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THE SOLVABILITY CONDITIONS OF MATRIX EQUATIONS WITH K-INVOLUTION*

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Abstract. Let $m \times m$ complex matrix P and $n \times n$ complex matrix Q be k-involutions, i.e., $P^{k-1} = P^{-1}$, $Q^{k-1} = Q^{-1}$ for some integer $k \geq 2$. An $m \times n$ complex matrix A is (P, Q, β)-symmetric if $PAQ^{-1} = \lambda^{\beta} A$, or (P, Q, α, β)-symmetric if $PAQ^{-\alpha} = \lambda^{\beta} A$, where $\lambda = e^{2\pi i/k}$ and $\alpha, \beta \in \{1, 2, \ldots, k\}$. In this paper, for given matrices X, Y, E, F with appropriate sizes, the solvability of matrix equations $AX = E$ and $Y^* A = F$ under (P, Q, β)- and (P, Q, α, β)-constraints, respectively, are investigated. Meanwhile, the associated optimal approximation problem is also considered when the above P and Q are unitary.

Key words. k-Involution, (P, Q, β)-Symmetric matrices, (P, Q, α, β)-Symmetric matrices, Matrix equations, Optimal approximation.

AMS subject classifications. 65F05, 15A24.

1. Introduction. The matrix inverse problem, which arises in the design of Hopfield neural networks [16, 17], has important applications in vibration theory [3], control theory [7], civil engineering and aviation [10]. The constrained matrix equation(s) problem, as the natural extension of the matrix inverse problem, plays an important role in matrix theory, and has been widely studied [5, 6, 11, 13, 20, 21, 27].

In this paper, we will discuss the solvability conditions of matrix equations

$$AX = E,\ Y^* A = F \tag{1.1}$$

with (P, Q, β)- and (P, Q, α, β)-constraints, respectively.

For convenience, we first introduce some notations: $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices, $\mathbb{UC}^{m\times m}$ is the set of all $m \times m$ orthogonal matrices. I_n represents identity matrix with order n. A^T, A^*, A^+ and $\|A\|$ stand for the transpose, conjugate transpose, Moore-Penrose generalized inverse and Frobenius norm of matrix A, respectively.

A complex matrix $R \in \mathbb{C}^{m\times m}$ is called k-involutory if its minimal polynomial

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is $x^k - 1$ for some integer $k \geq 2$. It is easy to know that k-involutory R satisfies $R^{k-1} = R^{-1}$, and its k eigenvalues are $\lambda,\ \lambda^2,\ \ldots,\ \lambda^k$, where $\lambda = e^{2\pi i/k}$.

Definition 1.1. Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be k-involutory matrices, we say that $A \in \mathbb{C}^{m\times n}$ is $(P,\, Q,\, \beta)$-symmetric if $PAQ^{-1} = \lambda^\beta A$, or $(P,\, Q,\, \alpha,\, \beta)$-symmetric if $PAQ^{-\alpha} = \lambda^\beta A$, where $\alpha,\, \beta \in \mathbb{Z}_k = \{1,\, 2,\, \ldots,\, k\}$.

The $(P,\, Q,\, \beta)$-symmetric matrices and $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices were put forward by William F. Trench in [23] and [24], respectively. Obviously, the centrosymmetric matrices [1] and generalized reflexive matrices [13, 18, 27] are the extensions of the $(P,\, Q,\, \beta)$- and $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices.

In [23] and [24], the author investigates the structural properties, the expressions of generalized inverse, singular value decompositions of the $(P,\, Q,\, \beta)$- and $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices, respectively. The Procrustes problem for $(P,\, Q,\, \beta)$-symmetric matrices has been considered by Jia in [9]. Moreover, the least squares problem of matrix equation $AX = B$ with $(P,\, Q,\, \beta)$- and $(P,\, Q,\, \alpha,\, \beta)$-constraints have been discussed by Li [15]. However, to our knowledge, the constrained matrix equations problem (1.1) has not been investigated yet.

Denote by $\mathscr{S}$ the set of $(P,\, Q,\, \beta)$-symmetric matrices or $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices. The first problem to be solved in the present paper can be expressed concretely as follows:

Problem 1.2. For given $X \in \mathbb{C}^{n\times p},\ Y \in \mathbb{C}^{m\times q},\ E \in \mathbb{C}^{m\times p},\ F \in \mathbb{C}^{q\times n}$ and k-involutory matrices $P \in \mathbb{C}^{m\times m},\ Q \in \mathbb{C}^{n\times n}$, find $A \in \mathscr{S}$ which satisfies the simultaneous equations (1.1).

Actually, if $m = n$ and let $E = X\Lambda,\ F = \mu Y^*$, where Λ and μ are all diagonal matrices with appropriate sizes, then the above constrained matrix equations problem becomes the corresponding left and right inverse eigenvalue problem [12, 14, 18] which mainly arises in perturbation analysis of matrix eigenvalue [25] and recursive matters [2].

The second problem to be considered here is the so-called optimal approximation problem.

Problem 1.3. Suppose that Problem 1.2 is consistent, i.e., the solution set $\mathscr{S}_E$ of which is nonempty. For given $\overline{A} \in \mathbb{C}^{m\times n}$, find $\widehat{A} \in \mathscr{S}_E$ such that

$$\|\widehat{A} - \overline{A}\| = \min_{A\in\mathscr{S}_E} \|A - \overline{A}\|.$$

This kind of problems plays an important role in practice, and has been discussed far and wide (see, e.g., [4, 8, 9, 12, 13, 14, 15, 18, 19, 20, 21, 26, 27] and therein).

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The remainder of this paper is outlined as follows. In Section 2, we will introduce some preliminary knowledge about the k-involutory matrices, the (P, Q, β)- and (P, Q, α, β)-symmetric matrices. Furthermore, the necessary and sufficient conditions for the consistency of (1.1) will be investigated, and the general solution will be represented in Section 3. The optimal approximation Problem 1.3 will be solved in Section 4.

2. Preliminaries. For k-involutory matrices P and Q as in Definition 1.1, let $p_t = \dim \Phi_P(\lambda^t)$, $q_t = \dim \Phi_Q(\lambda^t)$ $(t = 1, 2, \ldots, k)$ be the dimensions of eigenspaces to λ^t with respect to P and Q, respectively, which reveal that $m = \sum_{t=1}^k p_t$, $n = \sum_{t=1}^k q_t$. Hence, we know that there exist $P_t \in \mathbb{C}^{m\times p_t}$ and $Q_t \in \mathbb{C}^{n\times q_t}$ such that

$$PP_t = \lambda^t P_t,\ QQ_t = \lambda^t Q_t \text{ with } P_t^* P_t = I_{p_t},\ Q_t^* Q_t = I_{q_t}.$$

Moreover, there exist matrices $\widehat{P}_t \in \mathbb{C}^{p_t\times m}$, $\widehat{Q}_t \in \mathbb{C}^{q_t\times n}$ (see [23]) such that

$$P = U \operatorname{diag}(\lambda I_{p_1},\ \lambda^2 I_{p_2},\ \ldots,\ \lambda^k I_{p_k})\, \widehat{U},\ Q = V \operatorname{diag}(\lambda I_{q_1},\ \lambda^2 I_{q_2},\ \ldots,\ \lambda^k I_{q_k})\, \widehat{V}, \tag{2.1}$$

where $U = (P_1\ P_2\ \cdots\ P_k)$, $\widehat{U} = (\widehat{P}_1^T\ \widehat{P}_2^T\ \cdots\ \widehat{P}_k^T)^T$ with $\widehat{U}U = I_m$, and $V = (Q_1\ Q_2\ \cdots\ Q_k)$, $\widehat{V} = (\widehat{Q}_1^T\ \widehat{Q}_2^T\ \cdots\ \widehat{Q}_k^T)^T$ with $\widehat{V}V = I_n$.

In particular, if P and Q are unitary, then $\widehat{U} = U^* \in \mathbb{UC}^{m\times m}$, $\widehat{V} = V^* \in \mathbb{UC}^{n\times n}$, so are $\mathbb{U}_\beta$, $\mathbb{U}_{\beta,\alpha}$, here

$$\mathbb{U}_\beta = (P_{1+\beta}\ P_{2+\beta}\ \cdots\ P_{k+\beta}), \quad \widehat{\mathbb{U}}_\beta = (P_{1+\beta}^T\ P_{2+\beta}^T\ \cdots\ P_{k+\beta}^T)^T,$$

and

$$\mathbb{U}_{\beta,\alpha} = (P_{\alpha+\beta}\ P_{2\alpha+\beta}\ \cdots\ P_{\alpha k+\beta}).$$

In this case,

$$P = U \operatorname{diag}(\lambda I_{p_1},\ \lambda^2 I_{p_2}, \ldots,\ \lambda^k I_{p_k})\, U^*,\ Q = V \operatorname{diag}(\lambda I_{q_1},\ \lambda^2 I_{q_2}, \ldots,\ \lambda^k I_{q_k})\, V^*. \tag{2.2}$$

Remark 2.1. All arithmetic operations in subscripts are modulo k, i.e., $P_{k\alpha+\beta} = P_s$ predicates that $s \equiv k\alpha + \beta\,(mod\, k)$.

Lemma 2.2. ([23]) *Let P and Q be k-involutions as in (2.1). Then $A \in \mathbb{C}^{m\times n}$ is (P, Q, β)-symmetric if and only if*

$$A = \sum_{t=1}^k P_{t+\beta} G_t \widehat{Q}_t, \tag{2.3}$$

where $G_t = P_{t+\beta}^* A Q_t \in \mathbb{C}^{p_{t+\beta}\times q_t}$.

For (P, Q, α, β)-symmetric matrices, we have similar conclusion.

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Lemma 2.3. ([24]) *If P and Q are k-involutions as in (2.1), then $A \in \mathbb{C}^{m\times n}$ is (P, Q, α, β)-symmetric if and only if $A = \sum_{t=1}^{k} P_{t\alpha+\beta} G_t \widehat{Q}_t$ with $G_t = P^*_{t\alpha+\beta} A Q_t \in \mathbb{C}^{p_{t\alpha+\beta}\times q_t}$.*

In Lemma 2.3, the substitution $t \to t\alpha + \beta (mod\, k)$ is a permutation of $\mathbb{Z}_k$ if $gcd(\alpha, k) = 1$, but $\mathbb{U}_{\beta,\alpha}$ is not invertible if $gcd(\alpha, k) > 1$. In fact, when $gcd(\alpha, k) = a > 1$, the first $b = k/a$ columns of $\mathbb{U}_{\beta,\alpha}$ are repeated a times as brought forth in Example 1 of [15]. Furthermore, this lemma can be reformed in the following convenient form.

Lemma 2.4. ([24]) *Suppose that $gcd(\alpha, k) = a$ and $b = k/a$. For $1 \le l \le b$, denote*

$$\mathbb{V}_{l,\alpha} = (Q_{l+b}\ Q_{l+2b}\ \cdots\ Q_{l+ab}),\ \ \widehat{\mathbb{V}}_{l,\alpha} = (\widehat{Q}^T_{l+b}\ \widehat{Q}^T_{l+2b}\ \cdots\ \widehat{Q}^T_{l+ab})^T,$$

$$\mathscr{U}_{\beta,\alpha} = (P_{\alpha+\beta}\ P_{2\alpha+\beta}\ \cdots\ P_{b\alpha+\beta}),\ \ \widehat{\mathscr{U}_{\beta,\alpha}} = (\widehat{P}^T_{\alpha+\beta}\ \widehat{P}^T_{2\alpha+\beta}\ \cdots\ \widehat{P}^T_{b\alpha+\beta})^T,$$

and

$$\mathbb{G}_{l,\alpha} = (G_{l+b}\ G_{l+2b}\ \cdots\ G_{l+ab}),\ \ \mathscr{V}_\alpha = (\mathbb{V}_{1,\alpha}\ \mathbb{V}_{2,\alpha}\ \cdots\ \mathbb{V}_{b,\alpha}).$$

Then the columns of invertible matrix $\mathscr{V}_\alpha$ are a rearrangement of Q_t ($t \in \mathbb{Z}_k$), and $\widehat{\mathscr{U}_{\beta,\alpha}}\mathscr{U}_{\beta,\alpha} = I_{p_{\beta+\alpha}+p_{2\alpha+\beta}+\cdots+p_{b\alpha+\beta}}$. Then, the (P, Q, α, β)-symmetric matrix A can be represented as

$$A = \sum_{l=1}^{b} P_{l\alpha+\beta}\mathbb{G}_{l,\alpha}\widehat{\mathbb{V}}_{l,\alpha}. \tag{2.4}$$

3. The solvability conditions and general solution for Problem 1.2. In this section, we will obtain the solvability conditions for Problem 1.2 with (P, Q, β)- and (P, Q, α, β)-constraints, and the general solution of which will be derived.

The next lemma quoted from [28] is essential for our main results.

Lemma 3.1. *Given matrices M, N, C, D with appropriate orders, then matrix equations $MZ = N$ and $ZC = D$ are consistent, if and only if*

$$MD = NC,\ MM^+N = N,\ DC^+C = D.$$

Moreover, the expression of its general solution is

$$X = M^+N + (I - M^+M)DC^+ + (I - M^+M)H(I - CC^+)$$

with arbitrary matrix H.

Denote $X_t = \widehat{Q}_t X$, $E_{t+\beta} = \widehat{P}_{t+\beta}E$, $Y_{t+\beta} = Y^* P_{t+\beta}$, $F_t = FQ_t$. Then the solvability condition of matrix equation (1.1) in (P, Q, β)-symmetric matrices set is obtained by the following theorem.

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THEOREM 3.2. *Given matrices $X \in \mathbb{C}^{n\times p}$, $Y \in \mathbb{C}^{m\times q}$, $E \in \mathbb{C}^{m\times p}$, $F \in \mathbb{C}^{q\times n}$. Then matrix equations (1.1) with $(P,\, Q,\, \beta)$-symmetric matrix A are consistent if and only if*

$$Y_{t+\beta}E_{t+\beta} = F_tX_t,\ Y_{t+\beta}Y_{t+\beta}^{+}F_t = F_t \ \text{ and } \ E_{t+\beta}X_t^{+}X_t = E_{t+\beta} \tag{3.1}$$

hold for $t \in \mathbb{Z}_k$ at the same time. And the general solution can be expressed as

$$A = \sum_{t=1}^{k} P_{t+\beta}G_t\widehat{Q}_t, \tag{3.2}$$

where $G_t = \widetilde{G}_t + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})Z_t(I_{q_t} - X_tX_t^{+})$,

$$\widetilde{G}_t = Y_{t+\beta}^{+}F_t + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})E_{t+\beta}X_t^{+}, \quad Z_t \in \mathbb{C}^{p_{t+\beta}\times q_t}.$$

Proof. From Lemma 2.2, any $m\times n$ $(P,\, Q,\, \beta)$-symmetric matrix A can be written as in (2.3). Substituting it into (1.1), yields

$$\begin{cases} \sum_{t=1}^{k} P_{t+\beta}G_t\widehat{Q}_tX = E, \\ Y^*\sum_{t=1}^{k} P_{t+\beta}G_t\widehat{Q}_t = F. \end{cases} \tag{3.3}$$

Noting that the previous notations and pre-multiplying the both side of (3.3) from the left by $\widehat{\mathbb{U}}_\beta$ on the first equation, and right by V on the second one, then

$$\begin{cases} G_tX_t = E_{t+\beta}, \\ Y_{t+\beta}G_t = F_t, \end{cases} \qquad t = 1,\, 2,\, \ldots,\, k, \tag{3.4}$$

which imply that the solvability of matrix equations (3.3) is in accord with (3.4). While the matrix equations (3.4) are consistent for G_t, from Lemma 3.1, if and only if (3.1) holds, and the general expression of the solution is

$$G_t = Y_{t+\beta}^{+}F_t + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})E_{t+\beta}X_t^{+} + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})Z_t(I_{q_t} - X_tX_t^{+}).$$

Connecting with (2.3), we get (3.2). □

REMARK 3.3.

- If P and Q are unitary matrices, then $\widehat{Q}_t$ and $\widehat{P}_{t+\beta}$ in Theorem 3.2 become Q_t^*, and $P_{t+\beta}^*$.
- Let $E = X\Lambda$ and $F = \mu Y^*$ in Theorem 3.2. Then we can obtain the solvability conditions and the general soluiton of the left and right inverse eigenvalue problem for $(P,\, Q,\, \beta)$-symmetric matrices.

Analogously, we can establish the consistent criterions of matrix equations (1.1) with $(P,\, Q,\, \alpha,\, \beta)$-constraint.

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THEOREM 3.4. *Given matrices X, Y, E, F as in Theorem 3.2. Let $\mathbb{E}_{l\alpha+\beta} = \widehat{P}_{l\alpha+\beta}E$, $\mathbb{X}_{l,\alpha} = \widehat{\mathbb{V}}_{l,\alpha}X$, $\mathbb{Y}_{l\alpha+\beta} = Y^* P_{l\alpha+\beta}$, $\mathbb{F}_{l,\alpha} = F\mathbb{V}_{l,\alpha}$. Then matrix equations (1.1) are solvable with (P, Q, α, β)-symmetric A if and only if*

$$\mathbb{Y}_{l\alpha+\beta}\mathbb{E}_{l\alpha+\beta} = \mathbb{F}_{l,\alpha}\mathbb{X}_{l,\alpha},\ \mathbb{Y}_{l\alpha+\beta}\mathbb{Y}^+_{l\alpha+\beta}\mathbb{F}_{l,\alpha} = \mathbb{F}_{l,\alpha} \text{ and } \mathbb{E}_{l\alpha+\beta}X_t^+\mathbb{X}_{l,\alpha} = E_{l\alpha+\beta} \tag{3.5}$$

hold for $t \in \mathbb{Z}_k$, and the general solution is

$$A = \sum_{l=1}^{b} P_{l\alpha+\beta}\mathbb{G}_{l,\alpha}\widehat{\mathbb{V}}_{l,\alpha}, \tag{3.6}$$

where $\mathbb{G}_{l,\alpha} = \overline{\mathbb{G}}_{l,\alpha} + (I_{p_{l\alpha+\beta}} - \mathbb{Y}^+_{l\alpha+\beta}\mathbb{Y}_{l\alpha+\beta})\mathscr{Z}_{l,\alpha}(I_{\sum_{r=1}^{a} q_{l+rb}} - \mathbb{X}_{l,\alpha}\mathbb{X}^+_{l,\alpha})$,
$\overline{\mathbb{G}}_{l,\alpha} = \mathbb{Y}_{l\alpha+\beta}\mathbb{F}_{l,\alpha} + (I_{p_{l\alpha+\beta}} - \mathbb{Y}^+_{l\alpha+\beta}\mathbb{Y}_{l\alpha+\beta})\mathbb{E}_{l\alpha+\beta}\mathbb{X}^+_{l,\alpha}$, $\mathscr{Z}_{l,\alpha} \in \mathbb{C}^{p_{l\alpha+\beta}\times\sum_{r=1}^{a} q_{l+rb}}$.

Proof. Similarly to the proof of Theorem 3.2, substituting (2.4) into (1.1), we obtain

$$\begin{cases} \sum_{l=1}^{b} P_{l\alpha+\beta}\mathbb{G}_{l,\alpha}\widehat{\mathbb{V}}_{l,\alpha}X = E, \\ Y^*\sum_{l=1}^{b} P_{l\alpha+\beta}\mathbb{G}_{l,\alpha}\widehat{\mathbb{V}}_{l,\alpha} = F. \end{cases} \tag{3.7}$$

Pre-multiplying both sides of (3.7) from the left by $\widehat{\mathscr{U}_{\beta,\alpha}}$ on the first equation, and right by $\mathscr{V}_\alpha$ on the second one, imply that (3.7) can be transformed equivalently as

$$\begin{cases} \mathbb{G}_{l,\alpha}\mathbb{X}_{l,\alpha} = \mathbb{E}_{l\alpha+\beta}, \\ \mathbb{Y}_{l\alpha+\beta}\mathbb{G}_{l,\alpha} = \mathbb{F}_{l,\alpha}, \end{cases} \quad l = 1, 2, \ldots, b. \tag{3.8}$$

Applying Lemma 3.1 to (3.8) reveals that (3.5) and (3.6) hold. □

REMARK 3.5.

- If P and Q are unitary matrices in Theorem 3.4, we need only write $\widehat{\mathbb{V}}_{l,\alpha}$ as $\mathbb{V}^*_{l,\alpha}$, and $\widehat{P}_{t\alpha+\beta}$ as $P^*_{t\alpha+\beta}$ in (3.6).
- Let $E = X\Lambda$ and $F = \mu Y^*$ in Theorem 3.4. Then the left and right inverse eigenvalue problem for the (P, Q, α, β)-symmetric matrices will also be solved.

4. The solution to the optimal approximation problem. In this section, let the k-involutory matrices P and Q be unitary, that is, they possess the forms as in (2.2), and U, V, $\widehat{\mathscr{U}_\beta} = \mathscr{U}^*_\beta$, $\widehat{\mathscr{U}_{\beta,\alpha}} = \mathscr{U}^*_{\beta,\alpha}$ are also unitary.

The following lemma derived from [22] is necessary for solving the approximation Problem 1.3.

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LEMMA 4.1. *Let* $L \in \mathbb{C}^{q\times m}$, $\Delta \in \mathbb{C}^{q\times q}$, $\Gamma \in \mathbb{C}^{m\times m}$ *and* $\Delta^2 = \Delta = \Delta^*$, $\Gamma^2 = \Gamma = \Gamma^*$. *Then*

$$\| L - \Delta L\Gamma \| = \min_{G\in\mathbb{C}^{q\times m}} \| L - \Delta G\Gamma \|$$

if and only if $\Delta(L-G)\Gamma = 0$.

THEOREM 4.2. *Suppose that (3.1) holds in Theorem 3.2, i.e., the solution set* $\mathscr{S}_{E_1}$ *of matrix equations (1.1) is nonempty. Then for given* $\overline{A} \in \mathbb{C}^{m\times n}$, *there exists uniquely* $\widehat{A}_1 \in \mathscr{S}_{E_1}$ *such that*

$$\|\widehat{A}_1 - \overline{A}\| = \min_{A\in\mathscr{S}_{E_1}} \|A - \overline{A}\|$$

and

$$\widehat{A}_1 = \sum_{t=1}^{k} P_{t+\beta}\widehat{G}_t Q_t^*, \tag{4.1}$$

where $\widehat{G}_t = \widetilde{G}_t + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})(\widetilde{G}_t - P_{t+\beta}^*\overline{A}Q_t)(I_{q_t} - X_tX_t^{+})$, $\widetilde{G}_t$ *as in (3.2).*

Proof. The uniqueness of the solution dues to the closed and convex set $\mathscr{S}_{E_1}$ when it is nonempty. Hence, it is enough to prove (4.1). In fact,

$$\mathscr{U}_\beta^*(A-\overline{A})V = \begin{pmatrix} G_1 - P_{1+\beta}^*\overline{A}Q_1 & \cdots & -P_{1+\beta}^*\overline{A}Q_k \\ \vdots & \ddots & \vdots \\ -P_{k+\beta}^*\overline{A}Q_1 & \cdots & G_k - P_{k+\beta}^*\overline{A}Q_k \end{pmatrix},$$

it follows from the unitary invariance of Frobenius norm that

$$\|(A-\overline{A})\|^2 = \textstyle\sum_{t=1}^k \|G_t - P_{t+\beta}^*\overline{A}Q_t\|^2 + \sum_{s,t=1,s\neq t}^k \|P_{t+\beta}^*\overline{A}Q_s\|^2,$$

which shows that the approximation Problem 1.3 can be changed equivalently into the following least residual problem

$$\min_{G_t\in\mathbb{C}^{p_{t+\beta}\times q_t}} \|G_t - P_{t+\beta}^*\overline{A}Q_t\|, \quad t = 1, 2, \dots, k.$$

That is,

$$\min_{Z_t\in\mathbb{C}^{p_{t+\beta}\times q_t}} \|\widetilde{G}_t - P_{t+\beta}^*\overline{A}Q_t + (I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta})Z_t(I_{q_t} - X_tX_t^{+})\|.$$

We can verify that $I_{p_{t+\beta}} - Y_{t+\beta}^{+}Y_{t+\beta}$ and $I_{q_t} - X_tX_t^{+}$ satisfy the requirements in Lemma 4.1, which implies that

$$(I_{t+\beta} - Y_{t+\beta}^{+}Y_{t+\beta})(\widetilde{G}_t - P_{t+\beta}^*\overline{A}Q_t - Z_t)(I_{q_t} - X_tX_t^{+}) = 0.$$

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Combining it with (3.2), we know that (4.1) holds. □

For the approximation problem of (P, Q, α, β)-symmetric matrices, we have the following conclusion.

THEOREM 4.3. *Suppose that (3.5) holds in Theorem 3.4, i.e., the solution set* $\mathscr{S}_{E_2}$ *of (1.1) is nonempty. Then, for given matrix* $\overline{A} \in \mathbb{C}^{m\times n}$*, the unique solution* $\widehat{A}_2 \in \mathscr{S}_{E_2}$ *to Problem 1.2 is*

$$\widehat{A}_2 = \sum_{l=1}^{b} P_{l\alpha+\beta}\widehat{\mathbb{G}}_{l,\alpha}\widehat{\mathbb{V}}_{l,\alpha}, \tag{4.2}$$

$\widehat{\mathbb{G}}_{l,\alpha} = \overline{\mathbb{G}}_{l,\alpha} + (I_{p_{l\alpha+\beta}} - \mathbb{Y}^{+}_{l\alpha+\beta}\mathbb{Y}_{l\alpha+\beta})(\overline{\mathbb{G}}_{l,\alpha} - P_{l\alpha+\beta}\overline{A}\mathbb{V}_{l,\alpha})(I_{\sum_{r=1}^{a}(q_{l+rb})}) - \mathbb{X}_{l,\alpha}\mathbb{X}^{+}_{l,\alpha})$, $\overline{\mathbb{G}}_{l,\alpha}$ *as in (3.6).*

Proof. We only prove (4.2). Since

$$\mathscr{U}^{*}_{\beta,\alpha}(A-\overline{A})\mathscr{V}_{\alpha} = \begin{pmatrix} \mathbb{G}_{1,\alpha} - P^{*}_{\alpha+\beta}\overline{A}\mathbb{V}_{1,\alpha} & \cdots & -P^{*}_{\alpha+\beta}\overline{A}\mathbb{V}_{b,\alpha} \\ \vdots & \ddots & \vdots \\ -P^{*}_{b\alpha+\beta}\overline{A}\mathbb{V}_{1,\alpha} & \cdots & \mathbb{G}_{b,\alpha} - P^{*}_{b\alpha+\beta}\overline{A}\mathbb{V}_{b,\alpha} \end{pmatrix},$$

then we have

$$\|(A-\overline{A})\|^2 = \textstyle\sum_{l=1}^{b} \|\mathbb{G}_{l,\alpha} - P^{*}_{l\alpha+\beta}\overline{A}\mathbb{V}_{l,\alpha}\|^2 + \sum_{s,l=1,s\neq l}^{k} \|P^{*}_{l\alpha+\beta}\overline{A}\mathbb{V}_{s,\alpha}\|^2.$$

Hence Problem 1.3 is equivalent to the least residual problem

$$\min_{\mathbb{G}_{l,\alpha}\in\mathbb{C}^{p_{l\alpha+\beta}\times\sum_{r=1}^{a}q_{l+rb}}} \|\mathbb{G}_{l,\alpha} - P^{*}_{l\alpha+\beta}\overline{A}\mathbb{V}_{l,\alpha}\|, \quad t = 1,\, 2,\, \ldots,\, k.$$

That is,

$$\min_{\mathbb{Z}_{l,\alpha}\in\mathbb{C}^{p_{l\alpha+\beta}\times\sum_{r=1}^{a}q_{l+rb}}} \|\overline{\mathbb{G}}_{l,\alpha} - P^{*}_{t\alpha+\beta}\overline{A}Q_t + (I_{p_{l\alpha+\beta}}$$

$$- \mathbb{Y}^{+}_{l\alpha+\beta}\mathbb{Y}_{l\alpha+\beta})\mathcal{Z}_{l,\alpha}(I_{\sum_{r=1}^{a}q_{l+rb}} - \mathbb{X}_{l,\alpha}\mathbb{X}^{+}_{l,\alpha})\|.$$

By Lemma 4.1, we know that (4.2) holds. □

REMARK 4.4.

- If P and Q are unitary, E and F are as in Remark 3.5, we can also obtain the optimal approximation problem of the left and right inverse eigenvalue problem for the (P, Q, β)-symmetric matrices and (P, Q, α, β)-symmetric matrices from Theorem 4.2 and Theorem 4.3, respectively.

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5. Conclusion. In this paper, we have obtained the solvability conditions of matrix equations (1.1) over two kinds of generalized symmetric matrices sets, that is, $(P,\, Q,\, \beta)$- and $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices sets, and the general solutions of which have been represented when the solvability conditions were satisfied. Particularly, the left and right inverse eigenvalue problems for the $(P,\, Q,\, \beta)$-symmetric matrices and $(P,\, Q,\, \alpha,\, \beta)$-symmetric matrices were solved as special cases. Furthermore, when the k-involutory matrices P and Q were unitary, the approximation problem for some given matrix was also considered.

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