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NEW IDENTITY FOR CAYLEY'S FIRST HYPERDETERMINANT WITH APPLICATIONS TO SYMMETRIC TENSORS AND ENTANGLEMENT*

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Abstract. In this article, a new formula for computing Cayley's first hyperdeterminant in terms of the Levi-Civita symbol is given. It is then shown that this formula can be used to compute the hyperdeterminant of symmetric tensors in polynomial time with respect to their order (assuming fixed side length). Applications to quantifying the entanglement of states of bosonic quantum systems are then discussed. Additionally, in order to obtain the fast calculation of the hyperdeterminant on symmetric tensors, generalized elimination and duplication matrices are defined, and their explicit formulas are derived.

Key words. Hyperdeterminant, Levi-Civita symbol, Symmetric tensors, Quantum entanglement.

AMS subject classifications. 15A69, 15A72.

1. Introduction. Originally introduced in 1844 by Sir Arthur Cayley [5], Cayley's first hyperdeterminant (also known as the combinatorial hyperdeterminant) was nearly forgotten for the next 150 years. Indeed, only in the last 30 years has there been significant research interest into the applications of Cayley's first hyperdeterminant, and in this time, numerous applications to both mathematics and physics have been uncovered! For example, in 1997, P. Zappa expressed Cayley's first hyperdeterminant in terms of the difference in even and odd Latin squares of order N [24], in 2003, J. G. Luque and J. Y. Thibon relate Cayley's first hyperdeterminant of Hankel hypermatrices to Selberg Integrals [15], in 2008, S. Matsumoto expresses a particular type of Jack function in terms of Cayley's first hyperdeterminant [18], in 2021, P. Lammers utilizes Cayley's first hyperdeterminant to characterize a generalized Kasteleyn theory [13], also in 2021, A. Amanov and D. Yeliussizov used Cayley's first hyperdeterminant to establish lower bounds of certain tensor ranks [2], and in 2024/2025, I. Dobes and N. Jing prove that Cayley's first hyperdeterminant generalizes the concurrence, an important entanglement measure in quantum information [7, 8]. Therefore, while one may initially consider Cayley's first hyperdeterminant to be a naive generalization of the usual determinant, it is in fact an important mathematical object with a wide variety of applications.

In this paper, we derive a new formula for computing Cayley's first hyperdeterminant in terms of the Levi-Civita symbol. For an order N tensor $\mathcal{A} \in \mathbb{K}^{d\times\cdots\times d}$, **Cayley's first hyperdeterminant** (also known as the **combinatorial hyperdeterminant**) on $\mathcal{A}$ is typically defined as

$$\operatorname{hdet}(\mathcal{A}) := \frac{1}{d!} \sum_{\sigma_1,\dots,\sigma_N \in S_d} \left(\prod_{k=1}^{N} \operatorname{sgn}(\sigma_k) \right) \prod_{i=1}^{d} a_{\sigma_1(i)\dots\sigma_N(i)}.$$

If N is odd, then $\operatorname{hdet}(\mathcal{A})$ is identically 0 [14]; however, if N is even, then $\operatorname{hdet}(\mathcal{A})$ is nontrivial in general, and furthermore, we have that

$$\operatorname{hdet}(\mathcal{A}) = \sum_{\sigma_2,\dots,\sigma_N \in S_d} \left(\prod_{k=2}^{N} \operatorname{sgn}(\sigma_k) \right) \prod_{i=1}^{d} a_{i\sigma_2(i)\dots\sigma_N(i)}.$$

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The d-dimensional Levi-Civita symbol $\varepsilon = [\varepsilon_{i_1 i_2 \dots i_d}] \in \mathbb{K}^{d\times\cdots\times d}$, which represents an important antisymmetric tensor in geometry and physics (e.g., see [11]), is the order d tensor such that

$$\varepsilon_{i_1 i_2 \dots i_d} = \left\{ \begin{matrix} 1 & \text{if } (i_1 i_2 \dots i_d) \text{ is an even permutation of the tuple } (12\dots d) \\ -1 & \text{if } (i_1 i_2 \dots i_d) \text{ is an odd permutation of the tuple } (12\dots d) \\ 0 & \text{otherwise} \end{matrix} \right\} .$$

In this article, we show that if $\mathcal{A} \in \mathbb{K}^{d\times\cdots\times d}$ is an arbitrary order N tensor, then Cayley's first hyperdeterminant of $\mathcal{A}$ is equal to the multilinear matrix product

$$\left(\frac{1}{d!} \bigotimes_{k=1}^{N} \varepsilon \right) * (\mathrm{hvec}(\mathcal{A}), \dots, \mathrm{hvec}(\mathcal{A})),$$

where ε denotes the d-dimensional Levi-Civita symbol, "$\bigotimes$" denotes the tensor Kronecker product, "$*$" denotes multilinear matrix multiplication, and $\mathrm{hvec}(\mathcal{A})$ denotes a vectorization of the tensor $\mathcal{A}$. The major benefit to this new identity for computing Cayley's first hyperdeterminant applies to symmetric tensors.

In general, the run time cost for computing Cayley's first hyperdeterminant (which we denote as hdet) of an arbitrary cubical tensor of order N with side length d is VNP-hard [12]. Indeed, with current state-of-the-art methods, the run time cost for computing hdet is exponential in both N and d; even after fixing one of the variables, it is still exponential with respect to the other [1]. However, after generalizing the notions of half-vectorization and duplication matrices to apply to symmetric tensors of order N with side length d, it is then shown in this article that the run time cost for computing hdet on symmetric tensors is in fact polynomial with respect to N, assuming d fixed. We then discuss how this result can be used to efficiently calculate the $2n$-way entanglement of states of bosonic quantum systems. Lastly, in the appendix, we derive an explicit formula for the generalized duplication matrix, which the fast calculation of hdet on symmetric tensors depends on.

2. Hypermatrix algebra.

2.1. Preliminaries. In this paper, we will use $\mathbb{K}$ to denote an arbitrary field of characteristic 0, and unless otherwise specified, we will always assume that the base field is $\mathbb{K}$. The material in this and the next section holds for any field F of any characteristic; however, from Section 4 onward, we will need to assume the characteristic is 0, so for notational consistency and convenience, we just go ahead and write everything in terms of $\mathbb{K}$. Also, we utilize the convention in combinatorics that for a positive integer n, $[n]$ denotes the set $\{1, 2, \dots, n\}$.

A **hypermatrix** of order N with dimensions $n_1 \times n_2 \times \cdots \times n_N$ is an N-dimensional array; specifically, an element in $\mathbb{K}^{n_1\times n_2\times\cdots\times n_N}$. Since every higher-dimensional array may be viewed as the coordinate array of some tensor in a fixed basis, we identify these arrays with their corresponding tensors and adopt the more common convention of referring to the arrays themselves as tensors. A tensor $A \in \mathbb{K}^{n_1\times n_2\times\cdots\times n_N}$ is called **cubical** if $n_1 = n_2 = \cdots = n_N =: d$, and in such case, we call the number d the **side length** of the tensor. An order N cubical tensor $\mathcal{A} = [a_{i_1\dots i_N}] \in \mathbb{K}^{d\times\cdots\times d}$ is called **symmetric** if $a_{i_1\dots i_N} = a_{i_{\sigma(1)}\dots i_{\sigma(N)}}$ for every $\sigma \in S_N$.

Definition 2.1 (Segre Outer Product). *Let $\mathcal{A} = [a_{i_1\dots i_N}] \in \mathbb{K}^{n_1\times\cdots\times n_N}$ and $\mathcal{B} = [b_{j_1\dots j_M}] \in \mathbb{K}^{m_1\times\cdots\times m_M}$ be tensors of order N and M, respectively. The **Segre outer product** of $\mathcal{A}$ with $\mathcal{B}$, denoted $\mathcal{A}\circ\mathcal{B}$, is the order $N+M$ tensor whose $(i_1,\dots,i_N,j_1,\dots,j_M)$-coordinate is given by $a_{i_1\dots i_N} b_{j_1\dots j_M}$.*

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Definition 2.2 (Multilinear Matrix Multiplication [14]). *Suppose* $\mathcal{A} = [a_{i_1 \cdots {}_N}] \in \mathbb{K}^{n_1 \times \cdots \times n_N}$*, and* $X^{(k)} = [x^{(k)}_{ij}] \in \mathbb{K}^{n_k \times m_k}$ *for* $k = 1, \dots, N$*. The* ***(right) multilinear matrix multiplication*** *of* $\mathcal{A}$ *with the tuple* $(X^{(1)}, \dots, X^{(N)})$ *is given by the tensor* $A * (X^{(1)}, \dots, X^{(N)}) =: \mathcal{A}'$ *in* $\mathbb{K}^{m_1 \times \cdots \times m_N}$ *whose* $(j_1, \dots, j_N)$*-coordinate is given by*

$$a'_{j_1 \dots j_N} \ = \sum_{k_1, \dots, k_N = 1}^{n_1, \dots, n_N} a_{k_1 \dots k_N} x^{(1)}_{k_1 j_1} \dots x^{(N)}_{k_N j_N}.$$

Note also that it is common in the literature for multilinear matrix multiplication to be defined as a left action; however, we define it as a right action since this convention leads to slightly cleaner formulas (at least in this article) and makes our subsequent generalizations of known objects (namely, elimination and duplication matrices) more faithful to their classical counterparts.

Proposition 2.1 ([14]). *Let* $\alpha, \beta \in \mathbb{K}$*;* $X^{(1)}, Y^{(1)} \in \mathbb{K}^{m_1 \times n_1}$*,... m* $X^{(N)}, Y^{(N)} \in \mathbb{K}^{m_N \times n_N}$*; and* $\mathcal{A}, \mathcal{B} \in \mathbb{K}^{n_1 \times \cdots \times n_N}$*. Multilinear matrix multiplication satisfies the following properties.*

1. *Multilinearity:*

$$(\alpha \mathcal{A} + \beta \mathcal{B}) * (X^{(1)}, \dots, X^{(N)}) = \alpha \mathcal{A} * (X^{(1)}, \dots, X^{(N)}) + \beta \mathcal{B} * (X^{(1)}, \dots, X^{(N)}),$$

and

$$\mathcal{A} * [\alpha(X^{(1)}, \dots, X^{(N)}) + \beta(Y^{(1)}, \dots, Y^{(N)})] = \alpha \mathcal{A} * (X^{(1)}, \dots, X^{(N)}) + \beta \mathcal{A} * (Y^{(1)}, \dots, Y^{(N)}).$$

2. *Multiplicative: if* $Z^{(1)} \in \mathbb{K}^{n_1 \times p_1}$*,...,* $Z^{(N)} \in \mathbb{K}^{n_N \times p_N}$*, then*

$$\mathcal{A} * (X^{(1)} Z^{(1)}, \dots, X^{(N)} Z^{(N)}) = [\mathcal{A} * (X^{(1)}, \dots, X^{(N)})] * (Z^{(1)}, \dots, Z^{(N)}).$$

Definition 2.3 (Tensor Kronecker Product). *The* ***tensor Kronecker product*** *of two order* N *tensors* $\mathcal{A} = [a_{i_1 \cdots {}_N}] \in \mathbb{K}^{m_1 \times \cdots \times m_N}$ *and* $\mathcal{B} = [b_{i_1 \cdots {}_N}] \in \mathbb{K}^{n_1 \times \cdots \times n_N}$ *is defined as the order* N *block tensor* $\mathcal{A} \otimes \mathcal{B} \in \mathbb{K}^{m_1 n_1 \times \cdots \times m_N n_N}$ *whose* $(i_1, \dots, i_N)$*-block is the tensor* $a_{i_1 \dots i_N} \mathcal{B}$*.*

The tensor Kronecker product is compatible with multilinear matrix multiplication in the following way.

Proposition 2.2 ([20]). *Let* $\mathcal{A} \in \mathbb{K}^{m_1 \times \cdots \times m_N}$ *and* $\mathcal{B} \in \mathbb{K}^{n_1 \times \cdots \times n_N}$ *be two tensors of order* N*, and suppose* $X^{(i)} \in \mathbb{K}^{m_i \times p_i}$ *and* $Y^{(i)} \in \mathbb{K}^{n_i \times q_i}$ *for each* $i \in [N]$*. Then,*

$$(\mathcal{A} * (X^{(1)}, \dots, X^{(N)})) \otimes (\mathcal{B} * (Y^{(1)}, \dots, Y^{(N)})) = (\mathcal{A} \otimes \mathcal{B}) * (X^{(1)} \otimes Y^{(1)}, \dots, X^{(N)} \otimes Y^{(N)}).$$

2.2. Vectorizing tensors. Note that if $e_{i_k} \in \mathbb{K}^{n_k}$ for $k = 1, \dots N$ are standard basis vectors, then $e_{i_1} \circ \cdots \circ e_{i_N} \in \mathbb{K}^{n_1 \times \cdots \times n_N}$ is the order N tensor with a 1 in its $(i_1, \dots, i_N)$-entry and 0's elsewhere, hence very tensor $\mathcal{A} \in \mathbb{K}^{n_1 \times \cdots \times n_N}$ may be uniquely written as

$$\mathcal{A} = \sum_{i_1, \dots i_N = 1}^{n_1, \dots n_N} a_{i_1 \dots i_N} e_{i_1} \circ \dots e_{i_N}.$$

Moreover, $e_{i_1} \otimes \cdots \otimes e_{i_{i_N}} \in \mathbb{K}^{n_1 \cdot \cdots \cdot n_N}$ is the $(i_N + n_N(i_{N-1} - 1) + \cdots + n_N n_{N-1} \cdot \cdots \cdot n_1(i_1 - 1))^{th}$ element in the standard ordered basis of $\mathbb{K}^{n_1 \cdot \cdots \cdot n_N}$. Therefore, extending linearly the map on basis elements

$$\begin{aligned} \text{hvec} : \mathbb{K}^{n_1 \times \cdots \times n_N} &\longrightarrow \mathbb{K}^{n_1 \cdot \cdots \cdot n_N} \\ e_{i_1} \circ \cdots \circ e_{i_N} &\longmapsto e_{i_1} \otimes \cdots \otimes e_{i_N} \end{aligned}$$

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is a linear isomorphism; we denote this map as hvec (for **hypermatrix vectorization**).[1]

Next, note that if $\mathcal{A} \in \mathbb{K}^{d\times\cdots\times d}$ is symmetric, there is a natural equivalence relation on the coordinates of $\mathcal{A}$: $a_{i_1\ldots i_N} \sim a_{j_1\ldots j_N} \iff j_k = i_{\sigma(k)}$ for each $k \in [N]$, for some $\sigma \in S_N$. Hence, we may recover all of the coordinates of $\mathcal{A}$ by considering only the equivalence class representatives $a_{i_1\ldots i_N}$ such that $1 \leq i_1 \leq \cdots \leq i_N \leq d$. This motivates the following definition.

Definition 2.4. *Let $\mathcal{A} \in \mathbb{K}^{d\times\cdots\times d}$ be a symmetric order N tensor. The* ***1/N-hypermatrix vectorization*** *of $\mathcal{A}$, denoted as $\mathrm{hvec}_{1/N}(\mathcal{A})$, is defined to be the vector consisting of the equivalence class representatives mentioned above, ordered in the lexicographical order from least to greatest. In other words,*

$$\mathrm{hvec}_{1/N}(\mathcal{A}) := \begin{bmatrix} a_{11\ldots11} \\ a_{11\ldots12} \\ \vdots \\ a_{11\ldots1d} \\ a_{11\ldots22} \\ \vdots \\ a_{11\ldots2d} \\ a_{11\ldots33} \\ \vdots \\ \vdots \\ a_{d-1,d\ldots dd} \\ a_{dd\ldots dd} \end{bmatrix}.$$

Since $\dim(\mathrm{Sym}^N(\mathbb{K}^d)) = \binom{d+N-1}{N}$, it follows that $\mathrm{hvec}_{1/N}(\mathcal{A}) \in \mathbb{K}^{\binom{d+N-1}{N}}$.

3. Main result.

3.1. Cayley's first hyperdeterminant in terms of the Levi-Civita symbol.

First, we note the following observation.

Lemma 3.1. *Let $\mathcal{A} = [a_{i_1\ldots i_N}] \in \mathbb{K}^{d\times\cdots\times d}$ be a cubical tensor of order N with side length d. Then,*

$$\mathrm{hdet}(\mathcal{A}) = \frac{1}{d!} \sum_{\substack{i_k^{(1)},\ldots,i_k^{(d)}=1 \\ k\in[N]}}^{d} \prod_{k=1}^{N} \varepsilon_{i_k^{(1)},\ldots,i_k^{(d)}} \prod_{l=1}^{d} a_{i_1^{(l)},\ldots,i_N^{(l)}}. \tag{1}$$

Proof. The product $\prod_{k=1}^{N} \varepsilon_{i_1^{(k)},\ldots,i_d^{(k)}}$ is nonzero only if for each $k \in [N]$, $(i_1^{(k)},\ldots,i_d^{(k)})$ is a permutation of S_d. Therefore, assuming $(i_1^{(k)},\ldots,i_d^{(k)})$ is in fact a permutation of S_d, and denoting said permutation as σ_k, it then follows that the sum in equation (1) can be rewritten as

$$\frac{1}{d!} \sum_{\substack{\sigma_k\in S_d \\ k\in[N]}} \left(\prod_{k=1}^{N} \varepsilon_{\sigma_k(1)\ldots\sigma_k(d)} \right) \prod_{l=1}^{d} a_{\sigma_1(l)\ldots\sigma_N(l)}.$$

[1]Note that in the special case of $N = 2$, in which case $\mathcal{A}$ may be thought of as a matrix, $\mathrm{hvec}(\mathcal{A}) = \mathrm{vec}(\mathcal{A}^T)$, where vec is the typical vectorization map defined on matrices.

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By definition $\varepsilon_{\sigma_k(1)\dots\sigma_k(d)} = \operatorname{sgn}(\sigma_k)$, hence the sum in equation (1) is in fact equal to

$$\frac{1}{d!}\sum_{\substack{\sigma_k\in S_d\\ k\in[N]}}\left(\prod_{k=1}^{N}\operatorname{sgn}(\sigma_k)\right)\prod_{l=1}^{d}a_{\sigma_1(l)\dots\sigma_N(l)},$$

which is precisely the formula for $\operatorname{hdet}(\mathcal{A})$. ◻

With the above lemma, we may now prove the main result of this article.

THEOREM 3.1. *Let $\mathcal{A} = [a_{i_1\dots i_N}] \in \mathbb{K}^{d\times\cdots\times d}$ be a cubical tensor of order N with side length d. Then,*

$$\operatorname{hdet}(\mathcal{A}) = \left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon\right) * \big(\operatorname{hvec}(\mathcal{A}),\dots,\operatorname{hvec}(\mathcal{A})\big),$$

where ε denotes the d-dimensional Levi-Civita symbol, $\bigotimes$ denotes the tensor Kronecker product, and the tuple $\big(\operatorname{hvec}(\mathcal{A}),\dots,\operatorname{hvec}(\mathcal{A})\big)$ has length d.

Proof. Observe

$$\begin{aligned}
&\left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon\right) * \big(\operatorname{hvec}(\mathcal{A}),\dots,\operatorname{hvec}(\mathcal{A})\big)\\
&= \left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon\right) * \Bigg(\sum_{i_1^{(1)},\dots,i_N^{(1)}=1}^{d} a_{i_1^{(1)}\dots i_N^{(1)}} e_{i_1^{(1)}}\otimes\cdots\otimes e_{i_N^{(1)}},\dots,\\
&\qquad\qquad \sum_{i_1^{(d)},\dots,i_N^{(d)}=1}^{d} a_{i_1^{(d)}\dots i_N^{(d)}} e_{i_1^{(d)}}\otimes\cdots\otimes e_{i_N^{(d)}}\Bigg), \quad \text{by definition}\\
&= \frac{1}{d!}\sum_{\substack{i_1^{(l)},\dots,i_N^{(l)}=1\\ l\in[d]}}^{d}\prod_{l=1}^{d}a_{i_1^{(l)}\dots i_N^{(l)}}\left(\bigotimes_{k=1}^{N}\varepsilon * \big(e_{i_1^{(1)}}\otimes\cdots\otimes e_{i_N^{(1)}},\dots,e_{i_1^{(d)}}\otimes\cdots\otimes e_{i_N^{(d)}}\big)\right), \quad \text{by multilinearity}\\
&= \frac{1}{d!}\sum_{\substack{i_1^{(l)},\dots,i_N^{(l)}=1\\ l\in[d]}}^{d}\prod_{l=1}^{d}a_{i_1^{(l)}\dots i_N^{(l)}}\big(\varepsilon * (e_{i_1^{(1)}},\dots,e_{i_1^{(d)}})\big)\otimes\cdots\otimes\big(\varepsilon * (e_{i_N^{(1)}},\dots,e_{i_N^{(d)}})\big), \quad \text{by Proposition 2.2}\\
&= \frac{1}{d!}\sum_{\substack{i_1^{(l)},\dots,i_N^{(l)}=1\\ l\in[d]}}^{d}\prod_{l=1}^{d}a_{i_1^{(l)}\dots i_N^{(l)}}\varepsilon_{i_1^{(1)}\dots i_1^{(d)}}\cdots\varepsilon_{i_N^{(1)}\dots i_N^{(d)}}\\
&= \frac{1}{d!}\sum_{\substack{i_k^{(1)},\dots,i_k^{(d)}=1\\ k\in[N]}}^{d}\prod_{k=1}^{N}\varepsilon_{i_k^{(1)}\dots i_k^{(d)}}\prod_{l=1}^{d}a_{i_1^{(l)}\dots i_N^{(l)}}\\
&= \operatorname{hdet}(\mathcal{A}), \quad \text{by Lemma 3.1.}
\end{aligned}$$

◻

Note in particular that when N is odd, the anti-symmetry of ε implies that the product in Theorem 3.1 is identically 0, in agreement with the fact that Cayley's first hyperdeterminant is identically 0 on all odd order tensors.

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3.2. Run time cost of Cayley's first hyperdeterminant via Levi-Civita symbol. Let ε denote the d-dimensional Levi-Civita symbol, which recall is a cubical tensor of order d with side length d. It then follows that the run time cost for computing the tensor Kronecker product $\bigotimes_{k=1}^{N} \varepsilon$ is $\mathcal{O}(d^{Nd})$. Suppose also that $\mathcal{A}$ is a cubical tensor of order N with side length d. Then, $\text{hvec}(\mathcal{A}) \in \mathbb{K}^{d^N}$ and so the run time cost for preparing $\text{hvec}(\mathcal{A})$ is $\mathcal{O}\big(d^N\big)$. By Theorem 3.1,

$$\text{hdet}(\mathcal{A}) = \left(\frac{1}{d!}\bigotimes_{k=1}^{N} \varepsilon\right) * \big(\text{hvec}(\mathcal{A}), \ldots, \text{hvec}(\mathcal{A})\big), \tag{2}$$

which yields a run time cost of $\mathcal{O}(d^{Nd})$ due to the evaluation of the multilinear matrix product of the order d tensor $\bigotimes_{k=1}^{N} \varepsilon \in \mathbb{K}^{d^N \times \cdots \times d^N}$ with d copies of $\text{hvec}(\mathcal{A}) \in \mathbb{K}^{d^N}$. Now, since $\left(\frac{1}{d!}\bigotimes_{k=1}^{N} \varepsilon\right)$ depends only on d and N and not on the specific tensor $\mathcal{A}$, we may compute it once and reuse it for all other subsequent hyperdeterminant calculation. Therefore, the total run time cost for computing $\text{hdet}(\mathcal{A})$ via identity (2) from Theorem Theorem 3.1 is $\mathcal{O}(d^{Nd})$ and incurs a memory cost also of $\mathcal{O}(d^{Nd})$.

Current state-of-the-art methods for computing $\text{hdet}(\mathcal{A})$ have a run time cost of $\mathcal{O}\big(2^{d(N-1)}d^{N-1}\big)$, according to [1]. Consider the ratio of that run time cost with the aforementioned run time cost of $\mathcal{O}(d^{Nd})$:

$$\frac{d^{Nd}}{2^{d(N-1)}d^{N-1}} = d^{Nd-(N-1)}2^{-d(N-1)}.$$

Taking logarithms, we obtain

$$\ln\Big(d^{Nd-(N-1)} \cdot 2^{-d(N-1)}\Big) = \Big(\big(Nd-(N-1)\big)\log_2(d) - (Nd-d)\Big)\ln(2). \tag{3}$$

In general, (3) will be positive except for when $d = 2$ and $N \geq 3$. Consequently, computing $\text{hdet}(\mathcal{A})$ via identity (2) is in general slower than current state-of-the-art methods, except for in the special case when $d = 2$ and $N \geq 3$. However, when it comes to symmetric tensors, the benefit to computing Cayley's first hyperdeterminant via identity (2) yields a far more substantial advantage, which we explain in the next section.

4. Fast calculation for hdet on symmetric tensors.

4.1. Generalized elimination and duplication matrices. In [16], Magnus and Neudecker introduce the notion of elimination and duplication matrices, which they define as follows: if A is any $d \times d$ symmetric matrix, then the **elimination matrix**, denoted L_d, is the unique $\frac{d(d+1)}{2} \times d^2$ matrix such that

$$L_d\text{vech}(A) = \text{vec}(A);$$

on the other hand, the duplication matrix, denoted D_d, is the unique $d^2 \times \frac{d(d+1)}{2}$ matrix such that

$$D_d\text{vech}(A) = \text{vec}(A).$$

It was shown in [16] that for each positive integer $d \geq 2$,

$$L_d = \sum_{d \geq i \geq j \geq 1} u_{ij}\text{vec}(E_{ij})^T,$$

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and

$$D_d = \sum_{d \geq i \geq j \geq 1} \mathrm{vec}(T_{ij}) u_{ij}^T,$$

where u_{ij} is the $\frac{1}{2}d(d+1)$-dimensional unit vector with a 1 in its $\left((j-1)n + i - \frac{1}{2}j(j-1)\right)^{th}$-coordinate and 0's everywhere else, and T_{ij} is the $d \times d$ matrix with a 1 in its (i,j) and (j,i) coordinates, and 0's everywhere else (i.e., $T_{ij} = E_{ij} + E_{ji}$ when $i \neq j$, otherwise $T_{ii} = E_{ii}$).

We may extend the notion of elimination and duplication matrices to symmetric tensors, which we define as follows:

Definition 4.1. *The* ***generalized elimination matrix*** $L_d^{(N)}$ *is the unique* $\binom{d+N-1}{N} \times d^N$ *matrix such that*

$$L_d^{(N)} \mathrm{hvec}(\mathcal{A}) = \mathrm{hvec}_{1/N}(\mathcal{A}),$$

and the ***generalized duplication matrix*** $D_d^{(N)}$ *is the unique* $d^N \times \binom{d+N-1}{N}$ *matrix such that*

$$D_d^{(N)} \mathrm{hvec}_{1/N}(\mathcal{A}) = \mathrm{hvec}(\mathcal{A}),$$

where $\mathcal{A}$ *is any arbitrary symmetric tensor of order* N *and side length* d.

Proposition 4.1. *Let* $N, d \geq 2$ *be positive integers. The generalized elimination matrix is given by*

$$L_d^{(N)} = \sum_{1 \leq i_1 \leq \cdots \leq i_N \leq d} u_{i_1 \ldots i_N} \mathrm{hvec}(E_{i_1 \ldots i_N})^T,$$

and the generalized duplication matrix $D_d^{(N)}$ *is given by*

$$D_d^{(N)} = \sum_{1 \leq i_1 \leq \cdots \leq i_N \leq d} \mathrm{hvec}(T_{i_1 \ldots i_N}) u_{i_1 \ldots i_N}^T,$$

where $u_{i_1 \ldots i_N}$ *is the unit vector of length* $\binom{d+N-1}{N}$ *with a 1 in its* $\left(\binom{d+N-1}{N} - \sum_{k=1}^{N} \binom{d+N-k-i_k}{N-k+1}\right)^{th}$*-coordinate and 0's elsewhere,* $E_{i_1 \ldots i_N} := e_{i_1} \circ \cdots \circ e_{i_N}$*, and* $T_{i_1 \ldots i_N} := \sum_{\sigma \in \mathrm{Orb}(i_1, \ldots, i_N)} E_{i_{\sigma(1)} \ldots i_{\sigma(N)}}$.

Proof. It is well known the number of tuples $(i_1, \ldots, i_N)$ such that $1 \leq i_1 \leq \cdots \leq i_N \leq d$ is equal to $\binom{d+N-1}{N}$, hence it follows that the number of tuples $(i_1, \ldots, i_N)$ such that $k \leq i_1 \leq \cdots \leq i_N \leq d$ is equal to $\binom{d+N-k}{N}$. Listing all tuples $(i_1, \ldots, i_N)$ such that $1 \leq i_1 \leq \cdots \leq i_N \leq d$ in lexicographical order, $(j_1, \ldots, j_N) > (i_1, \ldots, i_N)$ if and only if there exists $k \in [N]$ such that $j_k > i_k$ and $j_l = i_l$ for all l strictly less than k; in particular, the last $N-k+1$ entries of $(j_1, \ldots, j_N)$ are subject only to the restriction that $i_k + 1 \leq j_k \leq j_{k+1} \leq \cdots \leq j_N \leq d$, from which it follows that there are exactly $\sum_{k=1}^{N} \binom{d+(N-k+1)-(i_k+1)}{N-k+1} = \sum_{k=1}^{N} \binom{d+N-k-i_k}{N-k+1}$ many tuples greater than $(i_1, \ldots, i_N)$. Thus, the placement of the tuple $(i_1, \ldots, i_N)$, when listed in lexicographical order from least to greatest, is given by $\binom{d+N-1}{N} - \sum_{k=1}^{N} \binom{d+N-k-i_k}{N-k+1}$.

Denote $u_{i_1 \ldots i_N} \in \mathbb{K}^{\binom{d+N-1}{N}}$ to be the unit vector with a 1 in the placement of $(i_1, \ldots, i_N)$ and 0's elsewhere. Then for any order N symmetric tensor $\mathcal{A} = [a_{i_1 \ldots i_N}] \in \mathbb{K}^{d \times \cdots \times d}$,

$$\mathrm{hvec}_{1/N}(\mathcal{A}) = \sum_{1 \leq i_1 \leq \cdots \leq i_N \leq d} a_{i_1 \ldots i_N} u_{i_1 \ldots i_N}.$$

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Setting $E_{i_1\dots i_N} := e_{i_1} \circ \cdots \circ e_{i_N}$ with each e_{i_k} a standard ordered basis element in $\mathbb{K}^d$, it follows by definition and linearity of Kronecker products that

$$\sum_{1\le i_1\le\cdots\le i_N\le d} u_{i_1\dots i_N} \operatorname{hvec}(E_{i_1\dots i_N})^T \operatorname{hvec}(\mathcal{A}) = \operatorname{hvec}_{1/N}(\mathcal{A}).$$

Similarly, if $T_{i_1\dots i_N} := \sum_{\sigma\in \operatorname{Orb}(i_1,\dots,i_N)} E_{i_{\sigma(1)}\dots i_{\sigma(N)}}$, summing over the orbit of each equivalence class representative yields

$$\sum_{1\le i_1\le\dots i_N\le d} T_{i_1\dots i_N} = \sum_{i_1,\dots i_N=1}^{d} E_{i_1\dots i_N},$$

hence by orthogonality and the equation above, we have that

$$\begin{aligned}\sum_{1\le i_1\le\cdots\le i_N\le d} \operatorname{hvec}(T_{i_1\dots i_N}) u^T_{i_1\dots i_N} \operatorname{hvec}_{1/N}(\mathcal{A}) &= \sum_{1\le i_1\le\cdots\le i_N\le d} \operatorname{hvec}(T_{i_1\dots i_N}) a_{i_1\dots i_N} \\ &= \sum_{i_1,\dots i_N=1}^{d} a_{i_1\dots i_N} \operatorname{hvec}(E_{i_1\dots i_N}) \\ &= \operatorname{hvec}(\mathcal{A}),\end{aligned}$$

completing the proof. □

4.2. Time complexity reduction via $D_d^{(N)}$. Now, let $\mathcal{A}$ be any symmetric order N tensor and side length d. Then, by Theorem 3.1, Cayley's first hyperdeterminant of $\mathcal{A}$ is given by

$$\begin{aligned}\left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon\right) * (\operatorname{hvec}(\mathcal{A}),\dots,\operatorname{hvec}(\mathcal{A})) &= \left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon\right) * \left(D_d^{(N)}\operatorname{hvec}_{1/N}(\mathcal{A}),\dots,D_d^{(N)}\operatorname{hvec}_{1/N}(\mathcal{A})\right) \\ &= \left(\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon * (D_d^{(N)},\dots,D_d^{(N)})\right) * (\operatorname{hvec}_{1/N}(\mathcal{A}),\dots,\operatorname{hvec}_{1/N}(\mathcal{A})).\end{aligned}$$

Denote the product $\frac{1}{d!}\bigotimes_{k=1}^{N}\varepsilon * \left(D_d^{(N)},\dots,D_d^{(N)}\right)$ as $\mathcal{E}_d^{(N)}$. Note that $\mathcal{E}_d^{(N)}$ is a cubical tensor with order d and side length $\binom{d+N-1}{N}$. Using this notation, it follows that

$$\operatorname{hdet}(A) = \mathcal{E}_d^{(N)} * (\operatorname{hvec}_{1/N}(A),\dots,\operatorname{hvec}_{1/N}(A)). \tag{4}$$

The matrix $D_d^{(N)}$, and hence also the tensor $\mathcal{E}_d^{(N)}$, depends only on d and N. Therefore, similarly as before, $\mathcal{E}_d^{(N)}$ may be computed once and reused for all subsequent hyperdeterminant calculations. Assuming $\mathcal{E}_d^{(N)}$ is precomputed and stored, the run time cost of hdet on symmetric tensors is $\mathcal{O}\left(\binom{d+N-1}{N}^d\right)$ due to the multilinear matrix product of $\mathcal{E}_d^{(N)}$ with the d copies of $\operatorname{hvec}_{1/N}(\mathcal{A})$ (noting again that asymptotically the initial cost of preparing $\operatorname{hvec}_{1/N}(\mathcal{A})$ is marginal and hence omitted in the cost calculation). Formally, we have the algorithm for computing $\operatorname{hdet}(\mathcal{A})$ for any order N symmetric tensor $\mathcal{A}$ with side length d:

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Algorithm 1 Fast Calculation of hdet on Symmetric Tensors

Precomputed: $\mathcal{E}_d^{(N)} \in \mathbb{K}^{\binom{d+N-1}{N}\times\cdots\times\binom{d+N-1}{N}}$
Input: Order N symmetric tensor $\mathcal{A} \in \mathbb{K}^{d\times\cdots\times d}$
Output: $\text{hdet}(\mathcal{A}) \in \mathbb{K}$

1: $\mathbf{a} \leftarrow \text{hvec}_{1/N}(\mathcal{A})$
2: $\text{hdet}(\mathcal{A}) \leftarrow \mathcal{E}_d^{(N)} * (\mathbf{a}, \ldots \mathbf{a})$
3: **return** $\text{hdet}(\mathcal{A})$

Now, recall that Stirling's approximation says that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for any positive integer n [19]. Therefore,

$$\binom{d+N-1}{N} = \frac{(d+N-1)!}{N!(d-1)!} \sim \sqrt{\frac{d+N-1}{2\pi N(d-1)}} \cdot \frac{\left(\frac{d+N-1}{e}\right)^{d+N-1}}{\left(\frac{N}{e}\right)^N \left(\frac{d-1}{e}\right)^{d-1}}.$$

For fixed d, we have that $d+N-1 \sim N$ and $\sqrt{\frac{d+N-1}{2\pi N(d-1)}} \sim \frac{1}{\sqrt{2\pi(d-1)}}$ as $N \to \infty$, hence it follows that

$$\binom{d+N-1}{N} \sim \frac{1}{\sqrt{2\pi(d-1)}} \cdot \left(\frac{e}{N}\right)^N \left(\frac{e}{d-1}\right)^{d-1} \left(\frac{N}{e}\right)^{d+N-1} \propto N^{d-1}.$$

Thus, the complexity of Algorithm 1 is $\mathcal{O}\left((N^{d-1})^d\right) = \mathcal{O}(N^{d(d-1)})$, which is polynomial in N. This is a significant improvement from the current state-of-the art method yielding time complexity $\mathcal{O}(2^{d(N-1)}d^{N-1})$, which is still exponential in N with fixed d. For emphasis, we summarize by way of the following theorem.

Theorem 4.1. *Assuming $\mathcal{E}_d^{(N)}$ is precomputed and d is fixed, the time complexity of Algorithm 1 is $\mathcal{O}(N^{d(d-1)})$ and hence polynomial in N.*

The table below provides a clear comparison/summary of the different methods for computing hdet:

Table 1
Time complexity comparison for computing Cayley's first hyperdeterminant

Method	General case	Symmetric case
Prior State-of-the-art [1]	$\mathcal{O}(2^{d(N-1)}d^{N-1})$	$\mathcal{O}(2^{d(N-1)}d^{N-1})$
Naive Application of Theorem 3.1	$\mathcal{O}(d^{dN})$	$\mathcal{O}(d^{dN})$
Algorithm 1 ($\mathcal{E}_d^{(N)}$ precomputed)	N/A	$\mathcal{O}(N^{d(d-1)})$

The space complexity also needs to be accounted for since we are assuming $\mathcal{E}_d^{(N)}$ is precomputed and stored. Since $\mathcal{E}_d^{(N)}$ is a cubical tensor of order d with side length $\binom{d+N-1}{N}$, its memory cost also grows $\mathcal{O}\left(N^{d(d-1)}\right)$, assuming d is fixed. Note also that $\bigotimes_{k=1}^{N} \varepsilon$ is a very sparse tensor, and so $\mathcal{E}_d^{(N)}$ is also likely to be quite sparse, in which the case the memory costs may potentially be reduced (possibly significantly). For the sake of brevity, we will not consider such potential reductions in space complexity, but we encourage interested readers to investigate this.

4.3. Application to bosonic quantum systems. One of the most elusive phenomena in quantum physics, and at the same time one of the most powerful resources in quantum information, is quantum

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entanglement. Entangled quantum states arise in composite quantum systems, whose state spaces are tensor products of Hilbert spaces. One of the problems considered in quantum physics and quantum information theory is the quantification of a state's entanglement. In general, this is a complicated endeavor because quantum states in a tensor product of 3 or more Hilbert spaces can be entangled in different, physically inequivalent ways [9]. Nonetheless, it is widely agreed throughout the literature (a few authoritative sources affirming this include [3, 4, 10, 21, 22]) that any proposed entanglement measure E should at a bare minimum satisfy the following criteria:

1. E vanishes on separable states,
2. E is invariant under local unitary transformations, and
3. E is nonincreasing on average under local operations and classical communications (LOCC).

The three properties listed above are sometimes referred to as the entanglement axioms.

Qubits are the simplest type of quantum systems and the fundamental building blocks of quantum computation. Qudits are higher-dimensional analogues of qubits. In particular, for any positive integers $n, d \geq 2$, the state of an n-qudit (n-qubit when $d = 2$) is a unit vector $\psi \in (\mathbb{C}^d)^{\otimes n}$, which can be written as a sum

$$\psi = \sum_{i_1,\ldots,i_n=0}^{d-1} \psi_{i_1\ldots i_n} i_1 \ldots i_n,$$

such that

$$\sum_{i_1,\ldots,i_n=0}^{d-1} |\psi_{i_1\ldots i_n}|^2 = 1,$$

where $i_1 \ldots i_n$ denotes the Kronecker product $i_1 \otimes \cdots \otimes i_n$ and i_k denotes the $(i_k+1)^{th}$ standard ordered basis element in $\mathbb{C}^d$. Replacing the Kronecker product $\otimes$ with the outer product $\circ$ yields a cubical tensor of order n with side length d:

$$\sum_{i_1,\ldots,i_n=0}^{d-1} \psi_{i_1\ldots i_n} i_1 \circ \cdots \circ i_n =: \widehat{\psi},$$

which has Frobenius norm 1. In particular, there is a bijection between the space of n-qudit states and the space of cubical tensors of order n with side length d and Frobenius norm 1 given by the map

$$\psi \mapsto \widehat{\psi}.$$

The concurrence is a popular entanglement measure on $2n$-qubits that measures the $2n$-way entanglement involving all qubits [6, 23]. In [7], Dobes & Jing prove that the concurrence C satisfies the equation

$$C(\psi) = 2|\text{hdet}(\widehat{\psi})|,$$

and in [8] they show that hdet more generally on $2n$-qudits hdet satisfies the entanglement axioms. Thus, Cayley's first hyperdeterminant may be considered a physically meaningful generalization of the concurrence.

A special class of n-qudit states of particular interest to physicists are those found in bosonic quantum systems (see [17] for a nice overview). Mathematically, the state space of n indistinguishable bosons is the Hilbert space $\text{Sym}^n(\mathbb{C}^d)$, and so states ψ in bosonic quantum systems satisfy the symmetry condition:

$$\psi = P_\pi \cdot \psi \qquad \forall \pi \in S_n,$$

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where

$$P_\pi \cdot \psi := \sum_{i_1,\ldots,i_n=0}^{d-1} \psi_{i_{\pi^{-1}(1)},\ldots,i_{\pi^{-1}(n)}} i_{\pi^{-1}(1)} \ldots i_{\pi^{-1}(n)}.$$

The bijection $\psi \mapsto \widehat{\psi}$ implies that ψ satisfies the symmetry condition if and only if $\widehat{\psi}$ is symmetric. Therefore, to calculate the $2n$-way entanglement of a symmetric $2n$-qudit state reduces to calculating the hyperdeterminant of its corresponding tensor. Algorithm 1, therefore, provides an efficient way for one to calculate the $2n$-way entanglement of $2n$-qudit states of bosonic quantum systems.

5. Conclusion/Future work. In summary, in this article, we have derived a new formula for computing Cayley's first hyperdeterminant hdet in terms of a multilinear matrix product involving the Levi-Civita symbol and the canonical vectorization operator. For symmetric tensors of order N with side length d, we may consider their $1/N$-hypermatrix vectorization $\mathrm{hvec}_{1/N}$, which contains the same amount of information as their vectorization but requires only $\binom{d+N-1}{N}$ entries rather than d^N. This, therefore, substantially reduces the time complexity for computing hdet in terms of N assuming d fixed. An important application for this is the efficient calculation of the $2n$-way entanglement for $2n$-qudit states of bosonic quantum systems.

Besides exploiting the sparsity of $\mathcal{E}_d^{(N)}$ to reduce memory costs and the burden of precomputation, another (more interesting) potential direction for future research could be to consider partially symmetric tensors and derive subgroup analogues of generalized duplication matrices. That is, for subgroups $G \leq S_N$ and ***G*-symmetric** tensors satisfying the property:

$$a_{i_{\sigma(1)}\ldots i_{\sigma(N)}} = a_{i_1\ldots i_N} \qquad \forall \sigma \in G,$$

define G-analogues of $\mathrm{hvec}_{1/N}$ and $D_d^{(N)}$. With such objects, one could then factor hdet on partially symmetric tensors in a similar manner to equation (4), and then for fixed d and precomputed G-analogue of $\mathcal{E}_d^{(N)}$, consider the run time cost as $N \to \infty$. For partially symmetric tensors, the run time cost will be at least as costly as in the fully symmetric case, but is it sill polynomial for certain proper subgroups (indeed, it trivially is for the alternating subgroup A_N, but this does not hold in general for all proper subgroups), and which subgroups yield quasi-polynomial or subexponential run-time costs? This seems like an interesting problem at the intersection of group theory and complexity theory. Additionally, G-analogues of generalized duplication matrices likely have a rich algebraic structure worth further investigating.

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Data availability statement. Any data that support the findings of this study are included within the article.

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