.e.,  $R(I - F) = R((I - F)^2)$ . Then, it is clear from the proof of the first part that  $R(A) = R(A^2)$ , i.e.,  $A$  is GP.  $\square$

The below diagram illustrates the relationship between GP matrices, EP matrices, and idempotent matrices (IM).



Observe that the Cayley transform of a GP/EP matrix is not necessarily a GP/EP matrix (see Example 3.23). Nevertheless, the Cayley transform of an idempotent matrix is an idempotent matrix and that is proved next.

**THEOREM 3.25.** *Let  $A \in M_n(\mathbb{R})$ . Then,  $A$  is idempotent if and only if  $C(A)$  is idempotent.*

*Proof.* Suppose that  $A$  is an idempotent matrix. It follows that

$$(3.1) \quad (I - A)^2 = (I + A)(I - A).$$

Since all the eigenvalues of an idempotent matrix are either 0 or 1, the eigenvalues of  $I + A$  are positive, i.e.,  $I + A$  is invertible. Now, pre-multiplying (3.1) by  $(I + A)^{-2}$ , we have  $(I + A)^{-2}(I - A)^2 = (I + A)^{-1}(I - A)$ , i.e.,  $C(A)^2 = C(A)$  from the fact that  $(I + A)^{-1}$  and  $(I - A)$  commute. Hence,  $C(A)$  is an idempotent matrix.

Conversely, if  $F = C(A)$  is an idempotent matrix, by Lemma 2.1, we have  $A = C(F)$  is an idempotent matrix.  $\square$

We proceed to study the Cayley transform of another extension of EP matrices called  $T$ -EP matrices. Recall that a matrix  $T \in M_{m \times n}(\mathbb{C})$  is said to be a *partial isometry* [3] if  $T = TT^*T$ . In relation to the partial isometry  $T$ , Hestenes [19] introduced the concepts of  $T$ -Hermitian and  $T$ -normal matrices in 1961 to develop spectral theory for rectangular matrices, which are recalled next.

**DEFINITION 3.26.** ([12, Definition 1]) *Let  $A, T \in M_{m \times n}(\mathbb{C})$ . Then,  $A$  is said to be  $T$ -Hermitian if  $A = TA^*T$ .*

DEFINITION 3.27. ([12, Definition 2]) Let  $A, T \in M_{m \times n}(\mathbb{C})$ . Then,  $A$  is said to be  $T$ -normal if  $A = TT^*A = AT^*T$  and  $AA^*T = TA^*A$ .

We know that the Cayley transform of a Hermitian matrix is a Hermitian matrix (see Theorem 3.1, [30]). But, this is not true in the case of  $T$ -Hermitian matrices, as shown by the following example.

EXAMPLE 3.28. ([12, Example 3]) Let  $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , and consider the partial isometry  $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Then,  $TA^*T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$ , i.e.,  $A$  is  $T$ -Hermitian. However,  $C(A) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$  is not  $T$ -Hermitian because  $TC(A)^*T = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \neq C(A)$ .

Recall that a matrix  $T \in M_n(\mathbb{C})$  satisfying  $T = T^{-1} = T^*$  is known as *involutionary Hermitian matrix* [12]. For an involutory Hermitian matrix  $T$ , the Cayley transform of a  $T$ -Hermitian matrix is  $T$ -Hermitian, as shown next.

THEOREM 3.29. Let  $A, T \in M_n(\mathbb{C})$  be such that  $C(A)$  is well-defined, where  $T$  is an involutory Hermitian matrix. Then,  $A$  is  $T$ -Hermitian if and only if  $C(A)$  is  $T$ -Hermitian.

*Proof.* Suppose that  $A$  is  $T$ -Hermitian, i.e.,  $A = TA^*T$ . We now have

$$\begin{aligned} TC(A)^*T &= T(I + A^*)^{-1}(I - A^*)T \\ &= T^{-1}(I + A^*)^{-1}T^{-1}T(I - A^*)T \quad (\because T = T^{-1}) \\ &= (T(I + A^*)T)^{-1}(T^2 - TA^*T) \\ &= (T^2 + TA^*T)^{-1}(T^2 - TA^*T) \\ &= (I + A)^{-1}(I - A) \\ &= C(A). \end{aligned}$$

Thus,  $C(A)$  is  $T$ -Hermitian.

Conversely, suppose  $C(A)$  is  $T$ -Hermitian. So,  $TC(A)^*T = C(A)$ , which implies

$$\begin{aligned} T(I + A^*)^{-1}(I - A^*)T &= (I + A)^{-1}(I - A) \\ \implies T^{-1}(I + A^*)^{-1}T^{-1}T(I - A^*)T &= (I + A)^{-1}(I - A) \\ \implies (I + TA^*T)^{-1}(I - TA^*T) &= (I - A)(I + A)^{-1} \end{aligned}$$

Pre-multiplying and post-multiplying in the last equation by  $I + TA^*T$  and  $I + A$ , respectively, we have  $(I - TA^*T)(I + A) = (I + TA^*T)(I - A)$ , which yields  $A = TA^*T$ . Hence,  $A$  is  $T$ -Hermitian.  $\square$

Next, we recall a sufficient condition for a matrix to be a  $T$ -normal matrix.

THEOREM 3.30. ([12, Theorem 1]) Let  $A, T \in M_{m \times n}(\mathbb{C})$ . If  $A$  is  $T$ -Hermitian, then  $A$  is  $T$ -normal.

From Theorems 3.29 and 3.30, we have the following.

THEOREM 3.31. Let  $A, T \in M_n(\mathbb{C})$  be such that  $C(A)$  is well-defined, where  $T$  is an involutory Hermitian matrix. If  $A$  is  $T$ -Hermitian, then  $C(A)$  is  $T$ -normal.

Ferreyra and Malik [12] extended the concept of EP matrices to rectangular matrices, which is recalled next.

DEFINITION 3.32. ([12, Definition 3]) *Let  $A, T \in M_{m \times n}(\mathbb{C})$ , where  $T$  is a partial isometry. Then,  $A$  is said to be relative EP (to  $T$ ) or  $T$ -EP, if  $R(A) = R(TA^*T)$  and  $A = AT^*T$ .*

Two equivalent conditions out of the eight conditions in [12, Theorem 17] are recalled next.

THEOREM 3.33. ([12, Theorem 17]) *Let  $A, T \in M_n(\mathbb{C})$ . Suppose  $T$  is an involutory Hermitian matrix. Then, the following are equivalent:*

1.  $A$  is  $T$ -EP;
2.  $R(A) = R(TA^*)$ .

The Cayley transform of  $T$ -EP matrix need not be a  $T$ -EP matrix.

EXAMPLE 3.34. Let  $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ . Then,  $TA^*T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$ , which implies that  $R(TA^*T) = R(A)$  and  $AT^*T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$ . Hence,  $A$  is  $T$ -EP but  $C(A) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$  is not  $T$ -EP because  $C(A)T^*T = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \neq C(A)$ .

In the following example, we demonstrate that if a matrix  $A \in M_n(\mathbb{C})$  is  $T$ -EP, then it is not necessary that the factors  $(I + F)^{-1}$  and  $I - F$  are  $T$ -EP.

EXAMPLE 3.35. Let  $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  be  $T$ -EP, where  $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ . Then, although  $I - F = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  is  $T$ -EP,  $(I + F)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$  is not  $T$ -EP because  $(I + F)^{-1}T^*T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \neq (I + F)^{-1}$ .

In contrast, if  $T$  is taken as an involutory Hermitian matrix, then both factors  $(I + F)^{-1}$  and  $I - F$  are  $T$ -EP matrices, as shown next.

THEOREM 3.36. *Let  $A, T \in M_n(\mathbb{C})$  be such that  $F = C(A)$  is well-defined, where  $T$  is an involutory Hermitian matrix. Then,  $A$  is  $T$ -EP if and only if  $(I + F)^{-1}$  and  $I - F$  are  $T$ -EP.*

*Proof.* Suppose that  $A$  is  $T$ -EP. So,  $R(A) = R(TA^*)$  by Theorem 3.33. As  $T$  and  $I + F$  are nonsingular,  $R((I + F)^{-1}) = R(T(I + F)^{-1})$ . By Theorem 3.33,  $(I + F)^{-1}$  is  $T$ -EP. From the proof of Theorem 3.3, we have  $I - F = 2A(I + A)^{-1}$ . Since  $A$  and  $(I + A)^{-1}$  commute, so

$$\begin{aligned} R(T(I - F)^*) &= R(T(2A(I + A)^{-1})^*) \\ &= R(2TA^*(I + A^*)^{-1}) \\ &= R(TA^*) \\ &= R(A) \end{aligned}$$

$$\begin{aligned}
 &= R(2A(I + A)^{-1}) \\
 &= R(I - F).
 \end{aligned}$$

Thus, by Theorem 3.33,  $I - F$  is  $T$ -EP.

Conversely, let  $I - F$  be  $T$ -EP. So,  $R(I - F) = R(T(I - F)^*)$  by Theorem 3.33. Now,

$$\begin{aligned}
 R(A) &= R(2A(I + A)^{-1}) \\
 &= R(I - F) \\
 &= R(T(I - F)^*) \\
 &= R(TA^*).
 \end{aligned}$$

Hence, by Theorem 3.33,  $A$  is  $T$ -EP. □

Malik *et al.* [26] introduced yet another extension of the class of EP matrices called  $m$ -EP in order to find an expression for the *Drazin inverse* [3] of an  $m$ -EP matrix (see Theorem 2.11, [26]). A matrix  $A \in M_n(\mathbb{C})$  is  $m$ -EP [26] and *bi-EP* if  $A^\dagger A^m = A^m A^\dagger$ , where  $m$  is the index of  $A$  and  $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$ , respectively. The authors also proved that a 2-EP matrix is always a bi-EP matrix. We next see that the Cayley transform of an  $m$ -EP matrix is not necessarily an  $m$ -EP matrix for  $m = 2$ .

EXAMPLE 3.37. Let  $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ . The index of  $A$  is 2, i.e., 2 is the smallest positive in-

teger such that  $r(A^2) = r(A^3) = 3$ , and  $A^\dagger A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A^2 A^\dagger$ . So,  $A$  is 2-EP matrix,

but  $C(A) = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  is not a 2-EP matrix because  $C(A)^\dagger C(A)^2 = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C(A)^2 C(A)^\dagger$ .

REMARK 3.38. We have seen that the Cayley transform of a unipotent matrix of index  $m$  yields a nilpotent matrix of the same index  $m$  (see Theorem 3.4, [37]). Since every nilpotent matrix of index  $m$  is trivially an  $m$ -EP matrix, it follows that the Cayley transform of a unipotent matrix of index  $m$  is also an  $m$ -EP matrix. We are led to ask, when is the Cayley transform of an  $m$ -EP again an  $m$ -EP? This is left unanswered.

Lee [24] introduced the concept of the secondary transpose of a matrix as  $A^S = VA^tV$ , where  $V$  is a permutation matrix with ones on the secondary diagonal. It is easy to verify that  $(A + B)^S = A^S + B^S$ ,

$(\lambda A)^S = \lambda A^S$ , and  $(A^S)^{-1} = (A^{-1})^S$ , where  $\lambda \in \mathbb{R}$ . In the same paper, the following types of matrices are defined.

DEFINITION 3.39. ([24, Definition 2.1]) *A matrix  $A \in M_n(\mathbb{R})$  is said to be  $S$ -symmetric and  $S$ -skew symmetric if  $A^S = A$  and  $A^S = -A$ .*

DEFINITION 3.40. ([24, Definition 2.2]) *A matrix  $A \in M_n(\mathbb{R})$  is said to be  $S$ -orthogonal if  $A^S A = A A^S = I$ .*

Notice that if  $T = V$ , then the concept of  $T$ -Hermitian coincides with  $S$ -symmetric for real square matrices. Consequently, the following result, which is a special case of Theorem 3.33, when  $T = V$ , can be stated.

COROLLARY 3.41. *Let  $A \in M_n(\mathbb{R})$ . Then,  $A$  is  $S$ -symmetric if and only if  $C(A)$  is  $S$ -symmetric.*

The Cayley transform of an  $S$ -skew symmetric matrix is always an  $S$ -orthogonal matrix. This is shown next.

THEOREM 3.42. *Let  $A \in M_n(\mathbb{R})$ . Then,  $A$  is  $S$ -skew symmetric if and only if  $C(A)$  is  $S$ -orthogonal.*

*Proof.* Suppose that  $A$  is  $S$ -skew symmetric, i.e.,  $V A^t V = -A$ . Then,

$$\begin{aligned} C(A)^S C(A) &= V C(A)^t V C(A) \\ &= V(I + A^t)^{-1} (I - A^t) V (I + A)^{-1} (I - A) \\ &= V(I + A^t)^{-1} V^2 (I - A^t) V (I + A)^{-1} (I - A) \quad (\because V^2 = I) \\ &= (I + V A^t V)^{-1} (I - V A^t V) (I + A)^{-1} (I - A) \\ &= (I - A)^{-1} (I + A) (I + A)^{-1} (I - A) \\ &= I. \end{aligned}$$

Similarly, we have  $C(A) C(A)^S = I$ . Therefore,  $C(A)$  is a  $S$ -orthogonal matrix. Conversely, if  $C(A)$  is  $S$ -orthogonal, i.e.,  $C(A)^S C(A) = I$ . Then,

$$\begin{aligned} V(I + A^t)^{-1} (I - A^t) V (I + A)^{-1} (I - A) &= I \\ \implies V(I + A^t)^{-1} V^2 (I - A^t) V (I + A)^{-1} (I - A) &= I \\ \implies (I + V A^t V)^{-1} (I - V A^t V) (I - A) (I + A)^{-1} &= I. \end{aligned}$$

Pre and post multiplying the last equation by  $I + V A^t V$  and  $I + A$ , respectively, we get  $(I - V A^t V)(I - A) = (I + V A^t V)(I + A)$ , which yields

$$V A^t V = -A.$$

Hence,  $A$  is an  $S$ -skew symmetric matrix. □

The definition of secondary normal [23] is presented next.

DEFINITION 3.43. *A matrix  $A$  is said to be secondary normal if  $A A^S = A^S A$ .*

The following example demonstrates the fact that if a matrix  $A$  is  $S$ -normal, then it is not necessary that it is  $T$ -normal for  $T = V$ .

EXAMPLE 3.44. Let  $A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ ,  $T = V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ . Then,  $A^S = VA^tV = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ .  
 So,  $A^S A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^S$ , i.e.,  $S$ -normal but  $A$  is not  $T$ -normal since  $AA^tT = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \neq$   
 $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = TA^tA$ .

The following result shows that the Cayley transform of  $S$ -normal matrix is again an  $S$ -normal matrix.

THEOREM 3.45. Let  $A \in M_n(\mathbb{R})$ . Then,  $A$  is an  $S$ -normal matrix if and only if  $F = C(A)$  is a  $S$ -normal matrix.

*Proof.* Suppose that  $A$  is a  $S$ -normal matrix, i.e.,  $A^S A = AA^S$ . Then,

$$\begin{aligned} ((I + A)^{-1})^S (I + A)^{-1} &= (I + A^S)^{-1} (I + A)^{-1} (\because (A + B)^S = A^S + B^S \text{ and } (A^S)^{-1} = (A^{-1})^S) \\ &= [(I + A)(I + A^S)]^{-1} \\ &= [(I + A)^S (I + A)]^{-1} \\ &= (I + A)^{-1} ((I + A)^{-1})^S. \end{aligned}$$

So,  $(I + A)^{-1}$  is a  $S$ -normal matrix. Thus, by Lemma 2.1,  $I + F = 2(I + A)^{-1}$  is an  $S$ -normal matrix. Therefore,  $F = C(A)$  is a  $S$ -normal matrix, which follows by simplifying  $(I + F)^S (I + F) = (I + F)(I + F)^S$ . Conversely, if  $F = C(A)$  is a  $S$ -normal matrix, then  $(I + F)^{-1} = \frac{1}{2}(I + A)$  is a  $S$ -normal matrix, which imply that  $A$  is a  $S$ -normal matrix.  $\square$

Shenoy [10] defined the secondary range symmetric matrix as one that satisfies  $N(A) = N(A^S)$ . The author also established that the condition  $R(A) = R(A^S)$  is equivalent to being secondary range symmetric. Notably, when  $T = V$ , the definitions of  $T$ -EP and secondary range symmetric coincide. Therefore, the following result is a specific case of Theorem 3.36, which is stated as a corollary.

COROLLARY 3.46. Let  $A \in M_n(\mathbb{R})$  be such that  $F = C(A)$  is well-defined. Then,  $A$  is secondary range symmetric if and only if  $(I + F)^{-1}$  and  $I - F$  are secondary range symmetric.

A matrix  $A = [a_{ij}] \in M_n(\mathbb{C})$  is *centrosymmetric* [8] if  $a_{ij} = a_{n+1-i, n+1-j}$ . Equivalently,  $A$  is centrosymmetric if  $VA = AV$ , where  $V$  is a permutation matrix defined as in the definition of secondary transpose of a matrix.

Centrosymmetric matrices were studied thoroughly in the 1962 work of Collar [8] and appear in the numerical solution of certain differential equations [6].

Notice that if  $A \in M_n(\mathbb{R})$  is both symmetric and centrosymmetric, i.e.,  $A^t = A$  and  $VAV = A$ , then  $VA^tV = A$ , i.e.,  $A$  is an  $S$ -symmetric matrix. Now, we show that the Cayley transform of a centrosymmetric matrix is a centrosymmetric matrix.

THEOREM 3.47. Let  $A \in M_n(\mathbb{C})$  be such that  $C(A)$  is well-defined. Then,  $A$  is a centrosymmetric matrix if and only if  $C(A)$  is a centrosymmetric matrix.

*Proof.* Suppose that  $A$  is a centrosymmetric matrix, i.e.,  $VA = AV$ . Then, we have

$$\begin{aligned}
 VC(A) &= V(I + A)^{-1}(I - A) \\
 &= V^{-1}(I + A)^{-1}(I - A) \quad (\because V = V^{-1}) \\
 &= [(I + A)V]^{-1}(I - A) \\
 &= (V + AV)^{-1}(I - A) \\
 &= (V + VA)^{-1}(I - A) \\
 &= (I + A)^{-1}V(I - A) \\
 &= (I + A)^{-1}(V - VA) \\
 &= (I + A)^{-1}(V - AV) \\
 &= (I + A)^{-1}(I - A)V \\
 &= C(A)V.
 \end{aligned}$$

Thus,  $C(A)$  is a centrosymmetric matrix.

Conversely, if  $F = C(A)$  is a centrosymmetric matrix, then by Lemma 2.1,  $A = C(F)$  is a centrosymmetric matrix.  $\square$

**3.3. Tridiagonal and block triangular matrices.** A matrix  $A = [a_{ij}] \in M_n(\mathbb{C})$  is called a *tridiagonal matrix* [20] if  $a_{ij} = 0$  when  $|i - j| > 1$ . The Cayley transform of a tridiagonal matrix is not necessarily a tridiagonal matrix. This is illustrated in the following example.

EXAMPLE 3.48. Let  $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  be a tridiagonal matrix. Then,  $C(A) = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -1 & 1 & -1 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$  is not a tridiagonal matrix.

Barreras and Pea [1] introduced the following condition for a matrix  $A = [a_{ij}]$ .

$$(3.2) \quad a_{i,i-1} \neq 0 \Rightarrow a_{i+1,i} = 0, a_{i-1,i} \neq 0 \Rightarrow a_{i,i+1} = 0,$$

for any  $i = 2, \dots, n - 1$ . They provided a characterization of tridiagonal nonsingular  $M$ -matrix using the aforementioned condition, which is outlined next.

THEOREM 3.49. ([1, Theorem 1]) Let  $A \in M_n(\mathbb{R})$  be a nonsingular  $M$ -matrix. Then, the following conditions are equivalent:

1.  $A^{-1}$  is tridiagonal;
2.  $A$  is tridiagonal and satisfies (3.2).

The next result shows that a tridiagonal nonsingular  $M$ -matrix can be expressed as a product of two tridiagonal matrices in terms of the Cayley transform.

THEOREM 3.50. Let  $A \in M_n(\mathbb{R})$  be a nonsingular  $M$ -matrix such that (3.2) hold. Then,  $A$  is a tridiagonal matrix if and only if  $(I + F)^{-1}$  and  $I - F$  are tridiagonal matrices.

*Proof.* Suppose that  $A$  is a tridiagonal matrix. We have  $(I + F)^{-1} = \frac{1}{2}(I + A)$ , from Lemma 2.1. So,  $(I + F)^{-1}$  is a tridiagonal matrix. To show  $I - F$  is a tridiagonal matrix, we will apply Theorem 3.49 to  $C = I + A$ . First note that  $A$  being  $M$ -matrix implies that  $C$  is a nonsingular  $M$ -matrix (see Lemma

4.1, [2]). Now, the nondiagonal entries of  $C$  are same as  $A$ . So, it satisfies (ii) of Theorem 3.49. Hence,  $C^{-1} = (I + A)^{-1}$  is tridiagonal. Therefore,  $I - F = 2(I - (I + A)^{-1})$  is a tridiagonal matrix. Conversely, if  $(I + F)^{-1}$  is a tridiagonal matrix, then  $A = 2(I + F)^{-1} - I$  is a tridiagonal matrix.  $\square$

Consider next a *block upper triangular matrix* [20], namely,  $A \in M_n(\mathbb{C})$  of the form

$$A = \begin{bmatrix} A_{11} & \star & \star \\ & \ddots & \star \\ O & & A_{kk} \end{bmatrix},$$

in which  $A_{ii} \in M_{n_i}(\mathbb{C})$ ,  $i = 1, \dots, k$ ,  $\sum_{i=1}^k n_i = n$ , and all blocks below the block diagonal are zero. The inverse of a  $n \times n$  block upper triangular matrix is recalled next.

**THEOREM 3.51.** ([25, Corollary 3.2]) *Let  $A = \begin{bmatrix} B & D \\ O & E \end{bmatrix} \in M_n(\mathbb{C})$  be a block upper triangular matrix.*

*Then,  $A$  is invertible if and only if  $B$  and  $E$  are invertible, and its inverse is  $\begin{bmatrix} B^{-1} & -B^{-1}DE^{-1} \\ O & E^{-1} \end{bmatrix}$ .*

In the following result, we show that the Cayley transform of a block upper triangular matrix is again a block upper triangular matrix.

**THEOREM 3.52.** *Let  $A = \begin{bmatrix} B & D \\ O & E \end{bmatrix} \in M_n(\mathbb{C})$  be a block upper triangular matrix such that  $C(A)$  is well-defined, where  $B \in M_n(\mathbb{C})$  and  $E \in M_{n-r \times n-r}(\mathbb{C})$ . Then,  $C(A)$  is also a block upper triangular matrix and*

$$C(A) = \begin{bmatrix} C(B) & -2(I + B)^{-1}D(I + E)^{-1} \\ O & C(E) \end{bmatrix}.$$

*Proof.* It is easy to see that  $I + B$  and  $I + E$  are invertible matrices. Then, from Theorem 3.51, we get  $(I + A)^{-1} = \begin{bmatrix} (I + B)^{-1} & -(I + B)^{-1}D(I + E)^{-1} \\ O & (I + E)^{-1} \end{bmatrix}$ . Now, from  $C(A) = (I + A)^{-1}(I - A)$ , we have

$$\begin{aligned} C(A) &= \begin{bmatrix} (I + B)^{-1} & -(I + B)^{-1}D(I + E)^{-1} \\ O & (I + E)^{-1} \end{bmatrix} \begin{bmatrix} I - B & -D \\ O & I - E \end{bmatrix} \\ &= \begin{bmatrix} (I + B)^{-1}(I - B) & -(I + B)^{-1}D - (I + B)^{-1}D(I + E)^{-1}(I - E) \\ O & (I + E)^{-1}(I - E) \end{bmatrix} \\ &= \begin{bmatrix} (I + B)^{-1}(I - B) & -(I + B)^{-1}D((I + E)^{-1}(I + E) + (I + E)^{-1}(I - E)) \\ O & (I + E)^{-1}(I - E) \end{bmatrix} \\ &= \begin{bmatrix} (I + B)^{-1}(I - B) & -2(I + B)^{-1}D(I + E)^{-1} \\ O & (I + E)^{-1}(I - E) \end{bmatrix} \\ &= \begin{bmatrix} C(B) & -2(I + B)^{-1}D(I + E)^{-1} \\ O & C(E) \end{bmatrix}. \end{aligned}$$

Thus,  $C(A)$  is a block upper triangular matrix.  $\square$

Similarly, we can prove that if  $A = \begin{bmatrix} B & O \\ D & E \end{bmatrix}$  is a block lower triangular matrix, then  $C(A)$  is also a block lower triangular matrix and  $C(A) = \begin{bmatrix} C(B) & O \\ -2(I + E)^{-1}D(I + B)^{-1} & C(E) \end{bmatrix}$ .

Next, we show that the Cayley transform preserves similarity.

**THEOREM 3.53.** *Let  $A, B \in M_n(\mathbb{C})$  be such that  $C(A)$  is well-defined. Then,  $A$  is similar to  $B$ , i.e.,  $A = PBP^{-1}$  for some nonsingular matrix  $P$  if and only if  $C(A)$  is similar to  $C(B)$ .*

*Proof.* Suppose that  $A$  is similar to  $B$ , i.e.,  $A = PBP^{-1}$  for some nonsingular matrix  $P$ . So,  $C(B)$  is also well-defined because similar matrices have the same eigenvalue. Then,

$$\begin{aligned} C(A) &= C(PBP^{-1}) \\ &= (I + PBP^{-1})^{-1}(I - PBP^{-1}) \\ &= P(I + B)^{-1}(I - B)P^{-1} \\ &= PC(B)P^{-1}. \end{aligned}$$

Hence,  $C(A)$  is similar to  $C(B)$ .

Conversely, we have that  $C(A)$  is similar to  $C(B)$ , i.e.,  $C(A) = PC(B)P^{-1}$ . Then,

$$(3.3) \quad (I + A)^{-1}(I - A) = (I - PBP^{-1})(I + PBP^{-1})^{-1}.$$

Pre and post-multiplying (3.3) by  $I + A$  and  $I + PBP^{-1}$ , we have  $(I - A)(I + PBP^{-1}) = (I + A)(I - PBP^{-1})$ , which yields  $A = PBP^{-1}$ , i.e.,  $A$  is similar to  $B$ .  $\square$

It is easy to see that the Cayley transform of a diagonal matrix remains a diagonal matrix whenever it is well-defined. Therefore, the following result is a consequence of Theorem 3.53 and is stated as a remark.

**REMARK 3.54.** *If  $A$  is a diagonalizable matrix such that  $C(A)$  is well-defined, then  $C(A)$  is a diagonalizable matrix.*

Hartwig and Spindelbck [14] presented the following decomposition of a matrix.

**THEOREM 3.55.** ([14, Corollary 6]) *Let  $A \in M_n(\mathbb{C})$ . Then,*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} U^*,$$

where  $U$  is the unitary matrix, and  $K \in M_r(\mathbb{C})$  and  $L \in M_{r \times r-n}(\mathbb{C})$  such that  $KK^* + LL^* = I_r$ ,  $\Sigma = \text{diag}(\sigma_1 I_{r_1}, \dots, \sigma_t I_{r_t})$  is a diagonal matrix, the diagonal entries  $\sigma_i$  being singular values of  $A$ ,  $\sigma_1 > \dots > \sigma_t > 0$  and  $r_1 + \dots + r_t = r = r(A)$ .

We conclude our work with an expression of the Cayley transform of a matrix through Hartwig and Spindelbck decomposition.

**THEOREM 3.56.** *Let  $A \in M_n(\mathbb{C})$ . Then,*

$$C(A) = U \begin{bmatrix} (I + \Sigma K)^{-1}(I - \Sigma K) & -2(I + \Sigma K)^{-1}\Sigma L \\ O & I \end{bmatrix} U^*,$$

where  $U$  is the unitary matrix, and  $K \in M_r(\mathbb{C})$  and  $L \in M_{r \times r-n}(\mathbb{C})$  such that  $KK^* + LL^* = I_r$ ,  $\Sigma = \text{diag}(\sigma_1 I_{r_1}, \dots, \sigma_t I_{r_t})$  is a diagonal matrix, the diagonal entries  $\sigma_i$  being singular values of  $A$ ,  $\sigma_1 > \dots > \sigma_t > 0$  and  $r_1 + \dots + r_t = r = r(A)$ .

*Proof.* We have  $I + A = U \begin{bmatrix} I + \Sigma K & \Sigma L \\ O & I \end{bmatrix} U^*$  and  $I - A = U \begin{bmatrix} I - \Sigma K & -\Sigma L \\ O & I \end{bmatrix} U^*$  by Theorem 3.55. Since  $I + A$  is invertible, it follows that  $I + \Sigma K$  is invertible. So, by Theorem 3.51, we get

$$(I + A)^{-1} = U \begin{bmatrix} (I + \Sigma K)^{-1} & -(I + \Sigma K)^{-1} \Sigma L \\ O & I \end{bmatrix} U^*.$$

Now, substituting the values of  $(I + A)^{-1}$  and  $I - A$  in  $C(A) = (I + A)^{-1}(I - A)$ , we obtain

$$\begin{aligned} C(A) &= U \begin{bmatrix} (I + \Sigma K)^{-1} & -(I + \Sigma K)^{-1} \Sigma L \\ O & I \end{bmatrix} \begin{bmatrix} I - \Sigma K & -\Sigma L \\ O & I \end{bmatrix} U^* \\ &= U \begin{bmatrix} (I + \Sigma K)^{-1}(I - \Sigma K) & -(I + \Sigma K)^{-1} \Sigma L - (I + \Sigma K)^{-1} \Sigma L \\ O & I \end{bmatrix} U^* \\ &= U \begin{bmatrix} (I + \Sigma K)^{-1}(I - \Sigma K) & -2(I + \Sigma K)^{-1} \Sigma L \\ O & I \end{bmatrix} U^*. \quad \square \end{aligned}$$

**4. Review of Cayley transforms.** We conclude this paper with a comprehensive table, summarizing the known results for Cayley transforms of various matrix classes proved in the literature.

Table 1: Summary of results on Cayley transform of various matrix classes

	Class	Result	Where
1.	$P$ -matrix	Let $A \in M_n(\mathbb{C})$ be a $P$ -matrix. Then, $F = C(A)$ is well-defined, and both $I - F$ and $I + F$ are $P$ -matrices. In particular, $A = (I + F)^{-1}(I - F)$ is a factorization of a $P$ -matrix into (commuting) $P$ -matrices.	Theorem 3.1, [11]
2.	$M$ -matrix	Let $A \in M_n(\mathbb{R})$ . Then, $A$ is an $M$ -matrix if and only if $F = C(A)$ is well-defined, and $(I + F)^{-1}$ and $I - F$ are $M$ -matrices.	Theorem 4.1, [11], Theorem 5.2, [31]
3.	Positive definite matrix	<ol style="list-style-type: none"> <li>Let <math>A \in M_n(\mathbb{C})</math>. Then, <math>A</math> is a positive definite matrix if and only if <math>F = C(A)</math> is well-defined, and both <math>I + F</math> and <math>I - F</math> are both positive definite matrices.</li> <li>Let <math>A \in M_n(\mathbb{C})</math> be a positive definite matrix. Then, <math>C(A)</math> is a positive definite matrix if and only if <math>\sigma(A) \in (0, 1)</math>.</li> </ol>	Theorem 3.2, [31], Theorem 4.4, [11]
5.	Hermitian matrix	Let $A \in M_n(\mathbb{C})$ be such that $C(A)$ is well-defined. Then, $A$ is Hermitian if and only if $C(A)$ is Hermitian.	Theorem 3.1, [31]
6.	Skew-Hermitian matrix	Let $A \in M_n(\mathbb{C})$ be such that $C(A)$ is well-defined. Then, $A$ is skew-Hermitian if and only if $C(A)$ is unitary.	[5]

7.	Almost skew-Hermitian matrix	<ol style="list-style-type: none"> <li>1. Let <math>A \in M_n(\mathbb{R})</math> be an almost skew-symmetric matrix with <math>\delta(A) &gt; 0</math>. Then, <math>\text{rank}(I - FF^t) = 1</math>.</li> <li>2. Let <math>A \in M_n(\mathbb{C})</math> be such that <math>-1 \notin \sigma(A)</math>. Then, <math>A</math> is an almost skew-Hermitian matrix if and only if <math>F = C(A)</math> is well-defined and <math>F^*F</math> or <math>FF^*</math> is unitarily similar to <math>D = \text{diag}(1, -a, 1, \dots, 1)</math>, with <math>a = 0</math>. The scalar <math>a \in \mathbb{R}</math> is the sole nonzero eigenvalue of <math>I - F^*F</math>.</li> <li>3. Let <math>A \in M_n(\mathbb{C})</math> be an almost skew Hermitian matrix such that <math>-1 \notin \sigma(A)</math>. Then, <math>F = C(A)</math> is well-defined, and the following implications hold:                     <ol style="list-style-type: none"> <li>(a) <math>\delta(A) &gt; 0 \implies F^*F</math> or <math>FF^*</math> is diagonally congruent to a <math>Z</math>-matrix.</li> <li>(b) <math>\delta(A) &lt; 0 \implies F^*F</math> or <math>FF^*</math> is diagonally congruent to a nonnegative matrix.</li> </ol> </li> </ol>	Theorem 5.1, [7], Theorems 4.2 & 4.4, [31]
8.	$H$ -matrix	Let $A \in M_n(\mathbb{R})$ be invertible matrix such that $SAS = M(A)$ for some signature matrix $S$ . Then, $A \in H_I$ if and only if $F = C(A)$ is well-defined and both $(I + F)^{-1}$ and $I - F$ are in $H_I$ .	Theorem 5.3, [31]
9.	Inverse $H$ -matrix	Let $A \in M_n(\mathbb{R})$ be such that $SA^{-1}S = M(A^{-1})$ for some signature matrix $S$ . Then, $A$ is an inverse $H$ -matrix if and only if $F = C(A)$ is well-defined and both $(I + F)^{-1}$ and $I - F$ are inverse $H$ -matrices.	Theorem 5.5, [31]
10.	Semipositive matrix	Let $A \in M_n(\mathbb{R})$ be an invertible matrix such that $F = C(A)$ is well-defined if and only if $I + F$ and $I - F$ are semipositive with common semipositivity vector.	Theorem 6.5, [31]
11.	Mimes	Let $A \in M_n(\mathbb{R})$ be mime and let $F = C(A)$ be the Cayley transform of $A$ . Then, $I + F$ and $I - F$ are mimes.	Theorem 3.18, [27]
12.	Normal and $J$ -symplectic matrix	Let $A \in M_{2n}(\mathbb{C})$ be such that $\det(I_{2n} + A) \neq 0$ . Then, $A$ is normal and $J$ -symplectic if and only if $C(A)$ is normal and $J$ -Hamiltonian.	Lemma 4.1, [13]
13.	Toeplitz matrix	<ol style="list-style-type: none"> <li>1. Let <math>A \in M_n(\mathbb{R})</math> be a matrix of the form <math>M_n(a, b, c)</math>. Then, <math>A</math> is a Toeplitz matrix if and only if <math>(I + F)^{-1}</math> and <math>I - F</math> are Toeplitz matrices.</li> <li>2. Let <math>A \in M_n(\mathbb{R})</math> be such that <math>C(A)</math> is well-defined. If <math>A</math> is a Toeplitz matrix, then <math>(I + F)^{-1}</math> and <math>I - F</math> are both persymmetric matrices.</li> </ol>	Theorems 3.1 & 3.2, [37]
14.	Circulant matrix	Let $A \in M_n(\mathbb{R})$ be a matrix such that $C(A)$ is well-defined. Then, $A$ is circulant if and only if $C(A)$ is circulant.	Theorem 3.3, [37]
15.	Unipotent matrix	Let $A \in M_n(\mathbb{R})$ . Then, $A$ is a unipotent matrix if and only if $C(A)$ is a nilpotent matrix.	Theorem 3.4, [37]

16.	Dual matrix	<ol style="list-style-type: none"> <li>1. Let <math>\widehat{A} = A + \epsilon B</math> be a dual matrix such that <math>C(A)</math> is well-defined, where <math>A, B \in M_n(\mathbb{R})</math>. Then, the Cayley transform of <math>\widehat{A}</math>, i.e., <math>C(\widehat{A})</math> is well-defined and <math>C(\widehat{A}) = C(A) - \epsilon(2(I + A)^{-1}B(I + A)^{-1})</math>.</li> <li>2. Let <math>\widehat{A} = A + \epsilon B</math> be a dual matrix such that <math>C(A)</math> is well-defined, where <math>A, B \in M_n(\mathbb{R})</math>. Then, <math>\widehat{A}</math> is symmetric if and only if <math>C(\widehat{A})</math> is symmetric.</li> <li>3. Let <math>\widehat{A} = A + \epsilon B</math> be a dual matrix such that <math>C(A)</math> is well-defined, where <math>A, B \in M_n(\mathbb{R})</math>. Then, <math>\widehat{A}</math> is orthogonal if and only if <math>C(\widehat{A})</math> is skew-symmetric.</li> </ol>	Theorems 3.5, 3.9 & 3.10, [37]
17.	Convergent matrix	Let $A \in M_n(\mathbb{R})$ . Then, $A$ is a convergent matrix if and only if $A = C(B)$ , where $B$ is a stable matrix.	Theorem 8, [15]
18.	Totally non-negative matrix	Let $A \in M_n(\mathbb{R})$ be an irreducible matrix. Then, $A$ is essentially non-negative tridiagonal matrix with $\rho(A) < 1$ if and only if $I + A$ and $(I - A)^{-1}$ are totally nonnegative matrices. In particular, $C(-A) = (I - A)^{-1}(I + A)$ is a factorization into totally nonnegative matrices.	Theorem 4.5, [11]

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