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ON MEANS OF DETERMINANTS OF MATRIX PROJECTIONS*

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Abstract. We establish determinantal counterparts of classical integral-geometric representations of quermassintegrals of convex bodies in the case of mixed discriminants of positive semidefinite matrices and the identity matrix. In particular, we derive an analogue of the Cauchy–Kubota formulae for those mixed discriminants. This note is inspired by a result in [4], in which the average of the determinants of the projections of a positive semidefinite matrix onto $(n-1)$-linear subspaces is proven to be equal to a matrix analogue of the surface area of a convex body. Further, the one-to-one relation between positive semidefinite matrices and centered ellipsoids allows us to provide this notion of projection of a matrix with a geometrical insight.

Key words. Positive semidefinite matrices, Mixed discriminant, Ellipsoids, Surface area.

AMS subject classifications. Primary 15A15, 15A45; secondary 15B48, 52A20, 52A40.

1. Introduction. Aleksandrov introduced the mixed discriminant of positive semidefinite quadratic forms in his second proof of the Aleksandrov–Fenchel inequality for convex bodies in [1]. Inequalities for mixed volumes have been investigated in the context of mixed discriminants, see [2, 8, 9, 10, 12], and a characterization of the mixed discriminant was obtained in [7].

Let $\mathcal{S}^n_+$ denote the space of symmetric and positive semidefinite $n\times n$ real matrices. Let $Q_1,\dots,Q_m \in \mathcal{S}^n_+$, $m\in\mathbb{N}$, and let $\lambda_1,\dots,\lambda_m \geq 0$. The determinant of $\lambda_1 Q_1+\dots+\lambda_m Q_m$, when not zero, is a homogeneous polynomial of degree n in $\lambda_1,\dots,\lambda_m$ with coefficients depending only on $Q_1,\dots,Q_m$, i.e., there exists $\mathrm{D}\colon (\mathcal{S}^n_+)^n \to \mathbb{R}$, such that

$$\det\left(\sum_{i=1}^m \lambda_i Q_i\right) = \sum_{i_1,\dots,i_n=1}^m \lambda_{i_1}\cdots\lambda_{i_m}\mathrm{D}(Q_{i_1},\dots,Q_{i_m}). \tag{1.1}$$

We call the uniquely determined symmetric coefficient $\mathrm{D}(Q_1,\dots,Q_n)$ the mixed discriminant of $Q_1,\dots,Q_n$. We refer to Section 3 for further details and properties of the mixed discriminant D.

The identity (1.1) is a matrix counterpart of the well-known Minkowski Theorem (see Theorem 2.1) within the realm of convex geometry. Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$, i.e., nonempty convex and compact subsets of $\mathbb{R}^n$, and let $\mathrm{vol}_n(\cdot)$ denote the volume, i.e., the n-dimensional Lebesgue measure in $\mathbb{R}^n$. For $K_1,\dots,K_m\in\mathcal{K}^n$ and $\lambda_1,\dots,\lambda_m\geq 0$, $m\in\mathbb{N}$, the volume of the Minkowski combination $\lambda_1K_1+\dots+\lambda_mK_m$, when not zero, is a homogeneous polynomial of degree n in $\lambda_1,\dots,\lambda_m$ with coefficients given by the mixed volume $\mathrm{V}\colon(\mathcal{K}^n)^n\to\mathbb{R}$ (see Theorem 2.1), i.e.,

$$\mathrm{vol}_n\left(\sum_{i=1}^m \lambda_i K_i\right) = \sum_{i_1,\dots,i_n=1}^m \lambda_{i_1}\cdots\lambda_{i_m}\mathrm{V}(K_{i_1},\dots,K_{i_m}). \tag{1.2}$$

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The quermassintegrals of a convex body $K \in \mathcal{K}^n$ are particular instances of mixed volumes. We denote the n-dimensional closed Euclidean unit ball by $B_n \subseteq \mathbb{R}^n$ and its volume by κ_n. Then,

$$\mathrm{W}_k(K) = \mathrm{V}(\underbrace{K, \dots, K}_{(n-k)\text{-times}}, \underbrace{B_n, \dots, B_n}_{k\text{-times}}), \tag{1.3}$$

is the k-th quermassintegral of K for $0 \le k \le n$. We note that $\mathrm{W}_0(K) = \mathrm{vol}_n(K)$, and $n\mathrm{W}_1(K) = \mathrm{S}(K)$ is the surface area of K. The (Minkowski) surface area of a convex body K can be defined by the following variational formula [11, page 104] as follows (we refer the reader to Section 2 for further details),

$$\mathrm{S}(K) = \lim_{\rho \to 0^+} \frac{\mathrm{vol}(K + \rho B_n) - \mathrm{vol}(K)}{\rho}. \tag{1.4}$$

For $Q \in \mathcal{S}_+^n$ and I_n the $n \times n$ identity matrix, we consider the family of mixed discriminants defined as

$$\mathrm{D}_k(Q) := \mathrm{D}(\underbrace{Q, \dots, Q}_{(n-k)\text{-times}}, \underbrace{I_n, \dots, I_n}_{k\text{-times}}), \quad 0 \le k \le n, \tag{1.5}$$

and refer to D_k as the k-th quermassintegral of the matrix Q following (1.3).

As particular instances of mixed discriminants, the quermassintegrals of a matrix Q appear as coefficients in (1.1), when $m = 2$, and only the two matrices $Q, I_n \in \mathcal{S}_+^n$ are considered. More precisely, (1.1) and (1.5) yield

$$\det(Q + \lambda I_n) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mathrm{D}_k(Q). \tag{1.6}$$

This is a counterpart of Steiner's formula for convex bodies (see Section 2 for details). We observe that D_1 can be obtained from (1.6) by the following limit (cf. (1.4))

$$\lim_{\rho \to 0^+} \frac{\det(Q + \rho I_n) - \det(Q)}{\rho} = n\mathrm{D}_1(Q), \tag{1.7}$$

which provides us with a motivation to set $\mathrm{S}(Q) := n\mathrm{D}_1(Q)$ as a matrix counterpart of the surface area of a convex body, following (1.4).

Our primary aim in this note is to provide a matrix counterpart of the Cauchy–Kubota integral-geometric representation of the quermassintegrals of a convex body. The latter establishes that the quermassintegrals of a convex body are the averages of quermassintegrals of orthogonal projections of the convex body onto arbitrary linear subspaces. We write $\mathcal{L}_k^n$ for the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$. For $L \in \mathcal{L}_k^n$, we denote by p_L the orthogonal projection onto the subspace L. A particular case of the Cauchy–Kubota formula is the so-called Cauchy surface area formula (see Theorem 2.3 for details), which establishes

$$\mathrm{S}(K) = \frac{1}{\kappa_{n-1}} \int_{u \in \mathbb{S}^{n-1}} \mathrm{vol}_{n-1}(p_{u^\perp}(K))\, d\sigma(u), \tag{1.8}$$

where σ denotes the spherical Lebesgue measure on the $(n-1)$-dimensional Euclidean unit sphere $\mathbb{S}^{n-1}$, with $\sigma(\mathbb{S}^{n-1}) = n\kappa_n$, $\mathrm{vol}(B_n) = \kappa_n$, and, for $u \in \mathbb{S}^{n-1}$, $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to u.

In [4], Barvinok proved a matrix counterpart of the Cauchy surface area formula for positive semidefinite matrices, where the role of the volume is taken by the determinant. The result reads as follows:

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THEOREM 1.1. *[4] Let $Q \in \mathcal{S}_+^n$ be an $n \times n$ real, positive semidefinite matrix. Then,*

$$\mathrm{D}_1(Q) = \frac{1}{n\kappa_n} \int_{u \in \mathbb{S}^{n-1}} \det(P_{u^\perp}(Q))\, d\sigma(u). \tag{1.9}$$

Here, $P_{u^\perp}(Q)$ denotes the orthogonal projection of the matrix Q onto the $(n-1)$-dimensional subspace orthogonal to $u \in \mathbb{S}^{n-1}$, denoted by $u^\perp$.

Every centered ellipsoid $E \subseteq \mathbb{R}^n$ is uniquely determined by a positive semidefinite matrix Q_E. Moreover, every positive semidefinite matrix $Q \in \mathcal{S}_+^n$ uniquely determines an ellipsoid E_Q. If E is any centered ellipsoid, then $E_{Q_E} = E$. This bijection extends beyond the one-to-one correspondence to include additions: the L_2-addition of convex bodies (in particular, ellipsoids) and the usual sum of matrices. We refer the reader to Section 2 for details. For a subspace $L \in \mathcal{L}_k^n$, provided an orthonormal basis B of L has been fixed, the projection of a positive semidefinite matrix Q onto L is defined as the unique $k \times k$ positive semidefinite matrix, given in basis B, defining the ellipsoid $p_L(E_Q)$, which is the orthogonal projection of the ellipsoid E_Q onto L. For further details, we refer to Section 4.

Inspired by Theorem 1.1 and following the ideas of its proof, we state a matrix counterpart of the Cauchy–Kubota formula for convex bodies. Our main result is the following theorem. Let ν_k denotes the $O(n)$-invariant Haar measure on $\mathcal{L}_k^n$. For $m \neq n$, we write $\mathrm{D}^{(m)}$ for the mixed discriminant of matrices of size $m \times m$. If Q is positive semidefinite and of size $m \times m$, we write

$$\mathrm{D}_k^{(m)}(Q) = \mathrm{D}^{(m)}(\underbrace{Q, \dots, Q}_{(m-k)\text{-times}}, \underbrace{I_m, \dots, I_m}_{k\text{-times}}), \quad 0 \le k \le m.$$

THEOREM A. *Let $Q \in \mathcal{S}_+^n$ be a real, positive semidefinite matrix of size $n \times n$. Let $0 \le j \le k \le n$, then*

$$\mathrm{D}_{n-j}(Q) = \int_{L \in \mathcal{L}_k^n} \mathrm{D}_{k-j}^{(k)}(P_L(Q))\, d\nu_k(L). \tag{1.10}$$

First, we provide a different proof of the identity (1.9), i.e., of Theorem 1.1, following the proof of the Cauchy surface area formula for convex bodies given in [21]. In addition, we prove further results on $\mathrm{S}(Q)$ as in the above spirit, namely, connecting $\mathrm{S}(Q)$ to the projection of the matrix Q. We further consider a slightly more general notion, which can be seen as the matrix counterpart of the relative surface area of a convex body, with respect to a second one (see [19]). For $Q, R \in \mathcal{S}_+^n$, (1.1) yields

$$\lim_{\rho \to 0^+} \frac{\det(Q + \rho R) - \det(Q)}{\rho} = n\mathrm{D}(\underbrace{Q, \dots, Q}_{(n-1)\text{-times}}, R) =: \mathrm{S}(Q; R), \tag{1.11}$$

where we introduced the notation $\mathrm{S}(Q;R)$ in analogy to its convex body counterpart.

For the proof of Theorem A, we extend the methods used in [4, Lemma 2.3] to other mixed discriminants and to the projection of a matrix Q onto an arbitrary linear subspace of $\mathbb{R}^n$ of dimension $1 \le k \le n-1$.

We remark that Theorem A is the matrix counterpart of the so-called Cauchy–Kubota formula for convex bodies, which is an extension of the Cauchy surface area measure (1.8).

The following statement is a direct consequence of Theorem A. Its proof follows the lines of the convex bodies counterpart (see [13, Theorem 5.7]) and generalizes the case $k = 1$ shown in [4, Theorem 2.4].

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Corollary B. *Let $0 \le k \le n$ and $Q_1, \dots, Q_k \in \mathcal{S}^n_+$. Then,*

$$\mathrm{D}(Q_1, \dots, Q_k, \underbrace{I_n, \dots, I_n}_{(n-k)\text{-}times}) = \int_{L \in \mathcal{L}^n_k} \mathrm{D}^{(k)}(P_L(Q_1), \dots, P_L(Q_k))\, d\nu_k(L).$$

The paper is structured as follows. In Section 2, we gather several results on convex geometry, which motivate and inspire the main results in the paper. Among others, we state the Cauchy surface area formula and the Cauchy–Kubota formula for convex bodies, counterparts of which will be proven in the context of positive semidefinite matrices in Section 5. In Section 3, we deal with details and properties of mixed discriminants and of projections of matrices in $\mathcal{S}^n_+$, which are necessary for the rest of this note. In Section 4, we state and exploit the basics of the intertwining between the theory of symmetric and positive semidefinite matrices, and (centered) ellipsoids, presenting results and properties of matrix counterparts of classical magnitudes for convex bodies, as the surface area or, more generally, quermassintegrals, for matrices in $\mathcal{S}^n_+$. The main results of this work are contained in Section 5, where we first provide a new proof of Theorem 1.1, then we prove Theorem A, and finally obtain other results linking positive semidefinite matrices and ellipsoids.

2. Background results from convex geometry: motivation. Our ambient space is the n-dimensional Euclidean space $\mathbb{R}^n$, endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm $||\cdot||$. We denote by $\mathcal{K}^n$ the set of all nonempty convex and compact subsets, i.e., convex bodies, in $\mathbb{R}^n$. The Minkowski sum of the convex bodies K, L is defined as $K + L := \{k + l \ : \ k \in K, \ l \in L\}$. Moreover, if $\alpha \ge 0$, then $\alpha K := \{\alpha k \ : \ k \in K\}$. For every $K, L \in \mathcal{K}^n$ and $\alpha, \beta \ge 0$, we have that $\alpha K + \beta L$ is again a convex body. The dimension of $K \in \mathcal{K}^n$, denoted by $\dim(K)$, is defined as the dimension of the affine space $\mathrm{aff}(K)$ spanned by K. For a k-dimensional linear subspace $L \in \mathcal{L}^n_k$, we recall that $p_L(K)$ denotes the orthogonal projection of K onto L. Next, we introduce mixed volumes by means of the Minkowski theorem (we refer the reader to [20, Chapter 5] for equivalent definitions of this notion).

Theorem 2.1. *[20] There exists a unique nonnegative symmetric function $\mathrm{V}\colon (\mathcal{K}^n)^n \to \mathbb{R}$ such that for every $m \in \mathbb{N}$ and for every choice of $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \ge 0$, we have*

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{1 \le i_1, \dots, i_n \le m} \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{V}(K_{i_1}, \dots, \mathcal{K}_{i_n}).$$

The function V *is called mixed volume.*

We observe that the quermassintegrals of a convex body K introduced in (1.3)

$$\mathrm{W}_k(K) = \mathrm{V}(\underbrace{K, \dots, K}_{(n-k)\text{-times}}, \underbrace{B_n, \dots, B_n}_{k\text{-times}}),$$

are particular cases of mixed volumes. In particular, we recall that $\mathrm{W}_0(K) = \mathrm{vol}_n(K)$ and, by (1.4), $\mathrm{S}(K) = n\mathrm{W}_1(K)$ is the surface area of K. The quermassintegrals of K can also be introduced directly through the polynomiality given in Theorem 2.1, namely, via Steiner's formula.

Theorem 2.2 (Steiner's formula). *[20, Section 4.2] For every $\rho \ge 0$ and $K \in \mathcal{K}^n$*

$$\mathrm{vol}_n(K + \rho B_n) = \sum_{k=0}^{n} \binom{n}{k} \rho^k \mathrm{W}_k(K).$$

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The Cauchy surface area formula retrieves the surface area of a convex body as an average of the $(n-1)$-volume of its projections. Recall that κ_k stands for the k-Lebesgue measure of the Euclidean unit ball B_k in $\mathbb{R}^k$.

THEOREM 2.3. *[20, Section 5.3] Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$. Then,*

$$\mathrm{S}(K) = \frac{1}{\kappa_{n-1}} \int_{u\in\mathbb{S}^{n-1}} \mathrm{vol}_{n-1}(p_{u^\perp}(K))\, d\sigma(u). \tag{2.1}$$

The generalization of Theorem 2.3 expressing the quermassintegral of a convex body K by the mean of quermassintegrals of projections of K onto k-dimensional subspace of $\mathbb{R}^n$ is called Cauchy–Kubota formula. For $L \in \mathcal{L}_k^n$, we denote by $\mathrm{W}_j^{(k)}(p_L(K))$ the j-th quermassintegral of $p_L(K)$ defined on the ambient space L, with $0 \le j \le k \le n$.

THEOREM 2.4 (Cauchy–Kubota formula). *[20, Section 5.3] Let $K \in \mathcal{K}^n$, then*

$$\mathrm{W}_{n-j}(K) = \frac{\kappa_n}{\kappa_k} \int_{L\in\mathcal{L}_k^n} \mathrm{W}_{k-j}^{(k)}(p_L(K))\, d\nu_k(L), \tag{2.2}$$

for $0 \le j \le k \le n$.

Let $K \in \mathcal{K}^n$. The support function of K is defined by $h(K,x) = \max\{\langle x, y\rangle : y \in K\}$ for all $x \in \mathbb{R}^n$, which uniquely describes K. Indeed, if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a positively 1-homogeneous and subadditive function in $\mathbb{R}^n$, then there exists a unique convex body $K \in \mathcal{K}^n$, such that $f(x) = h(K,x)$, for every $x \in \mathbb{R}^n$, see [20, Theorem 1.7.1]. We further remark, that if K is a convex body and $L \in \mathcal{L}_k^n$, then

$$h(K,u) = h(p_L(K), u) \quad \text{for all } u \in L. \tag{2.3}$$

For two convex bodies K, L containing the origin, the so-called 2-mean of the support functions of K and L provides us with the support function of a new convex body, $K +_2 L$, called the 2-sum of K and L. More precisely, for $x \in \mathbb{R}^n$ and $K, L \in \mathcal{K}$ such that $0 \in K \cap L$, the function $f\colon \mathbb{R}^n \to \mathbb{R}$, given by

$$f(x) := (h(K,x)^2 + h(L,x)^2)^{\frac{1}{2}}, \tag{2.4}$$

is the support function of $K +_2 L$, see [20, Chapter 9]. Along with the 2-sum, the dilation $\lambda \cdot_2 K := \lambda^{1/2}K$, i.e., $h^2(\lambda \cdot_2 K, x) = \lambda h^2(K,x)$, for $\lambda \ge 0$ and every $x \in \mathbb{R}^n$ is introduced. The 2-sum natural framework is the L_2-theory. The latter is the case $p = 2$ of the so-called L_p Brunn–Minkowski theory, with $p \ge 1$, see [20, Section 1.9].

We will prove in Section 3 that $\mathrm{S}(Q)$, *the surface area of a matrix $Q \in \mathcal{S}_+^n$*, is closely related to the so-called L_2-surface of the associated ellipsoid E_Q. Within the L_2-Brunn–Minkowski theory, the L_2-surface area is introduced as the analogue notion of surface area of a convex body obtained by means of the variational quotient (1.4), replacing the Minkowski sum by the 2-sum, defined in (2.4). For convex bodies K and L containing the origin, the L_2-surface area of K relative to L is defined by the following limit expression:

$$\mathrm{S}_2(K;L) := \lim_{\rho\to 0^+} \frac{\mathrm{vol}(K +_2 \rho \cdot_2 L) - \mathrm{vol}(K)}{\rho}. \tag{2.5}$$

Along the proofs in the coming sections, we will make use of the surface area measure of a convex body $K \in \mathcal{K}^n$, denoted by $d\mathrm{S}_{n-1}(K,\cdot)$ (see e.g. [13, 20]). For completeness, we provide with the definition. In

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order to do so, we use the notation ν_x for the unit outward normal vector at any boundary point of $x \in K$. Then, the surface area measure of $K \in \mathcal{K}^n$ is the Borel measure on the unit sphere $\mathbb{S}^{n-1}$ defined, for any (measurable) $\omega \subseteq \mathbb{S}^{n-1}$, as

$$d\mathrm{S}_{n-1}(K,\omega) := \mathcal{H}^{n-1}(\{x \in \partial K \mid \nu_x \in \omega\}),$$

i.e., the $(n-1)$-Hausdorff measure of the (set of) boundary points of K with unit outward normal vector ν_x belonging to ω. We notice that $d\mathrm{S}_{n-1}(K,\cdot)$ depends on K. Indeed, we have $d\sigma = d\mathrm{S}_{n-1}(B_n,\cdot)$. We refer to [20, Chapters 4-5], for more details. The surface area measure $d\mathrm{S}_{n-1}(K,\cdot)$ of the convex body K provides us with an integral representation result for the mixed volume.

Theorem 2.5. *[20, Theorem 5.1.7] For every pair of convex bodies K and L, the following integral representation of the mixed volume*

$$\mathrm{V}(L,K,\dots,K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)d\mathrm{S}_{n-1}(K,u), \tag{2.6}$$

holds.

If $K = L$ in Theorem 2.5, the latter yields an integral representation of the volume of a convex body, namely

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K,u)d\mathrm{S}_{n-1}(K,u),$$

whereas if $L = B_n$, as $h(B_n,u) = 1$ for all $u \in \mathbb{S}^{n-1}$, we obtain an integral representation of the surface of a convex body by means of the surface area measure as follows:

$$\mathrm{S}(K) = n\mathrm{V}(B_n,K,\dots,K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} 1\, d\mathrm{S}_{n-1}(K,u).$$

The following theorem ensures an integral representation of the L_2-surface area, in a similar manner to the above integral representation of the (usual) surface area of a convex body, in terms of the surface area measure.

Theorem 2.6. *[20, Theorem 9.1.1] Let $K, L \in \mathcal{K}^n$ be two convex bodies containing the origin. Then, the following limit exists and it holds*

$$\mathrm{S}_2(K;L) = \lim_{\rho\to 0^+} \frac{\mathrm{vol}(K +_2 \rho \cdot_2 L) - \mathrm{vol}(K)}{\rho} = \frac{1}{2}\int_{u\in\mathbb{S}^{n-1}} h(L,u)^2 h(K,u)^{-1}\, d\mathrm{S}_{n-1}(K,u). \tag{2.7}$$

We observe that if $L = B_n$ in the above expression the L_2-surface area of K relative to B_n does only depend on K.

We finish this section with two technical results. The first one is key to representing the surface of a matrix in terms of the L_2-surface of the associated ellipsoid. It enables a change of variable via a non-degenerate linear transformation in an integral over the sphere w.r.t. surface area measure of a convex body, as long as the function admits a positively 1-homogeneous extension to $\mathbb{R}^n$.

Theorem 2.7. *[20, Section 5.1, formula (5.24)] Let $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$ be a continuous function, which admits a positively and 1-homogeneous extension on $\mathbb{R}^n$. Let $K \in \mathcal{K}^n$ be a convex body, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear, non-degenerate transformation. Then,*

$$\int_{u\in\mathbb{S}^{n-1}} f(u)\, d\mathrm{S}_{n-1}(TK,u) = |\det(T)| \int_{u\in\mathbb{S}^{n-1}} f(T^{-t}(u))\, d\mathrm{S}_{n-1}(K,u). \tag{2.8}$$

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The second one provides us with the direct value of an integral on the sphere, which happens to appear often along the calculations.

LEMMA 2.8. *[15, Lemma 3.12] Let $v \in \mathbb{S}^{n-1}$. Then,*

$$\int_{u\in\mathbb{S}^{n-1}} \langle u, v\rangle^2 \, d\sigma(u) = \kappa_n. \tag{2.9}$$

3. Symmetric and positive semidefinite matrices: mixed discriminants, projections, and ellipsoids. In the vector space of real symmetric matrices $\mathcal{M}^n$, the set of symmetric positive semidefinite ones, $\mathcal{S}^n_+$, is a closed, convex cone. In this note, we focus mainly on mixed discriminants of matrices in $\mathcal{S}^n_+$. However, mixed discriminants can be introduced for arbitrary square matrices. For completeness, we state also the definition of the mixed discriminant for arbitrary $n \times n$ matrices.

Following [3], let $n \in \mathbb{N}$, with $n \geq 1$, and $A_1, \ldots, A_n$ be arbitrary $n \times n$ matrices. If $A_j^{(i)}$ denotes the i-th column of the matrix A_j, then

$$\mathrm{D}(A_1, \ldots, A_n) := \frac{1}{n!} \sum_{\sigma\in S_n} \det\left(A^{(1)}_{\sigma(1)}, \ldots, A^{(n)}_{\sigma(n)}\right), \tag{3.1}$$

where S_n denotes the symmetric group of permutations of $\{1, \ldots, n\}$. In the literature, other approaches to mixed discriminants have been considered (we refer e.g. to [1, 14, 18]). In [3], also the following relation can be found (cf. (1.1)):

$$\mathrm{D}(A_1, \ldots, A_n) = \frac{1}{n!} \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \det(\lambda_1 A_1 + \ldots + \lambda_n A_n). \tag{3.2}$$

We observe that all apparently different definitions of mixed discriminants do coincide with the one given by (1.1), which we state as a theorem for completeness.

THEOREM 3.1. *[3] Let $n \in \mathbb{N}$, with $n \geq 1$. For every $m \geq 1$, $Q_1, \ldots, Q_m \in \mathcal{S}^n_+$ and $\lambda_1, \ldots, \lambda_m \geq 0$:*

$$\det(\lambda_1 Q_1 + \ldots + \lambda_m Q_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{D}(Q_{i_1}, \ldots, Q_{i_n}).$$

Observe that the latter provides us with the polynomial expansion of the determinant of the positive combination $\lambda_1 Q_1 + \ldots + \lambda_m Q_m$ in terms of $\lambda_1, \ldots, \lambda_m$.

Observe that m may not coincide with n. For connections of the mixed discriminant to other notions within the theory of matrix analysis, we refer to [4, 5] and the references therein.

We observe that the mixed discriminants D_k given by (1.5) can be seen as the counterpart of the quermassintegrals in convex geometry, see Section 2.

Theorem 3.1 yields (1.6), an analogue of the Steiner formula for matrices. We establish it as a theorem for completeness and further reference.

PROPOSITION 3.2. *For every $Q \in \mathcal{S}^n_+$ and $\rho \geq 0$ the following*

$$\det(Q + \rho I_n) = \sum_{k=0}^{n} \binom{n}{k} \rho^k \mathrm{D}_k(Q), \tag{3.3}$$

holds.

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Along this note, the mixed discriminant D_1 plays a crucial role. As mentioned in the introduction, we define $\mathrm{S}(Q)$, where the notation has been introduced because of its similarity with the surface area of a convex body, more precisely, (1.7), we have

$$\mathrm{S}(Q) := n\mathrm{D}_1(Q) = \lim_{\rho\to 0^+} \frac{\det(Q+\rho I_n) - \det(Q)}{\rho}, \tag{3.4}$$

which is an analogue of the limit expression defining the (Minkowski) surface area of a convex body in (1.4).

Using elementary symmetric polynomials, the mixed discriminants $\mathrm{D}_k(Q)$ can be directly given in terms of the eigenvalues of the matrix Q.

For completeness, we first recall the definition of elementary symmetric polynomials.

Definition 3.3. Let $1 \le k \le n$. The k-th elementary symmetric polynomial in n variables $x_1, \dots, x_n$ is

$$\mathrm{e}_k(x_1,\dots,x_n) = \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}.$$

We further set $\mathrm{e}_0(x_1,\dots,x_n) = 1$.

Definition 3.4. Let $Q \in \mathcal{S}^n_+$ and let $\lambda_1,\dots,\lambda_n \ge 0$ be the eigenvalues of Q. The k-th elementary symmetric function of the matrix Q is defined as the k-th elementary symmetric function of its eigenvalues. More precisely,

$$\mathrm{e}_k(Q) = \mathrm{e}_k(\lambda_1,\dots,\lambda_n),$$

for $k = 0,\dots,n$.

The following relation can be found scattered in the literature, see, e.g., [2]. As it seems a matter of folklore, for completeness, we provide all the details in this note.

Proposition 3.5. *Let $Q \in \mathcal{S}^n_+$ and $0 \le k \le n$. Then,*

$$\binom{n}{k}\mathrm{D}_k(Q) = \mathrm{e}_{n-k}(Q).$$

Proof. Let $\lambda_1,\dots,\lambda_n \ge 0$ be the eigenvalues of $Q \in \mathcal{S}^n_+$. We use Steiner's formula for matrices (3.3) and expand the characteristic polynomial of $-Q$. For $\rho \ge 0$, this gives

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\rho^k\mathrm{D}_k(Q) = \det(Q+\rho I_n) = (-1)^n\det(-Q-\rho I_n) = (-1)^n\prod_{k=1}^{n}(-\lambda_k-\rho)\\ = (-1)^n\sum_{k=0}^{n}(-\rho)^k(-1)^{n-k}\mathrm{e}_{n-k}(Q) = \sum_{k=0}^{n}\rho^k\mathrm{e}_{n-k}(Q).\end{aligned}$$

The claim follows by comparing the coefficients of ρ^k, $k = 0,\dots,n$. □

The following *extension* of Proposition 3.2 can also be found without further reference in the literature.

Proposition 3.6. *Let $Q \in \mathcal{S}^n_+$ and $\rho \ge 0$. Then,*

$$\mathrm{D}_k(Q+\rho I_n) = \sum_{j=0}^{n-k}\binom{n-k}{j}\rho^j\mathrm{D}_{k+j}(Q),$$

holds for $k = 0,\dots,n$.

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Proof. We apply (3.3) to the matrix $Q+\rho I_n+\lambda I_n$ for $\lambda\geq 0$ and obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\lambda^k \mathrm{D}_k(Q+\rho I_n) &= \det(Q+\rho I_n+\lambda I_n)=\det(Q+(\rho+\lambda)I_n)\\ &=\sum_{k=0}^{n}\binom{n}{k}(\rho+\lambda)^k\mathrm{D}_k(Q)=\sum_{k=0}^{n}\binom{n}{k}\mathrm{D}_k(Q)\sum_{j=0}^{k}\binom{k}{j}\rho^{k-j}\lambda^j\\ &=\sum_{k=0}^{n}\lambda^k\sum_{j=0}^{n-k}\binom{n}{k+j}\binom{k+j}{k}\mathrm{D}_{k+j}(Q)\rho^j.\end{aligned}$$

Comparing the coefficients of λ^k for $k=0,\dots,n$ yields

$$\mathrm{D}_k(Q+\rho I_n)=\sum_{j=0}^{n-k}\frac{\binom{n}{k+j}\binom{k+j}{j}}{\binom{n}{k}}\rho^j\mathrm{D}_{k+j}(Q)=\sum_{j=0}^{n-k}\binom{n-k}{j}\rho^j\mathrm{D}_{k+j}(Q),$$

which shows the claim. ∎

Next, we will deal with the projection of a matrix. In the Introduction section, we defined the projection of a positive semidefinite matrix $Q\in\mathcal{S}_+^n$ onto a subspace $L\in\mathcal{L}_k^n$, provided an orthonormal basis B of L has been fixed, as the unique $k\times k$ positive semidefinite matrix in basis B, defining the ellipsoid $p_L(E_Q)$, which is the orthogonal projection of the ellipsoid E_Q, associated to Q, onto L.

This definition of the projection of a matrix is equivalent, as we will see in Proposition 4.2, to the next one, which has been considered in [4] and [5], and consists of the restriction of the associated quadratic form to a linear subspace (see e.g. [17]). See also [6, 16, 22] for results concerning inequalities involving mixed discriminants and projections of a matrix. For a quadratic form q and a subspace L of $\mathbb{R}^n$, we denote by $q|L$ the restriction q to L.

Definition 3.7. Let L be a linear subspace of $\mathbb{R}^n$, let B be an orthonormal basis of L, let $Q\in\mathcal{S}_+^n$ be a positive semidefinite matrix, and let q be the quadratic form on $\mathbb{R}^n$ associated to Q, i.e., $q(x)=\langle x,Qx\rangle$. The projection of the matrix Q onto L is defined as the matrix associated with the restriction of q to the subspace $L\subset\mathbb{R}^n$, in basis B. It is denoted by $P_L(Q)\in\mathcal{S}_+^{\dim L}$.

Note that, with this definition, $P_L(Q)$ is positive semidefinite and well defined. For $Q\in\mathcal{S}_+^n$ and $1\leq k\leq n$, let $Q^{[k]}$ denote the $k\times k$ matrix obtained from Q by extracting the (common entries of the) first k columns and k rows. We write $e_1,\dots,e_n$ for the vectors of the standard basis of $\mathbb{R}^n$. If M is any matrix, and L is a linear subspace of $\mathbb{R}^n$, we denote by ML the image in $\mathbb{R}^n$ of the subspace L under the linear transformation given by M.

Proposition 3.8. *[4, 5] Let $Q\in\mathcal{S}_+^n$ be a positive semidefinite matrix, and let $L\in\mathcal{L}_k^n$ be a linear subspace of $\mathbb{R}^n$, $1\leq k\leq n$. Let $B=(u_1,\dots,u_n)$ be an orthonormal of $\mathbb{R}^n$ such that $B_L=(u_1,\dots,u_k)$ spans L. The following statements are equivalent:*

i) *Let $q\colon\mathbb{R}^n\to\mathbb{R}$ be the quadratic form $x\mapsto\langle x,Q\,x\rangle$. Then, the projection $P_L(Q)$ of the matrix Q onto the subspace L is the positive semidefinite $k\times k$ matrix of the restriction of q to the subspace L with respect to B_L.*
ii) *Let $M_L\in O(n)$ be the orthogonal matrix with $M_Le_i=u_i$, $i=1,\dots,n$. The projection of the matrix Q onto L is*

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$$(3.5)\qquad P_L(Q) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} M_L^T Q M_L \begin{pmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

where the identity submatrices in the left- and right-hand sides are of size $k \times k$.

We remark that (3.5) yields that $P_L(Q)$ is the $k \times k$ principal submatrix of $M_L^T Q M_L$ given by the first k (entries of the) columns and rows of $M_L^T Q M_L$.

The next lemma provides us with a connection between the projection of a matrix and its principal submatrices.

Lemma 3.9. *[6, Lemma 3.4] Let $Q \in \mathcal{S}_+^n$ be a positive semidefinite matrix and $1 \leq i \leq n$. Then,*

$$Q^{[i]} = P_L(Q),$$

holds for $L = \operatorname{lin}(e_1, \dots, e_i)$.

Remark 3.10. Let $Q \in \mathcal{S}_+^n$ and $L_1, L_2 \in \mathcal{L}_k^n$, $1 \leq k \leq n$. Let $M_i = M_{L_i} \in O(n)$ be as in Proposition 3.8, i.e., such that $M_i \operatorname{lin}(e_1, \dots, e_k) = L_i$, $i = 1, 2$. With $M = M_1 M_2^T \in O(n)$, we have

$$M L_2 = L_1 \quad \text{and} \quad P_{L_1}(Q) = P_{L_2}(M^T Q M).$$

Indeed, we have

$$P_{L_1}(Q) = (M_1^T Q M_1)^{[k]} = (M_2^T M_2 M_1^T Q M_1 M_2^T M_2)^{[k]} = (M_2^T M^T Q M M_2)^{[k]} = P_{L_2}(M^T Q M).$$

Let $Q_1, \dots, Q_{n-1}$ be $(n-1) \times (n-1)$ symmetric and positive semidefinite matrices. We denote by $\mathrm{D}^{(n-1)}(Q_1, \dots, Q_{n-1})$ the mixed discriminant for matrices in $\mathcal{S}_+^{n-1}$. Let $u \in \mathbb{S}^{n-1}$ and $L = u^\perp$. We recall that for any symmetric and positive semidefinite matrix Q, the projection $P_{u^\perp}(Q)$ is a $(n-1) \times (n-1)$ matrix. The following relation between the mixed discriminant of matrices in $\mathcal{S}_+^n$ and the mixed discriminant of their projections onto an $(n-1)$-dimensional subspace was proven in [5, Lemma 2.5].

Lemma 3.11. *[5, Lemma 2.5] Let $Q_1, \dots, Q_{n-1} \in \mathcal{S}_+^n$ and $u \in \mathbb{S}^{n-1}$. Then,*

$$n\mathrm{D}(Q_1, \dots, Q_{n-1}, uu^T) = \mathrm{D}^{(n-1)}(P_{u^\perp}(Q_1), \dots, P_{u^\perp}(Q_{n-1})).$$

This result has been generalized to other mixed discriminants and to projections onto lower dimensional subspaces in [22].

4. Results on ellipsoids. Let $\mathcal{E}^n$ be the set of centered ellipsoids in $\mathbb{R}^n$. Following [7], we introduce a bijection between $\mathcal{S}_+^n$ and $\mathcal{E}^n$. A centered ellipsoid $E \in \mathcal{E}^n$ is the image of the closed unit ball under a linear map. That is, there exists an $n \times n$ matrix A such that $E = AB_n$. We observe that

$$h(E, x) = \max\{\langle x, Ay\rangle \,:\, y \in B_n\} = \max\{\langle A^T x, y\rangle \,:\, y \in B_n\} = h(B_n, A^T x) = \langle x, AA^T x\rangle^{1/2},$$

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for $x \in \mathbb{R}^n$. In this way, E uniquely determines the positive semidefinite matrix $Q_E = AA^T \in \mathcal{S}^n_+$. On the other hand, any matrix $Q \in \mathcal{S}^n_+$ uniquely determines a centered ellipsoid $E_Q \in \mathcal{E}^n$ through its support function via

$$h(E_Q, x)^2 = \langle x, Qx \rangle, \quad x \in \mathbb{R}^n. \tag{4.1}$$

The notation Q_E for the positive semidefinite matrix corresponding to $E \in \mathcal{E}^n$ and E_Q for the ellipsoid corresponding to $Q \in \mathcal{S}^n_+$ will be used throughout.

We notice that if $Q \in \mathcal{S}^n_+$ is of rank one, namely, $Q = \lambda uu^T$ for some $u \in \mathbb{S}^{n-1}$ and $\lambda > 0$, then

$$h(E_{uu^T}, x) = \langle x, uu^T x \rangle^{1/2} = |\langle x, u \rangle| = \max\{\langle x, y \rangle \,:\, y \in [-u, u]\}, \tag{4.2}$$

which yields $E_{uu^T} = [-u, u] \in \mathcal{E}^n$. We also observe that

$$\mathrm{vol}(E_Q) = \mathrm{vol}(AB_n) = \kappa_n |\det(A)| = \kappa_n \sqrt{\det(Q)}. \tag{4.3}$$

Now that we have formally the correspondence between $\mathcal{S}^n_+$ and $\mathcal{E}^n$, we are able to deal with the equivalence of both mentioned definitions of the projection of a given matrix $Q \in \mathcal{S}^n_+$ onto a subspace. First, we make a remark considering the definition of projection of a matrix given in Proposition 3.8, which does provide the key connection to ellipsoids.

Remark 4.1. [16] Let $Q \in \mathcal{S}^n_+$ be a positive semidefinite matrix corresponding to the ellipsoid $E_Q \in \mathcal{E}^n$. Let $L \subseteq \mathbb{R}^n$ be a subspace. Let $q(x) = \langle x, Qx \rangle$, for all $x \in \mathbb{R}^n$, be the quadratic form associated to Q, as in Proposition 3.8. By (2.3), we have

$$h(p_L(E_Q), u) = q(u)^{\frac{1}{2}},$$

for every $u \in L$, where we do slightly abuse notation denoting by u also the image of u by embedding L into $\mathbb{R}^n$. The latter yields, by the equivalence in Proposition 3.8, that

$$h(p_L(E_Q), u) = q|L(u)^{\frac{1}{2}} = \langle u, P_L(Q)u \rangle^{\frac{1}{2}} = h(E_{P_L(Q)}, u),$$

for all $u \in L$. Hence,

$$p_L(E_Q) = E_{P_L(Q)} \in \mathcal{E}^n,$$

i.e., the projection of the ellipsoid E_Q onto L is the ellipsoid corresponding to the projection of the matrix Q onto L.

This remark enables a direct proof of the equivalence of both definitions of the projection of a matrix onto a subspace. We gather the equivalence in the following proposition, for completeness.

Proposition 4.2. *Let $Q \in \mathcal{S}^n_+$ be a positive semidefinite symmetric $n \times n$ real matrix, and let $L \in \mathcal{L}^n_k$, for $1 \leq k \leq n$ be a linear subspace of $\mathbb{R}^n$. Let $B = \{b_1, \ldots, b_k\}$ be an orthonormal basis of L, and let $B' = \{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ be any orthonormal basis of $\mathbb{R}^n$. The following $k \times k$ matrices coincide.*

i) *The matrix $P_L(Q)$ given by (3.5).*
ii) *The unique positive definite symmetric $k \times k$ matrix in basis B associated to the ellipsoid $p_L(E_Q)$, i.e., the orthogonal projection onto L of the ellipsoid E_Q.*

Proof. For the proof, we only need to reformulate the previous remark. From Remark 4.1 follows that $p_L(E_Q)$, the projection onto the subspace L of the ellipsoid E_Q, is described by the restriction of the quadratic

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form to L. Considering the matrix associated to this quadratic form in basis B, we obtain $P_L(Q)$, i.e., $P_L(Q)$ is the (unique) matrix in basis B giving $q|L$, which determines, by Remark 4.1, the projection of E_Q onto L, namely, $p_L(E_Q)$. □

For completeness, we notice also the following well-known fact.

Remark 4.3. Let $E \in \mathcal{E}^n$ be the ellipsoid $E = AB_n$ for A any real $n \times n$ matrix. There is no unique representation of the Q_E as the product $Q_E = AA^T$, i.e., there is no unique linear application that brings the ball into the ellipsoid E. However, among all the matrices A such that $Q_E = AA^T$, the matrix $Q^{1/2}$ is the unique positive semidefinite square root of Q. Moreover, we have $E_Q = Q^{1/2}B_n$.

The correspondence between positive semidefinite matrices and ellipsoids we are dealing with can be considered beyond the *objects*, namely, beyond matrices and ellipsoids.

There is a correspondence between positive semidefinite matrices and centered ellipsoids, which involves operations on both sides. On the side of positive semidefinite matrices, the operation is the usual matrix sum, while on the side of ellipsoids, it is the L_2-sum. This correspondence respects these operations: the sum of positive semidefinite matrices corresponds to the L_2-sum of the corresponding ellipsoids, as shown directly in equation (4.1).

We note that the L_2-sum $E_1 +_2 E_2$ of ellipsoids $E_1, E_2 \in \mathcal{E}^n$ is an ellipsoid. Indeed, for $x \in \mathbb{R}^n$, we have

$$\langle x, Q_{E_1+_2E_2}x\rangle = h(E_1 +_2 E_2, x)^2 = h(E_1, x)^2 + h(E_2, x)^2 = \langle x, (Q_{E_1} + Q_{E_2})x\rangle,$$

which yields $Q_{E_1+_2E_2} = Q_{E_1} + Q_{E_2}$.

We note that the positive semidefinite matrix corresponding to the closed unit ball is the identity matrix, i.e., $Q_{B_n} = I_n$.

This was the motivation to define the $\mathrm{D}_k(Q)$ in analogy to the quermassintegrals of a convex body K, given by (1.3).

Moreover, considering (3.4) we can provide with a connection of $\mathrm{S}(Q)$ to $\mathrm{S}_2(E_Q; B_n)$, i.e., the L_2-surface area of the associated ellipsoid E_Q, relative to the Euclidean ball (see (2.7)).

We extend the notion of *surface area* of a matrix following the notion of surface area of a convex body K relative to another convex body L (see e.g. [19]). We define, for $Q, R \in \mathcal{S}^n_+$,

$$\mathrm{S}(Q; R) := n\mathrm{D}(\underbrace{Q, \dots, Q}_{(n-1)\text{-times}}, R).$$

By an application of Proposition 3.2, we have

$$\mathrm{S}(Q; R) = \lim_{\rho\to 0^+} \frac{\det(Q + \rho R) - \det(Q)}{\rho}. \tag{4.4}$$

The latter yields the announced connection of the (relative) *surface area* of a matrix and the L_2-surface area of the associated ellipsoid.

Lemma 4.4. *Let $Q, R \in \mathcal{S}^n_+$ be positive semidefinite matrices. Then,*

$$\mathrm{S}(Q; R) = \frac{2}{\kappa_n}\sqrt{\det(Q)}\mathrm{S}_2(E_Q; E_R).$$

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Proof. The equality follows from the correspondence of positive semidefinite matrices with ellipsoids, using (4.4), (4.3), and (2.7) to write

$$\begin{aligned}\mathrm{S}(Q;R) &= \lim_{\rho\to 0^+}\frac{\det(Q+\rho R)-\det(Q)}{\rho} = \frac{1}{\kappa_n^2}\lim_{\rho\to 0^+}\frac{\mathrm{vol}(E_Q +_2 \sqrt{\rho}E_R)^2-\mathrm{vol}(E_Q)^2}{\rho}\\ &= \frac{1}{\kappa_n^2}\frac{d}{d\rho}\bigg|_{\rho=0}\mathrm{vol}(E_Q +_2 \sqrt{\rho}E_R)^2 = \frac{2}{\kappa_n^2}\mathrm{vol}(E_Q)\mathrm{S}_2(E_Q;E_R)\\ &= \frac{2}{\kappa_n}\sqrt{\det(Q)}\mathrm{S}_2(E_Q;E_R).\end{aligned}$$ ∎

In particular, taking $R = I_n$ in Lemma 4.4 yields

$$\mathrm{S}(Q) = \mathrm{S}(Q;I_n) = \frac{2}{\kappa_n}\sqrt{\det(Q)}\mathrm{S}_2(E_Q;B_n). \tag{4.5}$$

Using Theorem 2.7, we obtain an integral expression for $\mathrm{S}(Q;R)$, the *surface area* of a matrix $Q \in \mathcal{S}_+^n$ relative to another matrix $R \in \mathcal{S}_+^n$ (cf. (2.6)). In order to do so, we first notice that if $Q \in \mathcal{S}_+^n$ and E_Q is the ellipsoid associated to Q, then $E_Q = Q^{1/2}B_n$ (cf. Remark 4.3).

PROPOSITION 4.5. *Let $Q, R \in \mathcal{S}_+^n$, and let Q be invertible. Then,*

$$\begin{aligned}\mathrm{S}(Q;R) &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\frac{\langle u,Ru\rangle}{\sqrt{\langle u,Qu\rangle}}\,d\mathrm{S}_{n-1}(E_Q,u)\\ &= \frac{\det(Q)}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\langle Q^{-1/2}u, RQ^{-1/2}u\rangle\,d\sigma(u).\end{aligned}$$

Proof. The first claim follows from Lemma 4.4, (2.7) and (4.1) since

$$\begin{aligned}\mathrm{S}(Q;R) &= \frac{2}{\kappa_n}\sqrt{\det(Q)}\mathrm{S}_2(E_Q;E_R) = \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\frac{h(E_R,u)^2}{h(E_Q,u)}\,d\mathrm{S}_{n-1}(E_Q,u)\\ &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\frac{\langle u,Ru\rangle}{\sqrt{\langle u,Qu\rangle}}\,d\mathrm{S}_{n-1}(E_Q,u).\end{aligned}$$

For the second claim, we consider the continuous, positive homogeneous extension

$$x \mapsto \|x\|\frac{\langle x/\|x\|, Rx/\|x\|\rangle}{\sqrt{\langle x/\|x\|, Qx/\|x\|\rangle}}, \quad x\in\mathbb{R}^n,$$

of the integrand, $u \mapsto \langle u,Ru\rangle/\sqrt{\langle u,Qu\rangle}$, defined on $\mathbb{S}^{n-1}$. Applying (2.8) for $E_Q = Q^{1/2}B_n$, and using that both, $Q^{1/2}$ and its inverse $Q^{-1/2}$, are symmetric (and positive definite) matrices yields

$$\begin{aligned}\mathrm{S}(Q;R) &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\|u\|\frac{\langle u/\|u\|, Ru/\|u\|\rangle}{\sqrt{\langle u/\|u\|, Qu/\|u\|\rangle}}\,d\mathrm{S}(Q^{1/2}B_n,u)\\ &= \frac{\det(Q)}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\|Q^{-1/2}u\|\frac{\langle \frac{Q^{-1/2}u}{\|Q^{-1/2}u\|}, R\frac{Q^{-1/2}u}{\|Q^{-1/2}u\|}\rangle}{\sqrt{\langle \frac{Q^{-1/2}u}{\|Q^{-1/2}u\|}, Q\frac{Q^{-1/2}u}{\|Q^{-1/2}u\|}\rangle}}\,d\mathrm{S}(B_n,u)\\ &= \frac{\det(Q)}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\langle Q^{-1/2}u, RQ^{-1/2}u\rangle\,d\sigma(u).\end{aligned}$$ ∎

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We observe that if $R = I_n$, we obtain that

$$\begin{aligned}\mathrm{S}(Q) = \mathrm{S}(Q; I_n) &= \frac{\sqrt{\det(Q)}}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \frac{\langle u, u\rangle}{\sqrt{\langle u, Qu\rangle}}\, d\mathrm{S}_{n-1}(E_Q, u)\\ &= \frac{\det(Q)}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \langle Q^{-1/2}u, Q^{-1/2}u\rangle\, d\sigma(u).\end{aligned}$$

Proposition 4.5 yields a further geometric consequence, relating D_{n-1} and a sort of L_2 analogue of the mean width of the associated ellipsoid (see [20, Section 1.7] for the definition of mean width).

REMARK 4.6. Let $Q \in \mathcal{S}^n_+$. Proposition 4.5 yields

$$\mathrm{D}_{n-1}(Q) = \frac{1}{n}\mathrm{S}(I_n; Q) = \frac{1}{n\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \langle u, Qu\rangle\, d\sigma(u) = \frac{1}{n\kappa_n} \int_{u\in\mathbb{S}^{n-1}} h(E_Q, u)^2\, d\sigma(u).$$

Hence, $\mathrm{D}_{n-1}(Q)$ is the mean of the squared support function of E_Q, which coincides with the mean of the quadratic form associated to Q on $\mathbb{S}^{n-1}$.

In addition, we observe that, by Proposition 3.5, we have

$$\mathrm{D}_{n-1}(Q) = \frac{1}{n}\mathrm{e}_1(Q) = \frac{\mathrm{tr}(Q)}{n},$$

the surface area of Q is the trace of Q, as also pointed out in [2, Proof of Theorem 1.1].

Hence, we have the following relation between the trace of Q, the surface area of Q, and the ellipsoid associated to Q

$$\mathrm{tr}(Q) = \frac{1}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} h(E_Q, u)^2\, d\sigma(u).$$

The following observation, along with the notion of polarity for convex bodies containing the origin, provides us with another geometrical application of Proposition 4.5.

The polar body of a convex body $K \in \mathcal{K}^n$ containing the origin in its interior is defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\} \in \mathcal{K}^n.$$

We refer the reader to [20, Section 1.6] for properties of the polar body.

LEMMA 4.7. *Let $Q \in \mathcal{S}^n_+$ be invertible, and $E_Q \in \mathcal{E}^n$ be its associated ellipsoid. Then,*

$$\mathrm{S}(Q) = \frac{2}{\kappa_n}\det(Q)\mathrm{S}_2(B_n; E_Q^\circ) \quad \textit{and} \quad \mathrm{S}_2(E_Q; B_n) = \sqrt{\det(Q)}\mathrm{S}_2(B_n; E_Q^\circ). \tag{4.6}$$

Proof. We observe first, that $E_Q = Q^{1/2}B_n$ and

$$E_Q^\circ = (Q^{1/2}B_n)^\circ = Q^{-1/2}B_n = E_{Q^{-1}}.$$

Hence, by (4.1),

$$h(E_Q^\circ, u) = \langle Q^{-1/2}u, Q^{-1/2}u\rangle^{1/2}.$$

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Now, using Proposition 4.5, we obtain

$$\mathrm{S}(Q) = \frac{\det(Q)}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \langle Q^{-1/2}u, Q^{-1/2}u\rangle \, d\sigma(u) = \frac{\det(Q)}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} h(E_Q^\circ, u)^2 \, d\sigma(u)$$
$$= \frac{\det(Q)}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \frac{h(E_Q^\circ, u)^2}{h(B_n, u)} \, d\mathrm{S}_{n-1}(B_n, u) = \frac{2}{\kappa_n} \det(Q)\mathrm{S}_2(B_n; E_Q^\circ)$$

by (2.7). Thus, (4.5) gives

$$\mathrm{S}_2(E_Q; B_n) = \sqrt{\det(Q)}\mathrm{S}_2(B_n; E_Q^\circ).$$

∎

5. Integral means of mixed discriminants of matrix projections. In this section, we prove our main results expressing the quermassintegrals analogy for positive semidefinite matrices through means of quermassintegrals of matrix projections.

The first statement regards the counterpart of Cauchy surface area formula, (2.1), for matrices. In [4, Lemma 2.3], it was shown that

$$\mathrm{e}_{n-1}(Q) = \frac{1}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \det(P_{u^\perp}(Q)) \, d\sigma(u), \tag{5.1}$$

for any $Q \in \mathcal{S}_+^n$. The artful proof is based on the fact that the integral on the right-hand side is a homogeneous symmetric polynomial of degree $n-1$ in the eigenvalues of Q. Using the definition of $\mathrm{S}(Q)$ and Proposition 3.5, Theorem 1.1 follows. In the next, we present a different proof, following the ideas in [21] for a proof of (2.1) and using the correspondence of positive semidefinite matrices and ellipsoids. We remark that the following result is exactly Theorem 1.1; we provide an alternative proof. Note that by Proposition 3.5, we have also $\mathrm{e}_{n-1}(Q) = \mathrm{S}(Q)$.

THEOREM 5.1. *[4] Let $Q \in \mathcal{S}_+^n$. Then,*

$$\mathrm{S}(Q) = \frac{1}{\kappa_n} \int_{u\in\mathbb{S}^{n-1}} \det(P_{u^\perp}(Q)) \, d\sigma(u).$$

Alternative proof to the one in [4]. We will approach the integral by partitions P_m of $\mathbb{S}^{n-1}$ into measurable subsets of diameter less or equal to $1/m$ for all $m \in \mathbb{N}$. For subsets of the sphere $S \in P_m$, we choose a point $u(S) \in S \subseteq \mathbb{S}^{n-1}$. We use Lemma 3.11 together with the linearity (see [3, Lemma 2]) and continuity (c.f. (3.1)) of the mixed discriminants to obtain

$$\begin{aligned}
\int_{u\in\mathbb{S}^{n-1}} \det(P_{u^\perp}(Q)) \, d\sigma(u) &= n \int_{u\in\mathbb{S}^{n-1}} \mathrm{D}(Q[n-1], uu^T) \, d\sigma(u) \\
&= n \lim_{m\to\infty} \sum_{S\in P_m} \mathrm{D}(Q[n-1], u(S)u(S)^T)\sigma(S) \\
&= n \lim_{m\to\infty} \mathrm{D}\left(Q[n-1], \sum_{S\in P_m} \sigma(S)u(S)u(S)^T\right) \\
&= n\mathrm{D}\left(Q[n-1], \lim_{m\to\infty} \sum_{S\in P_m} \sigma(S)u(S)u(S)^T\right).
\end{aligned}$$

For $m \in \mathbb{N}$, we define the positive semidefinite matrix

$$Q_m = \sum_{S\in P_m} \sigma(S)u(S)u(S)^T \in \mathcal{S}_+^n.$$

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Let $x \in \mathbb{R}^n \setminus \{0\}$. Using (2.9), we have

$$\begin{aligned}\lim_{m\to\infty}\langle x, Q_m x\rangle &= \lim_{m\to\infty}\sum_{S\in P_m}\sigma(S)\langle u(S), x\rangle^2\\ &= \int_{u\in\mathbb{S}^{n-1}}\langle u, x\rangle^2\, d\sigma(u)\\ &= \|x\|^2\int_{u\in\mathbb{S}^{n-1}}\left\langle u, \frac{x}{\|x\|}\right\rangle^2 d\sigma(u)\\ &= \|x\|^2\kappa_n = \langle x, \kappa_n I_n x\rangle.\end{aligned}$$

Therefore, $\lim_{m\to\infty} Q_m = \kappa_n I_n$. We conclude that

$$\begin{aligned}\int_{u\in\mathbb{S}^{n-1}}\det(P_{u^\perp}(Q))\, d\sigma(u) &= n\mathrm{D}\left(Q[n-1], \lim_{m\to\infty} Q_m\right)\\ &= n\kappa_n\mathrm{D}(Q[n-1], I_n) = \kappa_n n\mathrm{D}_1(Q) = \kappa_n\mathrm{S}(Q),\end{aligned}$$

and the claim follows. ∎

Before we proceed to prove Theorem A, we show the following intermediate step, generalizing (5.1) to every D_k, $1 \le k \le n-1$, and thus, to means of determinants of projections onto $(n-k)$-dimensional subspaces. Recall that ν_k denotes the unique normalized $O(n)$-invariant Haar measure on $\mathcal{L}^n_k$. Since the spherical measure σ is also $O(n)$-invariant, uniqueness of ν_{n-1} together with Theorem 1.1 yields the following equivalent formulation of (5.1):

$$\mathrm{D}_1(Q) = \frac{1}{n}\mathrm{S}(Q) = \frac{1}{n\kappa_n}\int_{u\in\mathbb{S}^{n-1}}\det(P_{u^\perp}(Q))\, d\sigma(u) = \int_{L\in\mathcal{L}^n_{n-1}}\det(P_L(Q))\, d\nu_{n-1}(L), \tag{5.2}$$

for any $Q \in \mathcal{S}^n_+$. In order to get a similar representation for D_{n-k}, $1 \le k \le n-2$, the idea is to show that the integral is a symmetric, homogeneous polynomial in the eigenvalues of the matrix, of a suitable degree, following and generalizing the proof of Theorem 1.1 in [4]; cf. (5.1).

Proposition 5.2. *Let $Q \in \mathcal{S}^n_+$, and let $0 \le k \le n$. Then,*

$$\mathrm{D}_{n-k}(Q) = \int_{L\in\mathcal{L}^n_k}\det(P_L(Q))\, d\nu_k(L).$$

Proof. Let $Q \in \mathcal{S}^n_+$, and let

$$p(Q) := \int_{L\in\mathcal{L}^n_k}\det(P_L(Q))\, d\nu_k(L).$$

We prove that p is a symmetric, homogeneous polynomial of degree k in the eigenvalues of Q.

Let $\lambda_1, \dots, \lambda_n \ge 0$ be the eigenvalues of Q with corresponding eigenvectors $v_1, \dots, v_n \in \mathbb{S}^{n-1}$ forming an orthonormal basis of $\mathbb{R}^n$. For any subspace $L \in \mathcal{L}^n_k$, we choose an orthonormal basis $B_L = (b_1, \dots, b_k)$ of L. Then,

$$b_j = \sum_{i=1}^n \langle b_j, v_i\rangle v_i \text{ and } Qb_j = \sum_{i=1}^n \lambda_i\langle b_j, v_i\rangle v_i,$$

for $j = 1, \dots, k$. Therefore,

$$p_L(Qb_j) = \sum_{l=1}^k\left(\sum_{i=1}^n \lambda_i\langle b_j, v_i\rangle\right)\langle v_i, b_l\rangle b_l,$$

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for $j = 1, \ldots, k$. By Proposition 3.8, this shows that

$$P_L(Q) = \left(\sum_{i=1}^{n} \lambda_i \langle b_j, v_i \rangle \langle v_i, b_l \rangle \right)_{1 \le l, j \le k} \in \mathcal{S}_+^k.$$

We conclude that $\det(P_L(Q))$, and thus, also $p(Q)$, is a homogeneous polynomial of degree k in the eigenvalues of Q.

To show that $p(Q)$ is also symmetric, let $L_1, L_2 \in \mathcal{L}_k^n$ and $M \in O(n)$ be such that $ML_2 = L_1$. By Remark 3.10, we have $P_{L_1}(Q) = P_{L_2}(M^T QM)$. Since ν_k is $O(n)$-invariant, we obtain

$$\begin{aligned} p(M^T QM) &= \int_{L \in \mathcal{L}_k^n} \det(P_L(M^T QM))\, d\nu_k(L) \\ &= \int_{L \in \mathcal{L}_k^n} \det(P_{M^T L}(M^T QM))\, d\nu_k(L) \\ &= \int_{L \in \mathcal{L}_k^n} \det(P_L(Q))\, d\nu_k(L) = p(Q). \end{aligned}$$

In particular, by choosing M to be a permutation matrix, we get that $p(Q)$ is symmetric in the eigenvalues of Q.

Since $p(Q)$ is a symmetric, homogeneous polynomial of degree k in $\lambda_1, \ldots, \lambda_n$, we find coefficients $C_{n,k}(i_1, \ldots, i_k) \in \mathbb{R}$ for $1 \le i_1 \le \cdots \le i_k \le n$ such that

$$p(Q) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} C_{n,k}(i_1, \ldots, i_k) \lambda_{i_1} \cdots \lambda_{i_k}.$$

To calculate explicitly the coefficients, we assume next that $Q \in \mathcal{S}_+^n$ is of rank at most $k-1$, i.e., at least $n-k+1$ of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Q are zero. In this case, also $P_L(Q)$ is of rank at most $k-1$ for all $L \in \mathcal{L}_k^n$ and hence, $p(Q) = 0$. For indices $1 \le i_1 \le \cdots \le i_k \le n$ such that $|\{i_1, \ldots, i_k\}| < k$, we can choose a $Q \in \mathcal{S}_+^n$ of rank smaller than k, in such a way that exactly the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_k}$ are nonzero, which implies $C_{n,k}(i_1, \ldots, i_k) = 0$. Together with the symmetry, this implies that the polynomial p is of the form

$$p(Q) = C_{n,k} \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k},$$

for a constant $C_{n,k} \in \mathbb{R}$. If we now choose $Q = I_n \in \mathcal{S}_+^n$, using Proposition 3.8, we obtain that $P_L(I_n) = (M^T I_n M)^{[k]} = I_k$ is the $k \times k$ identity matrix for any $L \in \mathcal{L}_k^n$ and with a suitable $M \in O(n)$. Therefore, $p(I_n) = 1$. Thus,

$$C_{n,k} = p(I_n) \left(\sum_{1 \le i_1 < \cdots < i_k \le n} 1 \right)^{-1} = \binom{n}{k}^{-1}.$$

We conclude the proof by means of Proposition 3.5, as

$$p(Q) = \binom{n}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} = \binom{n}{n-k}^{-1} \mathrm{e}_k(Q) = \mathrm{D}_{n-k}(Q).$$ ∎

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Now, the matrix version of the Cauchy–Kubota formula, Theorem A, follows by application of the matrix version of Steiner's formula, (3.3), and Proposition 3.6.

THEOREM 5.3. *Let $Q \in \mathcal{S}_+^n$ and $0 \leq k \leq n$, then*

$$\mathrm{D}_{n-j}(Q) = \int_{L\in\mathcal{L}_k^n} \mathrm{D}_{k-j}^{(k)}(P_L(Q))\, d\nu_k(L),$$

holds for all $0 \leq j \leq k$.

Proof. Let $\rho \geq 0$. We use Proposition 3.6, Proposition 5.2, and Proposition 3.2 in dimension k, to obtain

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}\rho^i \mathrm{D}_{n-k+i}(Q) &= \mathrm{D}_{n-k}(Q+\rho I_n)\\ &= \int_{L\in\mathcal{L}_k^n} \det(P_L(Q+\rho I_n))\, d\nu_k(L)\\ &= \int_{L\in\mathcal{L}_k^n} \det(P_L(Q)+\rho I_k)\, d\nu_k(L)\\ &= \sum_{i=0}^{k}\binom{k}{i}\rho^i \int_{L\in\mathcal{L}_k^n} \mathrm{D}_i^{(k)}(P_L(Q))\, d\nu_k(L).\end{aligned}$$

Let $j = k-i$ for $i = 0, \dots, k$. Comparing coefficients yields

$$\mathrm{D}_{n-j}(Q) = \mathrm{D}_{n-k+i}(Q) = \int_{L\in\mathcal{L}_k^n} \mathrm{D}_i^{(k)}(P_L(Q))\, d\nu_k(L) = \int_{L\in\mathcal{L}_k^n} \mathrm{D}_{k-j}^{(k)}(P_L(Q))\, d\nu_k(L),$$

for $i, j = 0, \dots, k$, which shows the claim. □

The following consequence of Theorem 2.4 can be found in [13, Theorem 5.7]. For $0 \leq k \leq n$ and $K_1, \dots, K_k \in \mathcal{K}^n$, we have

$$\mathrm{V}(K_1,\dots,K_k,\underbrace{B_n,\dots,B_n}_{(n-k)\text{-times}}) = \frac{\kappa_n}{\kappa_k}\int_{L\in\mathcal{L}_k^n} \mathrm{V}^{(k)}(p_L(K_1),\dots,p_L(K_k))\, d\nu_k(L),$$

where, as for $\mathrm{D}^{(k)}$, the notation $V^{(k)}$ refers to the mixed volume of convex bodies in a k-dimensional subspace of $\mathbb{R}^n$, identified with $\mathbb{R}^k$. In the very same manner as for convex bodies, we obtain the analogue result for positive semidefinite matrices from Proposition 5.2.

COROLLARY 5.4. *Let $0 \leq k \leq n$ and $Q_1, \dots, Q_k \in \mathcal{S}_+^n$. Then,*

$$\mathrm{D}(Q_1,\dots,Q_k,\underbrace{I_n,\dots,I_n}_{(n-k)\text{-times}}) = \int_{L\in\mathcal{L}_k^n} \mathrm{D}^{(k)}(P_L(Q_1),\dots,P_L(Q_k))\, d\nu_k(L).$$

Proof. Let $\lambda_1, \dots, \lambda_k \geq 0$. Linearity of the mixed discriminant (see [3, Lemma 2]) gives

$$\sum_{i_1,\dots,i_k=1}^{k}\lambda_{i_1}\cdots\lambda_{i_k}\mathrm{D}(Q_{i_1},\dots,Q_{i_k},\underbrace{I_n,\dots,I_n}_{(n-k)\text{-times}}) = \mathrm{D}\left(\sum_{i=1}^{(k)}\lambda_i Q_i,\dots,\sum_{i=1}^{(k)}\lambda_i Q_i,\underbrace{I_n,\dots,I_n}_{(n-k)\text{-times}}\right),$$

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where $\sum_{i=1}^k \lambda_i Q_i$ appears k times in the latter expression. Thus, by Proposition 5.2 and Theorem 3.1, we have

$$\begin{aligned}\sum_{i_1,\ldots,i_k=1}^{k} \lambda_{i_1}\cdots\lambda_{i_k} \mathrm{D}(Q_{i_1},\ldots,Q_{i_k},\underbrace{I_n,\ldots,I_n}_{(n-k)\text{-times}}) &= \mathrm{D}_{n-k}\left(\sum_{i=1}^k \lambda_i Q_i\right)\\ &= \int_{L\in\mathcal{L}_k^n} \det\left(\sum_{i=1}^k \lambda_i P_L(Q_i)\right) d\nu_k(L)\\ &= \int_{L\in\mathcal{L}_k^n} \sum_{i_1,\ldots,i_k=1}^{k} \lambda_{i_1}\cdots\lambda_{i_k} \mathrm{D}^{(k)}(P_L(Q_{i_1}),\ldots,P_L(Q_{i_k}))\, d\nu_k(L)\\ &= \sum_{i_1,\ldots,i_k} \lambda_{i_1}\cdots\lambda_{i_k} \int_{L\in\mathcal{L}_k^n} \mathrm{D}^{(k)}(P_L(Q_{i_1}),\ldots,P_L(Q_{i_k}))\, d\nu_k(L).\end{aligned}$$

Comparing coefficients shows the claim. ∎

We finish the section with another counterpart of a result in the realm of convex bodies and integral geometric formulae: we prove a formula expressing the determinant of the projection of a positive semidefinite matrix by means of inner products on the sphere.

For a convex body $K \in \mathcal{K}^n$ and $v \in \mathbb{S}^{n-1}$, the following representation of the volume of the projection onto $v^\perp$ is known (see [13, Theorem 4.12])

$$\mathrm{vol}_{n-1}(p_{v^\perp}(K)) = \frac{1}{2}\int_{u\in\mathbb{S}^{n-1}} |\langle v,u\rangle|\, d\mathrm{S}_{n-1}(K,u). \tag{5.3}$$

For $Q \in \mathcal{S}_+^n$ and the corresponding ellipsoid $E_Q \in \mathcal{E}^n$, (2.8) thus shows that

$$\begin{aligned}\mathrm{vol}_{n-1}(p_{v^\perp}(E_Q)) &= \frac{1}{2}\int_{u\in\mathbb{S}^{n-1}} |\langle v,u\rangle|\, d\mathrm{S}_{n-1}(Q^{1/2}B_n,u)\\ &= \frac{\sqrt{\det(Q)}}{2}\int_{u\in\mathbb{S}^{n-1}} |\langle v,Q^{-1/2}u\rangle|\, d\sigma(u).\end{aligned}$$

THEOREM 5.5. *Let $Q \in \mathcal{S}_+^n$ be invertible and $v \in \mathbb{S}^{n-1}$. Then,*

$$\begin{aligned}\det(P_{v^\perp}(Q)) &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}} \|Q^{1/2}u\|^{-1}\langle u,v\rangle^2\, d\mathrm{S}_{n-1}(E_Q,u)\\ &= \frac{\det(Q)}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}} \langle v,Q^{-1/2}u\rangle^2\, d\sigma(u).\end{aligned}$$

Proof. First, Lemma 3.11 shows that

$$\begin{aligned}\det(P_{v^\perp}(Q)) &= \mathrm{D}^{(n-1)}(P_{v^\perp}(Q),\ldots,P_{v^\perp}(Q))\\ &= n\mathrm{D}(Q,\ldots,Q,vv^T) = \mathrm{S}(Q;vv^T).\end{aligned}$$

We apply Proposition 4.5 to obtain

$$\begin{aligned}\det(P_{v^\perp}(Q)) &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}} \frac{\langle u,vv^Tu\rangle}{\sqrt{\langle u,Qu\rangle}}\, d\mathrm{S}_{n-1}(E_Q,u)\\ &= \frac{\sqrt{\det(Q)}}{\kappa_n}\int_{u\in\mathbb{S}^{n-1}} \|Q^{1/2}u\|^{-1}\langle v,u\rangle^2\, d\mathrm{S}_{n-1}(E_Q,u),\end{aligned}$$

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which shows the first claim. The second equality of Theorem 5.5 follows from a direct application of (2.8) to the last integral of the above formula for $\det(P_{v^\perp}(Q))$. □

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