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ALGEBRAIC CONNECTIVITY AND SPECTRAL W-VARIATION OF UNICYCLIC GRAPHS[*]

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Abstract. This article characterizes the unicyclic graphs G with the following property: when a new edge with positive weight w is added between two nonadjacent vertices of G or when the weight of an existing edge is increased by w, exactly two Laplacian eigenvalues of G each increase by w, while all other eigenvalues remain unchanged. Furthermore, we identify the unicyclic graphs for which one of the altered eigenvalues is the algebraic connectivity.

Key words. Unicyclic graph, Laplacian eigenvalues, Algebraic connectivity, Spectral integral variation, Spectral w-variation.

AMS subject classifications. 05C50, 15A15.

1. Introduction.

Throughout this article, we restrict our attention to simple undirected graphs. Consider a graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The *adjacency matrix* of G, denoted by $A(G)$, is a matrix $[a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent in } G,\\ 0, & \text{otherwise.}\end{cases}$$

The Laplacian matrix $L(G)$ of G is determined by $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix with vertex degrees on the diagonal. It is well known that $L(G)$ is symmetric, positive, semi-definite, and singular. We write the eigenvalues of $L(G)$ in nonincreasing order as $\lambda_n(G) \geq \lambda_{n-1}(G) \geq \cdots \geq \lambda_1(G) = 0$. Fiedler [5] proved that $\lambda_2(G)$ is positive if and only if G is a connected graph. Based on this observation, Fiedler introduced the term *algebraic connectivity*, denoted by $a(G)$, to describe this eigenvalue. The eigenvectors associated with $a(G)$ are referred to as the *Fiedler vectors* of G. For further details on the fundamental properties of the Laplacian matrix, one may refer to [8] and its references.

By the weight of an edge, we mean a positive real number assigned to that edge. Consider two vertices $i, j \in V(G)$ and a real number $w > 0$. We construct a new graph which is a weighted graph, denoted by $G^w_{i,j}$, from G as follows: *if there is no edge between i and j in G, insert a new edge of weight w between them, otherwise, increase the weight of the existing edge $\{i, j\}$ by w.* Then, $L(G^w_{i,j}) = L(G) + w(e_i - e_j)(e_i - e_j)^t$, where e_i and e_j are the standard basis vectors. Under this operation if exactly two Laplacian eigenvalues of G increase by w and the remaining $n - 2$ eigenvalues remain unchanged, then the graph G is said to have *spectral w-variation in two places.* The notion of spectral w-variation of graphs was first introduced in [6]. Concisely, this phenomenon is expressed as $SV2[\{i, j\}, w]$ *occurs* in G. In [6], the authors characterized the weighted graphs that have spectral w-variation in two places. They determined the weighted trees that have spectral w-variation in two places. Note that the concept of spectral integral variation of graphs at two

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positions is a special case of the occurrence of SV2$[\{i,j\},w]$ in a graph G, when $w=1$ and the vertices i and j are not adjacent in G. The notion of spectral integral variation of the graph was first introduced by So [11], and its characteristics have been thoroughly investigated in subsequent research by many researchers; see [10, 12, 13, 14]. Fan [12, 13, 14] and Kirkland [10] have characterized the graphs within certain subclasses where the spectral integral variation occurs in two places. In particular, Fan et al. [12] identified the trees in which spectral integral variation occurs in two places and identified those in which one of the altered eigenvalues corresponds to algebraic connectivity. Barik et al. [9] identified graphs in which spectral integral variation occurs at one place, specifically when the altered eigenvalue is the algebraic connectivity. In this article, we present a complete characterization of unicyclic graphs in which SV2$[\{i,j\},w]$ occurs. We determine the admissible values of w and show that SV2$[\{i,j\},w]$ can occur in a unicyclic graph only when $w\in\{1,2,3\}$. For each of these values, we identify the corresponding classes of unicyclic graphs. In addition, we characterize the unicyclic graphs for which SV2$[\{i,j\},w]$ occurs and one of the altered eigenvalues is the algebraic connectivity. Since no prior work has addressed spectral integral variation in two places for unicyclic graphs, our results provide a complete classification of such graphs, including those where one of the modified eigenvalues equals the algebraic connectivity.

The article is structured as follows. Section 2 covers the preliminaries, including essential notation and definitions. In Section 3, we characterize the unicyclic graphs in which SV2$[\{i,j\},w]$ occurs. Section 4 addresses the special case where the algebraic connectivity is one of the altered eigenvalues.

2. Preliminaries.

Let G be a connected graph with vertex set $V=\{1,2,\ldots,n\}$. For any two vertices i and j in G, we write $i\sim j$ (respectively, $i\nsim j$) if i and j are adjacent (respectively, not adjacent) in G. Suppose that SV2$[\{1,2\},w]$ occurs in G. Then, throughout this article, the Laplacian matrix of G is considered in a partitioned form as follows:

$$L=\left[\begin{array}{cc|c|c|c|c} d_1 & -w' & -\mathbf{1}^t & \mathbf{0}^t & -\mathbf{1}^t & \mathbf{0}^t \\ -w' & d_2 & \mathbf{0}^t & -\mathbf{1}^t & -\mathbf{1}^t & \mathbf{0}^t \\ \hline -\mathbf{1} & \mathbf{0} & \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} & \mathbf{B}_{14} \\ \hline \mathbf{0} & -\mathbf{1} & \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} & \mathbf{B}_{24} \\ \hline -\mathbf{1} & -\mathbf{1} & \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{33} & \mathbf{B}_{34} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{B}_{41} & \mathbf{B}_{42} & \mathbf{B}_{43} & \mathbf{B}_{44} \end{array}\right], \tag{2.1}$$

where the value of w' is either 0 or 1 depending on whether $1\nsim 2$ or $1\sim 2$, respectively, and the sizes of the blocks $\mathbf{B}_{11},\mathbf{B}_{22},\ldots,\mathbf{B}_{44}$ are d_1-p-w', d_2-p-w', p and $n-2-d_1-d_2+p+2w'$, respectively. Here, p denotes the total number of vertices adjacent to both 1 and 2. We denote a column vector where every element is 1 by $\mathbf{1}$. The notations P_n and C_n represent the path and cycle graphs comprising n vertices, respectively.

The neighborhood of a vertex i in G, represented as $N(i)$, is defined as the collection $N(i)=\{v\in V(G)\mid v\sim i\}$. Given any pair of vertices i and j in G, we define the following: $\mathcal{N}_i=N(i)\backslash((N(i)\cap N(j))\cup\{j\}), \mathcal{N}_j=N(j)\backslash((N(i)\cap N(j))\cup\{i\}), \mathcal{N}_i^j=N(i)\cap N(j)$ and $\mathcal{N}_{ij}^c=V(G)\backslash(N(i)\cup N(j)\cup\{i,j\})$. For any vertex $i\in V(G)$ and any subset $S\subseteq V(G)$, we define the set $\Theta_i(S)$ as $\Theta_i(S)=N(i)\cap S$, and we use $|\Theta_i(S)|$ to denote the number of elements within the set $\Theta_i(S)$.

The following result can be derived from [[6], Lemma 2].

Lemma 2.1. *Let G be a graph with vertex set $\{1,2,\ldots,n\}$. Then, the spectra of $G_{i,j}^w$ and G coincide in $n-1$ places if and only if $N(i)\setminus\{j\}=N(j)\setminus\{i\}$.*

Using the same arguments presented in [[6], Theorem 3], the following result can be proved, which characterizes all graphs in which SV2$[\{i,j\},w]$ occurs.

Theorem 2.2. *Suppose G is a graph on vertices $1,2,\ldots,n$ and let its Laplacian matrix L be given by*

$$L=\left[\begin{array}{cc|c|c|c|c}
d_1 & -w' & -\mathbf{1}^t & \mathbf{0}^t & -\mathbf{1}^t & \mathbf{0}^t \\
-w' & d_2 & \mathbf{0}^t & -\mathbf{1}^t & -\mathbf{1}^t & \mathbf{0}^t \\ \hline
-\mathbf{1} & \mathbf{0} & \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} & \mathbf{B}_{14} \\ \hline
\mathbf{0} & -\mathbf{1} & \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} & \mathbf{B}_{24} \\ \hline
-\mathbf{1} & -\mathbf{1} & \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{33} & \mathbf{B}_{34} \\ \hline
\mathbf{0} & \mathbf{0} & \mathbf{B}_{41} & \mathbf{B}_{42} & \mathbf{B}_{43} & \mathbf{B}_{44}
\end{array}\right],$$

where the value of w' is either 0 or 1 depending on whether $1\nsim 2$ or $1\sim 2$, respectively, and the sizes of the blocks $\mathbf{B}_{11},\mathbf{B}_{22},\ldots,\mathbf{B}_{44}$ are d_1-p-w', d_2-p-w', p and $n-2-d_1-d_2+p+2w'$, respectively. Consider a real number $w>0$. Then, SV2$[\{1,2\},w]$ occurs in G with changed eigenvalues λ_{i_1} and λ_{i_2} with $\lambda_{i_1}\leq\lambda_{i_2}$ if and only if the following conditions hold:

$$d_1-d_2=(d_1-d_2)(w+2w'), \tag{2.2}$$

$$\mathbf{B}_{11}\mathbf{1}-\mathbf{B}_{12}\mathbf{1}=(d_2+w+w')\mathbf{1}, \tag{2.3}$$

$$\mathbf{B}_{21}\mathbf{1}-\mathbf{B}_{22}\mathbf{1}=-(d_1+w+w')\mathbf{1}, \tag{2.4}$$

$$\mathbf{B}_{31}\mathbf{1}-\mathbf{B}_{32}\mathbf{1}=(d_2-d_1)\mathbf{1}, \tag{2.5}$$

$$\mathbf{B}_{41}\mathbf{1}-\mathbf{B}_{42}\mathbf{1}=\mathbf{0}. \tag{2.6}$$

If conditions (2.2)- (2.6) are satisfied, the two eigenvalues of matrix L that will be changed are

$$\lambda_{i_1}=\frac{d_1+d_2+2w'+w-\sqrt{(d_1-d_2)^2+w^2+2(d_1+d_2-2p)}}{2},$$

and

$$\lambda_{i_2}=\frac{d_1+d_2+2w'+w+\sqrt{(d_1-d_2)^2+w^2+2(d_1+d_2-2p)}}{2},$$

and the vectors

$$u_1=\left[\begin{array}{c}
d_2+w+w'-\lambda_{i_1} \\
\lambda_{i_1}-d_1-w-w' \\ \hline
\mathbf{1} \\ \hline
-\mathbf{1} \\ \hline
\mathbf{0} \\ \hline
\mathbf{0}
\end{array}\right] \text{ and } u_2=\left[\begin{array}{c}
d_2+w+w'-\lambda_{i_2} \\
\lambda_{i_2}-d_1-w-w' \\ \hline
\mathbf{1} \\ \hline
-\mathbf{1} \\ \hline
\mathbf{0} \\ \hline
\mathbf{0}
\end{array}\right],$$

are eigenvectors of L associated with λ_{i_1} and λ_{i_2}, respectively.

Remark 2.3. In the scenario outlined in Theorem 2.2, there is a possibility that one or more of the values d_1-p-w', d_2-p-w', p, and $n-2-d_1-d_2+p+2w'$ could be zero. When this happens, the corresponding rows and columns in the partitioned structure of L are missing, which means that the related conditions (2.3)–(2.6) do not hold.

Observe that in the scenario described in Theorem 2.2, if SV2$[\{i,j\},w]$ occurs in G, then both d_1-p-w' and d_2-p-w' cannot be zero, otherwise we get $N(1)\setminus\{2\}=N(2)\setminus\{1\}$, which leads to a contradiction by Lemma 2.1. Consequently, one of the conditions (2.3) and (2.4) will always be applied. Hence, from the conditions (2.3) or (2.4) described in Theorem 2.2, we derive the following corollary.

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Corollary 2.4. *Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ such that $SV2[\{i, j\}, w]$ occurs in G. Then, w must be a positive integer.*

We derive the following result as a corollary of Theorem 2.2, which asserts that if SV2$[\{i, j\}, w]$ occurs for a graph G, then the set $N(i) \cup N(j)$ contains non-pendant vertices.

Corollary 2.5. *Let G be a graph with vertex set $\{1, 2, \dots, n\}$ such that $SV2[\{i, j\}, w]$ occurs in G. Then the set $N(i) \cup N(j)$ cannot contain pendant vertices.*

Proof. Without the loss of generality, take $i = 1, j = 2$ and consider the Laplacian matrix of G as presented in (2.1). On the contrary, assume that there exists a pendant vertex $v \in N(1)$. By condition (2.2) of Theorem 2.2, we have $v \neq 2$, and hence $v \in \mathcal{N}_1$. Then, the row sums of the blocks $\mathbf{B}_{11}$ and $\mathbf{B}_{12}$ corresponding to the vertex v are 1 and 0, respectively. By condition (2.3) of Theorem 2.2, we then have $d_2 + w + w' = 1$, which is a contradiction as $d_2 \geq 1$, $w > 0$ and $w' \in \{0, 1\}$. Using the same argument, we can show that the set $N(2)$ also cannot contain any pendant vertex. ◻

The subsequent result can be deduced as a corollary from Theorem 2.2.

Corollary 2.6. *Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ such that $SV2[\{i, j\}, w]$ occurs in G. Then, the following properties hold:*

$$\text{(1)} \quad \textit{For all } v \in \mathcal{N}_i, \quad 2|\Theta_v(\mathcal{N}_j)| + |\Theta_v(\mathcal{N}_i^j)| + |\Theta_v(\mathcal{N}_{ij}^c)| = d_j + w + w' - 1. \tag{2.7}$$

$$\text{(2)} \quad \textit{For all } v \in \mathcal{N}_j, \quad 2|\Theta_v(\mathcal{N}_i)| + |\Theta_v(\mathcal{N}_i^j)| + |\Theta_v(\mathcal{N}_{ij}^c)| = d_i + w + w' - 1.$$

$$\text{(3)} \quad \textit{For all } v \in \mathcal{N}_i^j, \quad |\Theta_v(\mathcal{N}_i)| - |\Theta_v(\mathcal{N}_j)| = d_i - d_j.$$

$$\text{(4)} \quad \textit{For all } v \in \mathcal{N}_{ij}^c, \quad |\Theta_v(\mathcal{N}_i)| = |\Theta_v(\mathcal{N}_j)|. \tag{2.8}$$

Proof. Without loss of generality, take $i = 1, j = 2$ and consider the Laplacian matrix of G as presented in (2.1). Since $L\mathbf{1} = \mathbf{0}$, we find that $\mathbf{B}_{11}\mathbf{1} - \mathbf{B}_{12}\mathbf{1} = -2\mathbf{B}_{12}\mathbf{1} - \mathbf{B}_{13}\mathbf{1} - \mathbf{B}_{14}\mathbf{1} + \mathbf{1}$. Applying condition (2.3) from Theorem 2.2, we deduce that for each vertex $v \in \mathcal{N}_1$,

$$2|\Theta_v(\mathcal{N}_2)| + |\Theta_v(\mathcal{N}_1^2)| + |\Theta_v(\mathcal{N}_{12}^c)| = d_2 + w + w' - 1,$$

which establishes property (1). Similarly, using conditions (2.4), (2.5), and (2.6), from Theorem 2.2, we derive the remaining properties (2), (3), and (4), corresponding to the vertices in $\mathcal{N}_j, \mathcal{N}_i^j$, and $\mathcal{N}_{ij}^c$, respectively. ◻

From condition (2.8) of Corollary 2.6, we derive the subsequent result.

Lemma 2.7. *Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ such that $SV2[\{i, j\}, w]$ occurs in G. Then, a vertex $v \in \mathcal{N}_{ij}^c$ is adjacent to a vertex in $\mathcal{N}_i$ if and only if it is adjacent to a vertex in $\mathcal{N}_j$.*

Applying Lemma 2.7, we obtain the subsequent result.

Lemma 2.8. *Suppose that G is a graph with $V(G) = \{1, 2, \dots, n\}$ such that $SV2[\{i, j\}, w]$ occurs in G. Let k denote either i or j. If $d_k \geq 2$, then for each $v \in N(k)$, there exists a cycle in G that contains both k and v.*

Proof. Let $P : i, v_1, v_2, \dots, v_l, j$ be a path between i and j, where $l \geq 0$. Without loss of generality, assume $k = i$. Let $u_1 \in N(i)$ such that $u_1 \neq v_1$. By Corollary 2.5, u_1 cannot be a pendant vertex. If $u_1 = j$ or $u_1 \in \mathcal{N}_i^j$, then we are done. Suppose $u_1 \neq j$ and $u_1 \notin \mathcal{N}_i^j$, which implies that $u_1 \in \mathcal{N}_i$. From condition (2.7) of Corollary 2.6, we have $2|\Theta_{u_1}(\mathcal{N}_j)| + |\Theta_{u_1}(\mathcal{N}_i^j)| + |\Theta_{u_1}(\mathcal{N}_{ij}^c)| = d_j + w + w' - 1$. Since

$d_j + w + w' - 1 \neq 0$, it follows that u_1 must be adjacent to at least one of the vertices of $\mathcal{N}_j \cup \mathcal{N}_i^j \cup \mathcal{N}_{ij}^c$. If u_1 is adjacent to a vertex in $\mathcal{N}_j \cup \mathcal{N}_i^j$, we get a cycle containing i and u_1. Also, if u_1 is adjacent to a vertex within $\mathcal{N}_{ij}^c$, from Lemma 2.7, we again get a cycle containing both i and u_1. ∎

3. Spectral w-variation of unicyclic graphs.

This section focuses on identifying unicyclic graphs for which SV2$[\{i,j\},w]$ occurs.

The subsequent result states that in a unicyclic graph U, SV2$[\{i,j\},w]$ does not occur if $i \sim j$ within U.

Lemma 3.1. *Let U be a unicyclic graph with $V(U) = \{1,2,\dots,n\}$. Suppose that $i,j \in V(U)$ be such that $i \sim j$. Then, SV2$[\{i,j\},w]$ does not occur in U for any value of w.*

Proof. Without loss of generality, take $i = 1, j = 2$ and consider the Laplacian matrix of U as presented in (2.1). From condition (2.2) described in Theorem 2.2, we have $d_1 - d_2 = (d_1 - d_2)(w+2)$, which implies that $d_1 = d_2$. Clearly, condition $d_1 = d_2 = 1$ is not possible. Since U contains exactly one cycle, by Lemma 2.8, we have $d_1 = d_2 = 2$. If $\mathcal{N}_1^2 \neq \emptyset$, then we have $N(1)\backslash\{2\} = N(2)\backslash\{1\}$, which is a contradiction by Lemma 2.1. Now assume that $\mathcal{N}_1^2 = \emptyset$ and let $v \in \mathcal{N}_1$. By condition 2.7 of Corollary 2.6, we have $2|\Theta_v(\mathcal{N}_2)| + |\Theta_v(\mathcal{N}_{12}^c)| = w+2$. By Corollary 2.5, the right-hand side satisfies $w + 2 \geq 3$ and hence $2|\Theta_v(\mathcal{N}_2)| + |\Theta_v(\mathcal{N}_{12}^c)| \geq 3$. Since $|\Theta_v(\mathcal{N}_2)| \leq 1$ and $|\Theta_v(\mathcal{N}_{12}^c)| \leq 1$ by Lemma 2.7, hence the only possibility is $|\Theta_v(\mathcal{N}_2)| = |\Theta_v(\mathcal{N}_{12}^c)| = 1$. This, however, implies that there exists more than one cycle in U, which is a contradiction. ∎

Using Lemma 2.8, we obtain the subsequent result, which determines the degrees of the vertices i and j in a unicyclic graph exhibiting SV2$[\{i,j\},w]$.

Lemma 3.2. *Let U be a unicyclic graph with $V(U) = \{1,2,\dots,n\}$. Suppose that SV2$[\{i,j\},w]$ occurs in U. Suppose Γ represents the unique cycle in U. Let k be either i or j. Then,*

1. *if $k \notin V(\Gamma)$, $d_k = 1$,*
2. *if $k \in V(\Gamma)$, $d_k = 2$.*

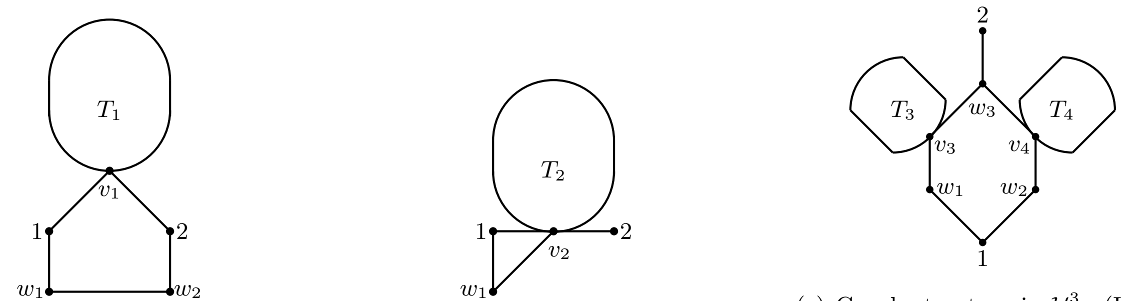
Proof. If $k \notin V(\Gamma)$, by Lemma 2.8, it follows that $d_k = 1$. Suppose $k \in V(\Gamma)$. On the contrary, assume that $d_k \geq 3$. Let $v_1, v_2, v_3 \in N(k)$ be distinct vertices. By Lemma 2.8, the vertices v_1, v_2, and v_3 are contained in a cycle that includes the vertex k. It follows that there are at least two distinct cycles in U, which is a contradiction. Hence, $d_k = 2$. ∎

Now we construct the following classes of unicyclic graphs. Let $\mathcal{U}_1^1$, $\mathcal{U}_1^2$, $\mathcal{U}_1^3$, $\mathcal{U}_1^4$, $\mathcal{U}_2$, and $\mathcal{U}_3$ denote the classes of unicyclic graphs with structure as shown in Figs. 1a, 1b, 1c, 2a, 2b and 2c, respectively.

The following theorem determines all the unicyclic graphs where SV2$[\{i,j\},w]$ occurs.

Theorem 3.3. *Suppose U is a unicyclic graph with $V(U) = \{1,2,\dots,n\}$. Then, SV2$[\{1,2\},w]$ occurs if and only if one of the following conditions is fulfilled:*

1. *$w = 1$ and either $U \in \mathcal{U}_1^1 \cup \mathcal{U}_1^2 \cup \mathcal{U}_1^3 \cup \mathcal{U}_1^4$ with vertices 1 and 2 as shown in Figs. 1a, 1b, 1c, 2a or U is a cycle of length 6 with vertices 1 and 2 at a distance 3.*
2. *$w = 2$ and $U \in \mathcal{U}_2$ with vertices 1 and 2 as shown in Fig. 2b.*
3. *$w = 3$ and $U \in \mathcal{U}_3$ with vertices 1 and 2 as shown in Fig. 2c.*

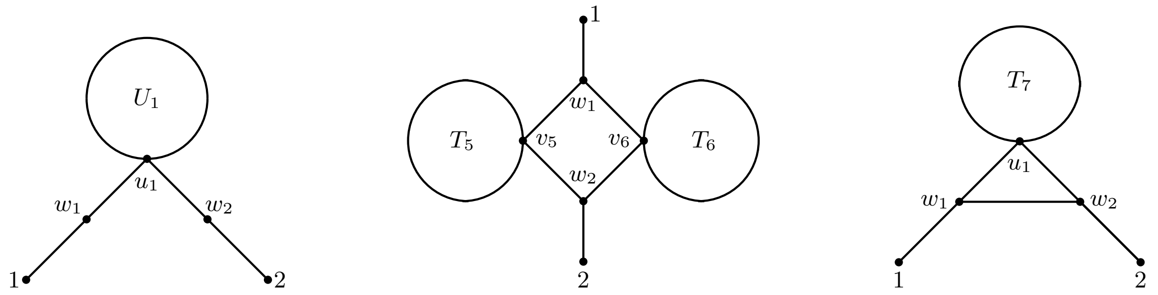


(a) Graph structure in $\mathcal{U}_1^1$. (Here, T_1 is a tree with some vertex identified with v_1.)

(b) Graph structure in $\mathcal{U}_1^2$. (Here, T_2 is a tree with some vertex identified with v_2.)

(c) Graph structure in $\mathcal{U}_1^3$. (Here, T_3 and T_4 are trees with some vertices identified with v_3 and v_4, respectively.)

Figure 1: Graph structures in the classes $\mathcal{U}_1^1$, $\mathcal{U}_1^2$, and $\mathcal{U}_1^3$.



(a) Graph structure in $\mathcal{U}_1^4$. (Here, U_1 is a unicyclic graph with some vertex identified with u_1.)

(b) Graph structure in $\mathcal{U}_2$. (Here, T_5 and T_6 are trees with some vertices identified with v_5 and v_6, respectively.)

(c) Graph structure in $\mathcal{U}_3$. (Here, T_7 is a tree with some vertex identified with u_1.)

Figure 2: Graph structures in the classes $\mathcal{U}_1^4$, $\mathcal{U}_2$, and $\mathcal{U}_3$.

Proof. Suppose that SV2$[\{1,2\},w]$ occurs in U. Without loss of generality, the Laplacian matrix $L(U)$ of U can be written as shown in (2.1). By Lemma 3.1, it implies that $1 \nsim 2$. Since U is a unicyclic graph, we have three cases: $p = 0, p = 1$, and $p = 2$.

Case 1. $p = 2$. It follows that $1, 2 \in V(\Gamma)$. By Lemma 3.2, we have $d_1 = d_2 = 2$, leading to the result $N(1) = N(2)$, which is not possible by Lemma 2.1.

Case 2. $p = 1$. Lemma 3.2 implies that both d_1 and d_2 can only hold values of either 1 or 2. When $d_1 = d_2 = 1$, it results in $N(1) = N(2)$, which leads to a contradiction by Lemma 2.1. Therefore, we have two possible situations: (a) $d_1 = d_2 = 2$ and (b) $d_1 = 2$ while $d_2 = 1$.(Here, the case $d_1 = 1$ and $d_2 = 2$ is omitted, as it is symmetric to the case (b) under relabeling of vertices 1 and 2).

At first, consider the situation where $d_1 = d_2 = 2$. Let $N(1) = \{v_1, w_1\}$ and $N(2) = \{v_1, w_2\}$, where $v_1 \in N(1) \cap N(2)$. As $w_1 \in \mathcal{N}_1$, by condition (2.7) of Corollary 2.6, we have

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$$2|\Theta_{w_1}(\mathcal{N}_2)| + |\Theta_{w_1}(\mathcal{N}_1^2)| + |\Theta_{w_1}(\mathcal{N}_{12}^c)| = d_2 + w - 1 = w + 1. \tag{3.9}$$

By Corollary 2.4, w is a positive integer. Therefore, (3.9) implies that

$$2|\Theta_{w_1}(\mathcal{N}_2)| + |\Theta_{w_1}(\mathcal{N}_1^2)| + |\Theta_{w_1}(\mathcal{N}_{12}^c)| \geq 2. \tag{3.10}$$

Since U is unicyclic, by Lemma 2.7, it follows that $|\Theta_{w_1}(\mathcal{N}_{12}^c)| \leq 1$. Again as $\mathcal{N}_1^2 = \{v_1\}$ and $\mathcal{N}_2 = \{w_2\}$, we have $|\Theta_{w_1}(\mathcal{N}_1^2)| \leq 1$ and $|\Theta_{w_1}(\mathcal{N}_2)| \leq 1$, respectively. Now if $|\Theta_{w_1}(\mathcal{N}_2)| = 0$, (3.10) implies that $|\Theta_{w_1}(\mathcal{N}_1^2)| = |\Theta_{w_1}(\mathcal{N}_{12}^c)| = 1$. It follows that w_1 is adjacent to v_1 and to a vertex $v_2 \in \mathcal{N}_{12}^c$, which further by Lemma 2.7, v_2 is adjacent to w_2. Consequently, we have two distinct cycles in U, which is a contradiction. Therefore, $|\Theta_{w_1}(\mathcal{N}_2)| = 1$ and hence w_1 is adjacent to w_2. Since U is unicyclic, w_1 cannot be adjacent to any other vertex in U. Hence, the structure of U is as shown in Fig. 1a and $U \in \mathcal{U}_1^1$. Furthermore, (3.9) implies that $w = 1$.

Now consider the situation where $d_1 = 2$ and $d_2 = 1$. Let $N(1) = \{v_2, w_1\}$ and $N(2) = \{v_2\}$, where $v_2 \in N(1) \cap N(2)$. As $p = 1$, w_1 cannot be adjacent to 2. Since $\mathcal{N}_2 = \emptyset$, by Lemma 2.7, it follows that w_1 cannot be adjacent to any vertex of $\mathcal{N}_{12}^c$. By Corollary 2.5, w_1 cannot be a pendant vertex. It follows that w_1 must be adjacent to v_2. As a result, the structure of U is as shown in Fig. 1b and $U \in \mathcal{U}_1^2$. Also, by condition (2.7) of Corollary 2.6, we have $w = 1$.

Case 3. $p = 0$. By Lemma 3.2, three possible situations exist: (*a*) $d_1 = d_2 = 2$, (*b*) $d_1 = 2$ while $d_2 = 1$, and (*c*) $d_1 = d_2 = 1$. (Here, the case $d_1 = 1$ and $d_2 = 2$ is omitted, as it is symmetric to the case (*b*) under relabeling of vertices 1 and 2.)

Initially, consider the situation where $d_1 = d_2 = 2$. Let $N(1) = \{w_1, w_2\}$ and $N(2) = \{w_3, w_4\}$. By Lemma 2.8, w_1 and w_2 each lie on a cycle that includes the vertex 1, and similarly, w_3 and w_4 each lie on a cycle that includes the vertex 2. Since U is unicyclic, this is possible only if the vertices $1, 2, w_1, w_2, w_3$ and w_4 all lie in a single cycle. By Lemma 2.7, it follows that $|\Theta_{w_k}(\mathcal{N}_{12}^c)| \leq 1$, where $w_k \in \{w_1, w_2, w_3, w_4\}$. Note that $\mathcal{N}_1^2 = \emptyset$. By condition (2.7) of Corollary 2.6, we have

$$2|\Theta_{w_1}(\mathcal{N}_2)| + |\Theta_{w_1}(\mathcal{N}_{12}^c)| = d_2 + w - 1 = w + 1. \tag{3.11}$$

Since by Corollary 2.4, w is a positive integer and also $|\Theta_{w_1}(\mathcal{N}_{12}^c)| \leq 1$, it follows from (3.11) that $|\Theta_{w_1}(\mathcal{N}_2)| \neq 0$. Since U is unicyclic, we have $|\Theta_{w_1}(\mathcal{N}_2)| = 1$, that is, w_1 is adjacent to a single vertex in $\mathcal{N}_2$. Similarly, it can be shown that w_2 is adjacent to a single vertex in $\mathcal{N}_2$. Hence, the possible structure of U is a cycle of order 6 with vertices 1 and 2 at a distance 3 and (3.11) implies that $w = 1$.

Now consider the situation where $d_1 = 2$ and $d_2 = 1$. Let $N(1) = \{w_1, w_2\}$ and $N(2) = \{w_3\}$. By condition (2.2) of Theorem 2.2, it follows that $w = 1$. By condition (2.7) of Corollary 2.6, we have $2|\Theta_{w_1}(\mathcal{N}_2)| + |\Theta_{w_1}(\mathcal{N}_{12}^c)| = 1$, which implies that $|\Theta_{w_1}(\mathcal{N}_2)| = 0$ and $|\Theta_{w_1}(\mathcal{N}_{12}^c)| = 1$. Thus, w_1 is adjacent to a vertex $v_3 \in \mathcal{N}_{12}^c$, which is further adjacent to w_3 by Lemma 2.7. Similarly, it can be shown that $|\Theta_{w_2}(\mathcal{N}_2)| = 0$ and $|\Theta_{w_2}(\mathcal{N}_{12}^c)| = 1$, that is, w_2 is adjacent to a vertex $v_4 \in \mathcal{N}_{12}^c$, which is further adjacent to w_3 by Lemma 2.7. Hence, the structure of U is as shown in Fig. 1c and $U \in \mathcal{U}_1^3$.

Now consider the situation where $d_1 = d_2 = 1$. Let $N(1) = \{w_1\}$ and $N(2) = \{w_2\}$. Since U contains exactly one cycle, Lemma 2.7 implies that $\Theta_{w_1}(\mathcal{N}_{12}^c) = \Theta_{w_2}(\mathcal{N}_{12}^c)$, and each of these sets contains at most two vertices. If $\Theta_{w_1}(\mathcal{N}_{12}^c) = \Theta_{w_2}(\mathcal{N}_{12}^c) = \{u_1\}$, then the possible structures of U are as shown in Figs. 2a and 2c belonging to the classes $\mathcal{U}_1^4$ and $\mathcal{U}_3$, respectively, and by condition (2.7), we obtain the corresponding values of w as 1 and 3, respectively. On the other hand, if $\Theta_{w_1}(\mathcal{N}_{12}^c) = \Theta_{w_2}(\mathcal{N}_{12}^c) = \{v_5, v_6\}$, then the structure

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of U is as shown in Fig. 2b belonging to the class $\mathcal{U}_2$, and by condition (2.7), we obtain the corresponding value of w as $w = 2$. This concludes the proof of the necessary part.

The sufficiency is easily verified by conditions (2.2)-(2.6) of Theorem 2.2. ◻

4. Spectral w-variation of unicyclic graphs with its algebraic connectivity. This section provides a complete characterization of unicyclic graphs where SV2$[\{i, j\}, w]$ takes place in such a way that one of its changed eigenvalues is the algebraic connectivity.

Let C_g^3 be a *lollipop graph*, which is constructed by identifying a vertex of a cycle C_g to a pendant vertex of a path P_3 (see Fig. 3). Let B_n denote the $n \times n$ matrix derived from the Laplacian matrix of P_{n+1} by removing the row and column associated with an end vertex of P_{n+1}. Similarly, define H_n as the $n \times n$ matrix resulting from the Laplacian matrix of P_{n+2} by removing the rows and columns associated with both end vertices of P_{n+2}. Define $\delta(M)$ to be the least eigenvalue of the matrix M.

The subsequent result provides the eigenvalues of the matrix H_n.

Lemma 4.1. *[1, Proposition 2.1] The eigenvalues of the matrix H_n are of the form $2(1+cos(z\pi/(n+1)))$, for $z = 1, 2, \ldots, n$.*

By Lemma 4.1, we have $\delta(H_n) = 2(1 + cos(\pi/(n + 1)))$. Observe that for all $m < n$, $\delta(H_m) > \delta(H_n)$. It can be verified that $\delta(B_2) = \frac{3-\sqrt{5}}{2}$. Since $\delta(H_5) = 2(1 + cos(\pi/6)) < \frac{3-\sqrt{5}}{2} = \delta(B_2)$, it follows that for all $g \geq 6$, $\delta(H_{g-1}) < \delta(B_2)$. So we have the subsequent result.

Lemma 4.2. *For all $g \geq 6$, $\delta(H_{g-1}) < \delta(B_2)$.*

The subsequent result reveals a significant relationship between the algebraic connectivity of a graph and the smallest eigenvalue of certain principal submatrices of its Laplacian matrix, as provided by Bapat and Pati [7]. Here, for a graph G and a subset $\mathcal{S}$ of $V(G)$, we use $G \setminus \mathcal{S}$ to indicate the disconnected graph formed by eliminating each vertex in $\mathcal{S}$ and its associated edges from G.

Lemma 4.3. *[7, Lemma 6] Let G be a connected graph. Consider a subset $\mathcal{S}$ of $V(G)$ such that $G \setminus \mathcal{S}$ is disconnected. Suppose M_1 and M_2 are principal submatrices of $L(G)$ associated with any two components of $G \setminus \mathcal{S}$. If $\delta(M_1) \leq \delta(M_2)$, then either $\delta(M_2) > a(G)$ or $\delta(M_1) = \delta(M_2) = a(G)$.*

The subsequent result is essential for further development.

Lemma 4.4. *For all $g \geq 6$, $a(C_g^3) < \frac{3-\sqrt{5}}{2}$.*

Proof. Suppose $v \in V(C_g^3)$ is such that $d_v = 3$. Then, $C_g^3 \setminus \{v\}$ is a disconnected graph containing two components. Observe that H_{g-1} and B_2 are the principal submatrices of $L(C_g^3)$ associated with the

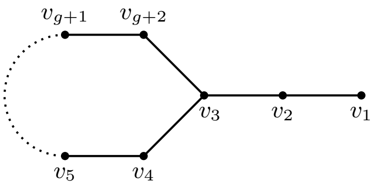


Figure 3: The lollipop graph C_g^3.

components of $C_g^3 \backslash \{v\}$. Given that $g \geq 6$. By Lemma 4.2, we have $\delta(H_{g-1}) < \delta(B_2)$. Since $\delta(H_{g-1}) \neq \delta(B_2)$, by Lemma 4.3, we have $a(C_g^3) < \delta(B_2) = \frac{3-\sqrt{5}}{2}$. $\square$

The subsequent lemma is essential for further development.

Lemma 4.5. [4] *Let G be a connected graph with a pendant vertex v. Then, the algebraic connectivity $a(G)$ of G satisfies $a(G) \leq a(G \setminus \{v\})$.*

The following lemma determines the possible cycle lengths in a unicyclic graph $U \in \mathcal{U}_1^4$ when $a(U) = \frac{3-\sqrt{5}}{2}$.

Lemma 4.6. *Suppose U is a unicyclic graph with $V(U) = \{1, 2, \ldots, n\}$ such that $U \in \mathcal{U}_1^4$ (see Fig. 2a). Let g be the length of the unique cycle in U. If $a(U) = \frac{3-\sqrt{5}}{2}$, then the possible values of g are $3, 4,$ or 5.*

Proof. By Lemma 4.5, through the successive removal of the pendant vertices of U, we have $a(U) \leq a(C_g^3)$. Furthermore, if $g \geq 6$, then by Lemma 4.4, it follows that $a(U) < \frac{3-\sqrt{5}}{2}$. Therefore, if $a(U) = \frac{3-\sqrt{5}}{2}$, then the possible values of g are $3, 4$, or 5. $\square$

Now we construct some classes of unicyclic graphs. To do this, we first introduce a class of trees denoted by $S(n_1, n_2)$, defined as follows:

For nonnegative integers n_1 and n_2 (with $n_2 \geq 2$), define $S(n_1, n_2)$ as the tree that is constructed from a single vertex v, n_1 copies of P_2 and n_2 copies of P_3 by identifying the vertex v with a pendant vertex of every path. We will refer to the vertex v of $S(n_1, n_2)$ as the center of the tree $S(n_1, n_2)$. For example, the tree $S(3, 4)$ with center v_1 is as shown in Fig. 4a.

Let $\bar{\mathcal{U}_1}$ be a class of unicyclic graphs constructed as follows:

1. Begin by forming a triangle $\mathcal{T}$ with vertices labeled as v_1, v_2, and v_3.
2. Next, create a tree $S(n_1, n_2)$ and identify its center to the vertex v_1 of $\mathcal{T}$.
3. Finally, add either a single pendant vertex or no pendant vertices at either vertex v_2 or v_3 or both.

Example 4.7. *The graph shown in Fig. 4b is an example of a graph in $\bar{\mathcal{U}_1}$.*

Define a class, denoted by $\bar{\mathcal{U}_2}$, consisting of unicyclic graphs constructed as follows:

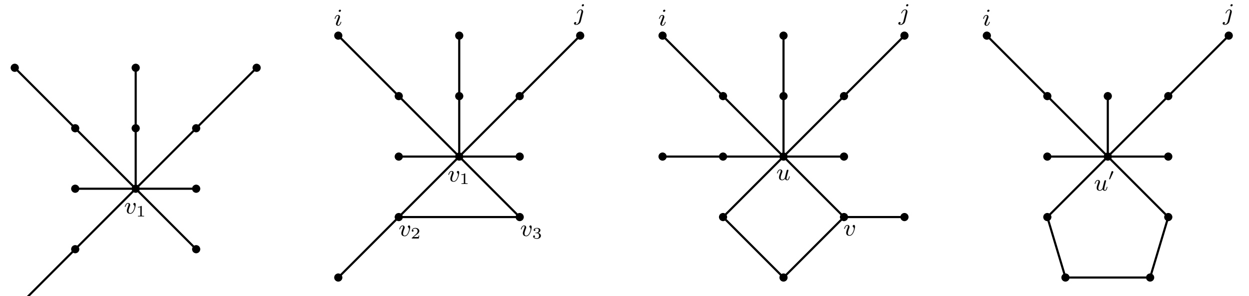
1. Begin by forming a cycle $\mathcal{C}_4$ of length 4. Introduce two adjacent vertices of C_4, denoted by u and v.
2. Next, create a tree $S(n_1, n_2)$ and identify its center to the vertex u of C_4.
3. Finally, add either a single pendant vertex or no pendant vertices at the vertex v.

Example 4.8. *The graph shown in Fig. 4c is an example of a graph in $\bar{\mathcal{U}_2}$.*

Let $\bar{\mathcal{U}_3}$ be a class of unicyclic graphs constructed as follows:

1. Begin by forming a cycle $\mathcal{C}_5$ of length 5. Introduce a vertex of C_5, denoted by u'.
2. Next, create a tree $S(n_1, n_2)$ and identify its center to the vertex u' of C_5.

Example 4.9. *The graph shown in Fig. 4d is an example of a graph in $\bar{\mathcal{U}_3}$.*



(a) The tree $S(3,4)$.

(b) Example of a graph in $\bar{\mathcal{U}}_1$

(c) Example of a graph in $\bar{\mathcal{U}}_2$

(d) Example of a graph in $\bar{\mathcal{U}}_3$

Figure 4: Examples of graphs belonging to the classes $S(n_1,n_2), \bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2$ and $\bar{\mathcal{U}}_3$.

The following definitions are given in [3]. Let G be a connected graph. A vertex v in G is said to be a *cut vertex* of G if $G \setminus \{v\}$ is disconnected. Let $v \in V(G)$ be a cut vertex of G. For $i = 1, 2, \ldots, m$, let C_i denote the components of $G \setminus \{v\}$ and let L_i be the principal submatrix of $L(G)$ associated with C_i. Then, a component C_k for some $k \in \{1, 2, \ldots .m\}$, is called a *Perron component* of G at v if $\delta(L_k) = \min_{1 \le j \le m} \delta(L_j)$.

By combining the results presented in Theorem 7.2.4 and Corollary 7.2.5 from [3], we deduce the following result.

LEMMA 4.10. *[3] Let G be a connected graph. Suppose G contains a unique cut vertex u at which there exists at least two Perron components. Then, algebraic connectivity $a(G)$ of G is given by $a(G) = \delta(L[C])$, where $L[C]$ denotes the principal submatrix of L corresponding to a Perron component C at u.*

The following lemma characterizes unicyclic graphs in $\mathcal{U}_1^4$ satisfying $a(U) = \frac{3-\sqrt{5}}{2}$.

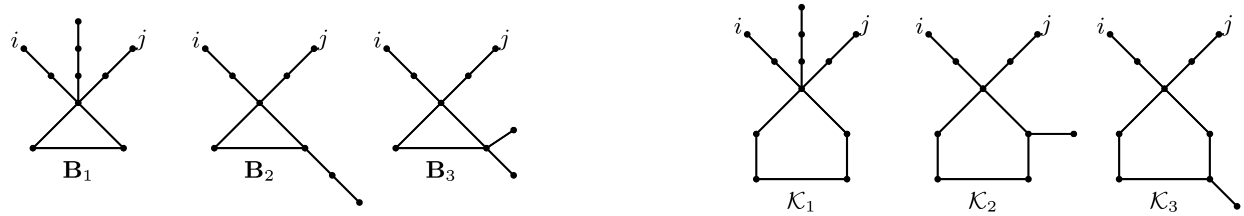
LEMMA 4.11. *Suppose U is a unicyclic graph with $V(U) = \{1, 2, \ldots, n\}$ such that $U \in \mathcal{U}_1^4$. Then, $a(U) = \frac{3-\sqrt{5}}{2}$ if and only if $U \in \bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_2 \cup \bar{\mathcal{U}}_3$.*

Proof. Suppose that $a(U) = \frac{3-\sqrt{5}}{2}$, and let g represent the length of the unique cycle present in U. By Lemma 4.6, the possible values of g are 3, 4, or 5. We claim that U belongs to one of the classes $\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2,$ or $\bar{\mathcal{U}}_3$ depending on whether $g = 3, 4,$ or 5, respectively.

Consider the unicyclic graphs $\mathbf{B}_1, \mathbf{B}_2,$ and $\mathbf{B}_3$, as shown in Fig. 5a. It can be verified that $a(\mathbf{B}_i) < \frac{3-\sqrt{5}}{2}$ for all $i = 1, 2, 3$. Now, suppose $g = 3$ and $U \notin \bar{\mathcal{U}}_1$. In this case, one of $\mathbf{B}_1, \mathbf{B}_2,$ or $\mathbf{B}_3$ must be a subgraph of U. Consequently, by Lemma 4.5, through the successive removal of the pendant vertices of U, we obtain $a(U) \le a(\mathbf{B}_i) < \frac{3-\sqrt{5}}{2}$ for some $i = 1, 2, 3$, which is a contradiction. Therefore, it is necessary that $U \in \bar{\mathcal{U}}_1$ when $g = 3$.

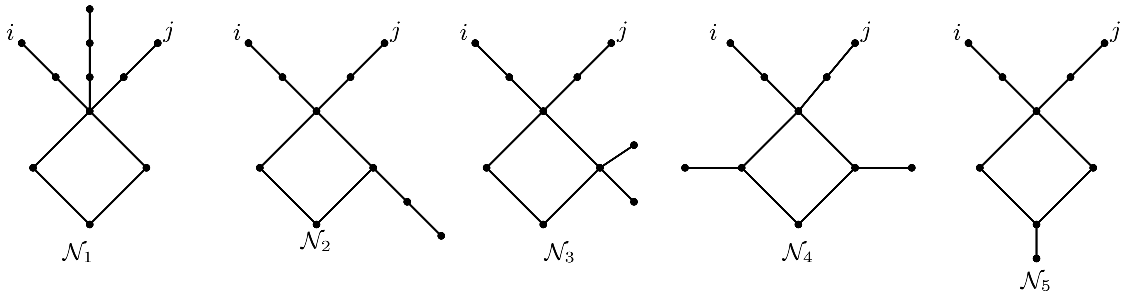
For the cases $g = 4$ and $g = 5$, we follow a similar argument as in the case $g = 3$, but with appropriate substitutions. Specifically, for $g = 4$, we replace the set $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ (Fig. 5a) with the set $\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5\}$ (Fig. 5c). Similarly, for $g = 5$, we replace it with the set $\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ (Fig. 5b). Applying the same reasoning, we conclude that $U \in \bar{\mathcal{U}}_2$ when $g = 4$ and $U \in \bar{\mathcal{U}}_3$ when $g = 5$.

Conversely, assume $U \in \bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_2 \cup \bar{\mathcal{U}}_3$. From the construction of the graphs belonging to one of the classes $\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2,$ and $\bar{\mathcal{U}}_3$, it can be observed that there exists exactly one vertex whose degree is at least 4. Let this



(a) $\mathbf{B}_i, i = 1, 2, 3$

(b) $\mathcal{K}_i, i = 1, 2, 3$



(c) $\mathcal{N}_i, i = 1, 2, 3, 4, 5$

Figure 5: The unicyclic graphs $\mathbf{B}_i (i = 1, 2, 3)$, $\mathcal{K}_i (i = 1, 2, 3)$, and $\mathcal{N}_i (i = 1, 2, \ldots, 5)$.

particular vertex be denoted by v. It can be observed that v is the unique cut vertex of U with at least two Perron components at v whose corresponding principal submatrix of $L(U)$ is B_2. By Lemma 4.10, we have $a(U) = \delta(B_2) = \frac{3-\sqrt{5}}{2}$ and the sufficiency holds. ☐

A vertex $v \in V(G)$ is said to be a *quasipendant vertex* if it is adjacent to a pendant vertex of G. The following result provides the number of eigenvalues of $L(G)$ that are less than one in terms of its quasipendant vertices.

LEMMA 4.12. *[3, Theorem 4.3.6] Consider a graph G with Laplacian matrix L containing q quasipendant vertices. Then, at least q many eigenvalues of L (counting multiplicity) lie in the interval* $[0, 1)$.

According to Theorem 3.3, when SV2$[\{i, j\}, w]$ occurs for a unicyclic graph U, the possible values for w are 1, 2, and 3. To characterize the unicyclic graphs in which SV2$[\{i, j\}, w]$ occurs, leading to one of its changed eigenvalues being its algebraic connectivity, we consider three cases corresponding to the values of $w = 1, 2,$ and 3.

In the case when $w = 1$, the subsequent result provides a characterization of unicyclic graphs where SV2$[\{i, j\}, w]$ occurs, resulting in one of its changed eigenvalues being its algebraic connectivity.

THEOREM 4.13. *Consider a unicyclic graph U with $V(U) = \{1, 2, \ldots, n\}$ and take $w = 1$. Assume that SV2$[\{i, j\}, w]$ occurs in U, producing altered eigenvalues λ_{i_1} and λ_{i_2} $(\lambda_{i_1} \leq \lambda_{i_2})$. Then, $\lambda_{i_1} = a(U)$ if and only if the following conditions are fulfilled:*

1. *U is one of C_5, (Fig. 6a) C_6, (Fig. 6b) G_6^1, (Fig. 6c) G_6^2, (Fig. 6d) or G_3^{n-3}, (Fig. 6e) and the vertices i and j correspond to those specifically indicated in the respective figures*
   or

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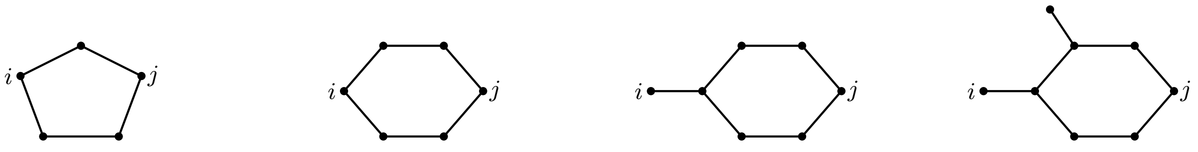
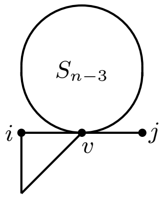


(a) C_5 (b) C_6 (c) G_6^1 (d) G_6^2



(e) G_3^{n-3}. (Here, S_{n-3} is a star on $n-3$ vertices with center identified with v.)

Figure 6: The unicyclic graphs C_5, C_6, G_6^1, G_6^2 and G_3^{n-3}.

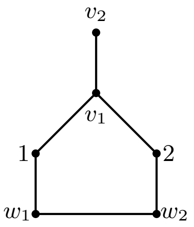


Figure 7: The unicyclic graph $\mathcal{Q}$.

2. *$U \in \bar{\mathcal{U}_1} \cup \bar{\mathcal{U}_2} \cup \bar{\mathcal{U}_3}$ and i and j are the pendant vertices of the corresponding tree $S(n_1, n_2)$ with a distance of 4.*

Proof. Suppose that SV2$[\{i,j\},w]$ occurs for the unicyclic graph U with changed eigenvalues λ_{i_1} and λ_{i_2} ($\lambda_{i_1} \leq \lambda_{i_2}$). By Theorem 3.3, $U \in \mathcal{U}_1^1 \cup \mathcal{U}_1^2 \cup \mathcal{U}_1^3 \cup \mathcal{U}_1^4$, where i and j are, respectively, the vertices 1 and 2 as depicted in Figs. 1a, 1b, 1c, 2a or U is a cycle C_6 with vertices i and j at a distance 3. If U is a cycle C_6 with vertices i and j as shown in Fig. 6b, then no further proof is required. Now assume $\lambda_{i_1} = a(U)$ and consider the following cases.

Case 1. $U \in \mathcal{U}_1^1$. By Theorem 2.2, we have $a(U) = \lambda_{i_1} = \frac{5-\sqrt{5}}{2}$. We claim that U must be a cycle C_5 with vertices i and j as depicted in Fig. 6a. On the contrary, suppose that U is not a cycle C_5. Then, the graph $\mathcal{Q}$ as shown in Fig. 7 must be a subgraph of U, which satisfies $a(\mathcal{Q}) < \frac{5-\sqrt{5}}{2}$. By Lemma 4.5, sequential deletion of the pendant vertices in U yields $a(U) \leq a(\mathcal{Q}) < \frac{5-\sqrt{5}}{2}$, contradicting $a(U) = \frac{5-\sqrt{5}}{2}$. Thus, U must be C_5.

Case 2. $U \in \mathcal{U}_1^2$. By Theorem 2.2, we have $a(U) = \lambda_{i_1} = 1$. Since $a(U) = 1$, Lemma 4.12 implies that U can have at most one quasipendant vertex. Hence, the structure of U must be G_3^{n-3}, as shown in Fig. 6e.

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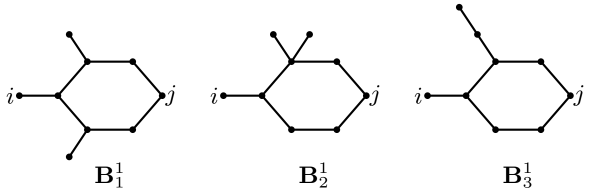


Figure 8: The unicyclic graphs $\mathbf{B}_i^1 (i = 1, 2, 3)$

Case 3. $U \in \mathcal{U}_1^3$. By Theorem 2.2, we have $a(U) = \lambda_{i_1} = 2 - \sqrt{2}$. Consider the unicyclic graphs G_6^1 and G_6^2 (Figs. 6c and 6d) and $\mathbf{B}_1^1$, $\mathbf{B}_2^1$ and $\mathbf{B}_3^1$ (Fig. 8). It can be determined that $a(G_6^1) = a(G_6^2) = 2 - \sqrt{2}$, while $a(\mathbf{B}_i^1) < 2 - \sqrt{2}$ for all $i = 1, 2, 3$. If $U \in \mathcal{U}_1^3$ but is not G_6^1 or G_6^2, then at least one of $\mathbf{B}_i^1$, $(i = 1, 2, 3)$ must be a subgraph of U, and consequently, by Lemma 4.5, sequential deletion of the pendant vertices in U leads to $a(U) \leq a(\mathbf{B}_i^1) < 2 - \sqrt{2}$ for some $i = 1, 2, 3$, contradicting $a(U) = 2 - \sqrt{2}$. Therefore, $\lambda_{i_1} = a(U)$ holds only if U is either G_6^1 or G_6^2.

Case 4. $U \in \mathcal{U}_1^4$. By Theorem 2.2, we have $a(U) = \lambda_{i_1} = \frac{3-\sqrt{5}}{2}$. By Lemma 4.11, it follows that necessarily $U \in \bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_2 \cup \bar{\mathcal{U}}_3$, where i and j are pendant vertices of the corresponding tree $S(n_1, n_2)$ at a distance of 4.

Conversely, suppose U satisfies all the given conditions. We show that $\lambda_{i_1} = a(U)$.

If U is one of C_5 (Fig. 6a), C_6 (Fig. 6b), G_6^1 (Fig. 6c), or G_6^2 (Fig. 6d) with vertices i and j as indicated in the respective figures, then by Theorem 2.2, we can determine the value of λ_{i_1} and then direct computation verifies that $\lambda_{i_1} = a(U)$.

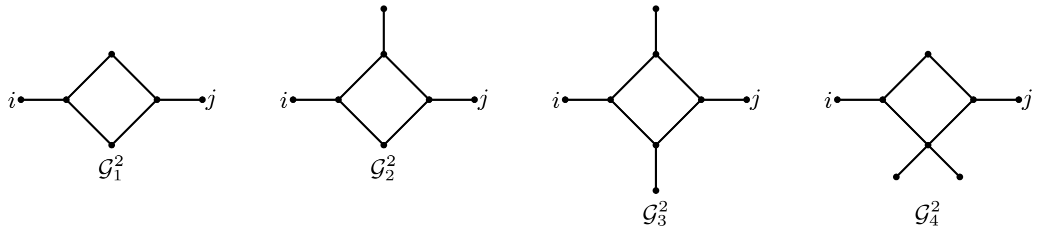
If U is G_3^{n-3} with vertices i and j as depicted in Fig. 6e, then by Theorem 2.2, we have $\lambda_{i_1} = 1$. Observe that G_3^{n-3} has exactly one vertex with a degree at least 3. Let v be such a vertex of G_3^{n-3}. Observe that all the components of $G_3^{n-3} \setminus \{v\}$ are Perron components of G_3^{n-3} at the vertex v, and for all of these components the smallest eigenvalue of the associated principal submatrix of $L(G_3^{n-3})$ is 1. By Lemma 4.10, we have $a(G_3^{n-3}) = 1 = \lambda_{i_1}$.

If $U \in \bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_2 \cup \bar{\mathcal{U}}_3$ with vertices i and j as pendant vertices of the corresponding tree $S(n_1, n_2)$ at a distance 4, then by Theorem 2.2, we have $\lambda_{i_1} = \frac{3-\sqrt{5}}{2}$. By Lemma 4.11, we have $a(U) = \frac{3-\sqrt{5}}{2} = \lambda_{i_1}$. ☐

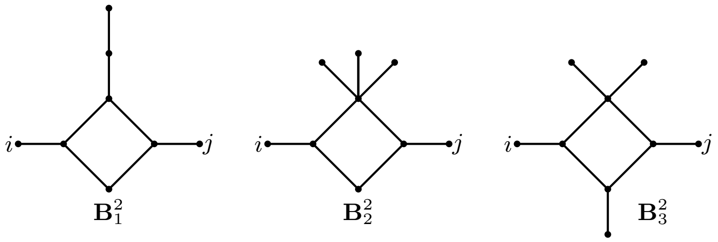
When $w = 2$, the subsequent result gives a characterization of the unicyclic graphs where SV2$[\{i, j\}, w]$ occurs, resulting in one of its altered eigenvalues being its algebraic connectivity.

THEOREM 4.14. *Consider a unicyclic graph U with $V(U) = \{1, 2, \ldots, n\}$ and take $w = 2$. Assume that SV2$[\{i, j\}, w]$ occurs in U, producing altered eigenvalues λ_{i_1} and λ_{i_2} $(\lambda_{i_1} \leq \lambda_{i_2})$. Then, $\lambda_{i_1} = a(U)$ if and only if U is one of $\mathcal{G}_1^2$, $\mathcal{G}_2^2$, $\mathcal{G}_3^2$, or $\mathcal{G}_4^2$ (Fig. 9a) with vertices i and j as shown in Fig. 9a.*

Proof. Suppose that $\lambda_{i_1} = a(U)$. By Theorem 3.3, it follows that $U \in \mathcal{U}_2$ and has structure shown in Fig. 2b, where vertices i and j correspond to vertices 1 and 2, respectively, as depicted in Fig. 2b. By applying the similar argument as in Case 3 of Lemma 4.13, replacing the sets $\{G_6^1, G_6^2\}$ and $\{\mathbf{B}_1^1, \mathbf{B}_2^1, \mathbf{B}_3^1\}$ with $\{\mathcal{G}_1^2, \mathcal{G}_2^2, \mathcal{G}_3^2, \mathcal{G}_4^2\}$ (Fig. 9a) and $\{\mathbf{B}_1^2, \mathbf{B}_2^2, \mathbf{B}_3^2\}$ (Fig. 9b), respectively, we can establish that $\lambda_{i_1} = a(U)$ only if U is one of $\mathcal{G}_1^2$, $\mathcal{G}_2^2$, $\mathcal{G}_3^2$, or $\mathcal{G}_4^2$. The proof for the converse part is straightforward. ☐

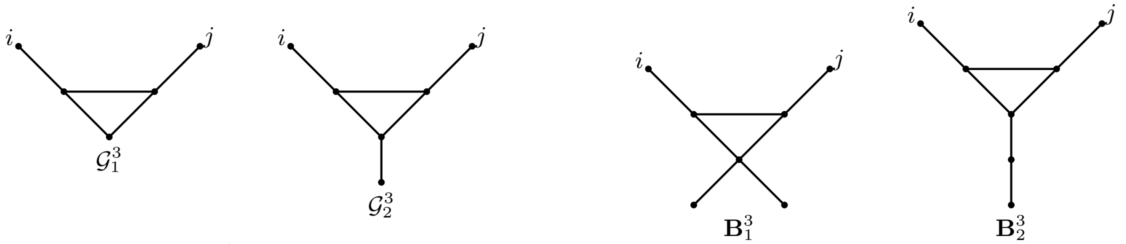


(a) $\mathcal{G}_i^2, i = 1, 2, 3, 4$



(b) $\mathbf{B}_i^2, i = 1, 2, 3$

Figure 9: The unicyclic graphs $\mathcal{G}_i^2 (i = 1, 2, 3, 4)$ and $\mathbf{B}_i^2 (i = 1, 2, 3)$.



(a) $\mathcal{G}_i^3, i = 1, 2$

(b) $\mathbf{B}_i^3, i = 1, 2$

Figure 10: The unicyclic graphs $\mathcal{G}_i^3 (i = 1, 2)$ and $\mathbf{B}_i^3 (i = 1, 2)$.

When $w = 3$, the subsequent result characterizes the unicyclic graphs in which SV2$[\{i, j\}, w]$ occurs, resulting in one of its altered eigenvalues being its algebraic connectivity.

Theorem 4.15. *Consider a unicyclic graph U with $V(U) = \{1, 2, \ldots, n\}$ and take $w = 3$. Assume that SV2$[\{i, j\}, w]$ occurs in U, producing altered eigenvalues λ_{i_1} and λ_{i_2} ($\lambda_{i_1} \leq \lambda_{i_2}$). Then, $\lambda_{i_1} = a(U)$ if and only if U is either $\mathcal{G}_1^3$ or $\mathcal{G}_2^3$, (Fig. 10a) with vertices i and j as shown in Fig. 10a.*

Proof. Suppose that $\lambda_{i_1} = a(U)$. By Theorem 3.3, $U \in \mathcal{U}_3$ and has structure shown in Fig. 2c, where vertices i and j correspond to vertices 1 and 2, respectively. By Theorem 2.2, we can compute λ_{i_1} as $\lambda_{i_1} = \frac{5-\sqrt{13}}{2}$. By following a similar approach as in Case 3 of Lemma 4.13, we replace the sets $\{G_6^1, G_6^2\}$ and $\{\mathbf{B}_1^1, \mathbf{B}_2^1, \mathbf{B}_3^1\}$ with $\{\mathcal{G}_1^3, \mathcal{G}_2^3\}$ (Fig. 10a) and $\{\mathbf{B}_1^3, \mathbf{B}_2^3\}$ (Fig. 10b), respectively. Using this substitution, we can establish that $\lambda_{i_1} = a(U)$ if U is either $\mathcal{G}_1^3$ or $\mathcal{G}_2^3$. The proof for the converse part is straightforward. ☐

By utilizing Theorem 4.13, Theorem 4.14, and Theorem 4.15 along with making use of Theorem 3.3, we arrive at the following result, which provides the complete characterization of the unicyclic graphs in which SV2$[\{i,j\},w]$ occurs, leading to one of its changed eigenvalues being its algebraic connectivity.

THEOREM 4.16. *Consider a unicyclic graph U with $V(U) = \{1, 2, \dots, n\}$. Assume that SV2$[\{i,j\},w]$ occurs in U, producing altered eigenvalues λ_{i_1} and λ_{i_2} ($\lambda_{i_1} \leq \lambda_{i_2}$). Then, $\lambda_{i_1} = a(U)$ if and only if the following conditions are fulfilled:*

1. *$w = 1$ and either U is one of C_5, (Fig. 6a) C_6, (Fig. 6b) G_6^1, (Fig. 6c) G_6^2, (Fig. 6d), or G_3^{n-3}, (Fig. 6e) where the vertices i and j correspond to those specifically indicated in the respective figures or $U \in \bar{\mathcal{U}_1} \cup \bar{\mathcal{U}_2} \cup \bar{\mathcal{U}_3}$ where i and j are the pendant vertices of the corresponding tree $S(n_1, n_2)$ with a distance of 4.*
2. *$w = 2$ and U is one of $\mathcal{G}_1^2$, $\mathcal{G}_2^2$, $\mathcal{G}_3^2$, or $\mathcal{G}_4^2$ (Fig. 9a) with vertices i and j as shown in Fig. 9a.*
3. *$w = 3$ and U is either $\mathcal{G}_1^3$ or $\mathcal{G}_2^3$, (Fig. 10a) with vertices i and j as shown in Fig. 10a.*

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