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THE ω-COMMUTING GRAPH OF THE UPPER-TRIANGULAR MATRIX ALGEBRA*

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Abstract. In this paper, the connectivity properties and the diameter of the ω-commuting graph Δ_ω of the upper-triangular matrix algebra over arbitrary fields are studied. For any $n \geq 3$ and $\omega \neq 0, \pm 1$, the directed graph Δ_ω is weakly connected with weak diameter 4. As a directed graph, it has one large strongly connected component of diameter 4 and a number of one-vertex components. In the special case of 2×2 matrices, it is established that the considered graph is disconnected with components of diameter at most 2.

Key words. Relation graphs for rings, Directed graphs, ω-commuting graph, Orthogonality graph, Matrix algebra, Upper-triangular matrices.

AMS subject classifications. 15A27, 05C12.

1. Introduction.

Binary relations on associative rings, in particular, on the matrix ring, are an important subject in contemporary mathematics that is important for numerous applications. At present, an efficient way to study a given relation is to consider its *relation graph*, vertices of which belong to a subset of the ring, and two vertices are connected by an edge if and only if the relation holds for them. Recently, commuting, zero-divisor, and orthogonality graphs have been studied in detail by many authors, see [1]–[6], [8]–[10], [23] and their bibliography. Symmetric relations, such as commutativity or orthogonality, are naturally represented by ordinary (undirected) graphs, while for nonsymmetric relations, it is more informative to use directed graphs (digraphs). This paper examines both approaches, which helps, in particular, to reveal the connections between ω-commutativity and orthogonality.

Throughout the paper, $M_n(\mathbb{F})$ denotes the algebra of $n \times n$ matrices over a field $\mathbb{F}$, and $T_n(\mathbb{F})$ denotes the subalgebra of upper-triangular matrices in $M_n(\mathbb{F})$.

Definition 1.1. Given a fixed $\omega \in \mathbb{F}$, we say that A, B in $M_n(\mathbb{F})$ *commute up to a factor* ω or *ω-commute* if $AB = \omega BA$.

Note that if $\omega = 1$, then 1-commutativity is the commutativity in the usual sense, so ω-commutativity as a relation generalizes commutativity. For $\omega = -1$, the term "*anti-commutativity*" is used. The cases $\omega = \pm 1$ are also special as the only cases when the ω-commutativity relation is symmetric. Another special case is $\omega = 0$, the relation $AB = 0BA$ implies that $AB = O$ and matrices A and B are zero divisors. Matrices commuting up to a factor ω are studied from different points of view, see, for instance, a survey [18] on possible normal forms for such pairs of matrices, [13, 14] for analogues of the double centralizer theorem with respect to this relation, [15, 16] for the connection with generating sets and their numerical characteristics. Moreover, this relation has various applications, for example, in quantum physics, see classical monographs

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[11, 19] and the references therein, in the representation theory of affine Hecke algebras, see [22] and in the coding theory [7].

The commutativity, ω-commutativity, and zero divisors relations are closely connected to the orthogonality relation. Recall that

Definition 1.2. Elements r, s of a ring R are said to be *orthogonal*, if $rs = sr = 0$.

Clearly, orthogonal elements are both two-sided zero divisors and commuting elements of a ring. The connection with ω-commutativity relation in the matrix algebra is described as follows.

Remark 1.3. Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F}$, $\omega \neq 0, \pm 1$ and $n \geq 2$. Given such matrices $A, B \in M_n(\mathbb{F})$ that both pairs (A, B) and (B, A) are ω-commuting, that is, $AB = \omega BA$ and $BA = \omega AB$. Then $AB = \omega^2 AB$ and $BA = \omega^2 BA$, but $\omega^2 \neq 1$, therefore, $AB = BA = O$. Consequently, the pairs of orthogonal matrices are exactly those pairs for which the ω-commutativity relation is symmetric.

The study of the ω-commutativity relation graph for the matrix algebra over a field was introduced by Raja and Vaezpour in [24].

Definition 1.4 ([24]). For a field $\mathbb{F}$, $\omega \in \mathbb{F}$, and $n \geq 2$, the *ω-commuting graph* of $M_n(\mathbb{F})$, denoted by $\Delta_\omega(M_n(\mathbb{F}))$, is a directed graph, which set of vertices consists of all nonzero matrices A such that A and ωA have a common eigenvalue, and there is an arc $A \to B$ if and only if A and B are ω-commutative matrices.

Since in the present paper both directed and ordinary graphs are considered, we will further refer to $\Delta_\omega(M_n(\mathbb{F}))$ as *ω-commuting digraph* to avoid ambiguity.

If $\mathcal{X}$ is a subset of $M_n(\mathbb{F})$, then $\Delta_\omega(\mathcal{X})$ denotes the induced directed subgraph of $\Delta_\omega(M_n(\mathbb{F}))$ on the intersection of $\mathcal{X}$ and the vertex set of $\Delta_\omega(M_n(\mathbb{F}))$.

Remark 1.5. The zero matrix ω-commutes with all matrices for any $\omega \in \mathbb{F}$. Conversely, if the matrices A and ωA do not share a common eigenvalue, then the equation $AB = \omega BA$ has only zero solution [18]. This is the reasoning for the choice of the vertex set for $\Delta_\omega(M_n(\mathbb{F}))$.

Raja and Vaezpour [24] determined several conditions which guarantee the strong connectivity of the digraph $\Delta_\omega(M_n(\mathbb{F}))$. Also they investigated induced directed subgraphs on the sets of all reducible matrices, triangularizable matrices, non-invertible, idempotent, and diagonizable matrices in $M_n(\mathbb{F})$.

Our paper continues research in this direction and is devoted to the study of the ω-commuting digraph $\Delta_\omega(T_n(\mathbb{F}))$ of the upper-triangular matrix algebra $T_n(\mathbb{F})$.

For the set of upper-triangular matrices, the commuting graph [6], zero divisor graph [12, 20, 21], and orthogonality graph [10] were investigated. In our research, we will exclude the symmetric cases $\omega = \pm 1$ that give commutativity and anti-commutativity graphs and $\omega = 0$ that defines the zero divisor graph. The condition $\omega \neq 0, \pm 1$ implies a restriction on the ground field cardinality, that is, $|\mathbb{F}| \geq 4$. Next, only such fields will be considered.

Our paper is organized as follows. Section 2 contains basic definitions and some auxiliary results regarding graph theory and ω-commuting matrices. Section 3 is devoted to the ω-commuting digraph for the upper-triangular matrix algebra. Namely, in Subsection 3.1, the vertex set of the ω-commuting digraph $\Delta_\omega(T_n(\mathbb{F}))$ is described. As a consequence, we deduce that ω-commuting digraph for the upper-triangular matrix algebra cannot be strongly connected. Subsection 3.2 is devoted to the special case of 2×2 matrices, for which the considered digraph is disconnected with connected components of diameter at most 2 (Theorem 3.12). In Subsection 3.3, the general case of $n \times n$ matrices with $n \geq 3$ is investigated. The main result of the paper is stated in Theorem 3.20: it is shown that for any $\omega \neq 0, \pm 1$ and $n \geq 3$, the directed graph $\Delta_\omega(T_n(\mathbb{F}))$ is weakly connected, with underlying connected graph of diameter 4, and as a directed graph, it has one large component of diameter 4 and a number of one-vertex components.

2. Notation, definitions, and known results. Throughout the paper, $M_n(\mathbb{F})$ denotes the algebra of $n \times n$ matrices over a field $\mathbb{F}$, $T_n(\mathbb{F})$ denotes the subalgebra of upper-triangular matrices in $M_n(\mathbb{F})$, and $\mathbb{F}^n$ denotes the column space of dimension n over the field $\mathbb{F}$.

Let E_{ij} denote the matrix unit with 1 at the position (i, j) and zeros elsewhere. Let I_n, or I if the size is seen from the context, denote the identity matrix in $M_n(\mathbb{F})$. Let O_n, or O, denote the zero matrix in $M_n(\mathbb{F})$, and $O_{k\times m}$ denote the $k \times m$ rectangular zero matrix. By $J_n(\lambda)$, we denote the Jordan block of size n with an eigenvalue λ.

2.1. Graphs. Recall some definitions from graph theory. The notions of graph theory used in this paper can be found, for example, in [17, Chapter 2].

Definition 2.1. A *directed graph* Δ is an ordered pair $\Delta = (V, A)$, where V is a nonempty vertex set and A is a (possibly empty) set of objects called *arcs* (or *directed edges*) such as $u \to v$, where u and v are two elements in the vertex set V of Δ. It differs from an *ordinary* or *undirected graph* $\Gamma = (V, E)$, in that the latter is defined in terms of unordered pairs of vertices, which are usually called *edges.* We do not allow a graph to have multiple arcs or edges, but loops are admissible.

Definition 2.2. *A symmetric digraph* is a digraph in which all edges appear twice, one in each direction (that is, for every arc $u \to v$ that belongs to the digraph, the corresponding inverse arc $v \to u$ also belongs to it).

Definition 2.3. Using the definition above, one may transform any undirected graph $\Gamma = (V, E)$ into a symmetric digraph $D\Gamma$ with the same vertex set V converting each edge $v - u$ into two arcs $u \to v$ and $v \to u$.

Reversely, with any digraph Δ, one may consider the undirected *underlying graph* $U\Delta$ obtained by replacing all directed edges of the graph with undirected edges. In this case, if for a pair of vertices u, v in the digraph there were two arcs $u \to v$ and $v \to u$, then in the underlying graph, one edge $u - v$ is left.

Definition 2.4. In a directed graph Δ, an arc (x, y) is considered to be directed from x to y; y is called the *head*, and x is called the *tail* of the arc. For a vertex, the number of head ends adjacent to a vertex is called the *indegree* of the vertex, and the number of tail ends adjacent to a vertex is its *outdegree*.

Definition 2.5. A *path* M in a graph Γ is a sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots e_k, v_k$, whose terms are alternately distinct vertices and distinct edges, such that for any i, $1 \leq i \leq k$, e_i is $v_{i-1} - v_i$. The number k is called the *length* of M. The graph is said to be connected if it is possible to establish a path from any vertex to any other vertex of the graph. The distance $\mathrm{d}(u, v)$ between two vertices u and v in a graph Γ is the length of the shortest path between them. If u and v are unreachable from each other, we define $\mathrm{d}(u, v) = \infty$. It is assumed that $\mathrm{d}(u, u) = 0$ for any vertex u. The *diameter* $\operatorname{diam}\Gamma$ of a graph Γ is the supremum of distances between vertices for all pairs of vertices in the graph.

Definition 2.6. In a directed graph Δ, a *directed path* P is a sequence $v_0, a_1, v_1, a_2, \ldots a_k, v_k$ whose terms are alternately distinct vertices and distinct arcs, such that for any i, $1 \leq i \leq k$, a_i is $v_{i-1} \to v_i$.

The number k is called the *length* of P. In a directed graph Δ, we say u is connected to v if there exists a directed path from u to v. For two vertices u and v in a directed graph Δ, the distance between u and v, denoted by $\mathrm{d}\,(u,v)$, is the length of the shortest directed path from u to v, if such a directed path exists; otherwise, we define $\mathrm{d}\,(u,v) = \infty$. The diameter of a directed graph Δ is defined as

$$\operatorname{diam}\Delta = \sup\{\mathrm{d}\,(u,v) \mid u, v \text{ are distinct vertices of } \Delta\}.$$

Definition 2.7. A *connected graph* is an undirected graph Γ in which every unordered pair of vertices in the graph is connected. Otherwise, it is called a *disconnected graph*.

Definition 2.8. In a directed graph Δ, an ordered pair of vertices (u,v) is called *strongly connected* if a directed path leads from u to v. The ordered pair is called *weakly connected* if an undirected path leads from u to v after replacing all of its directed edges with undirected edges. Otherwise, the ordered pair is called *disconnected*.

Definition 2.9. A *strongly connected digraph* is a directed graph in which every ordered pair of vertices is strongly connected. It is called a *weakly connected graph* if every ordered pair of vertices is weakly connected. Otherwise, it is called a *disconnected graph*.

Definition 2.10. A *connected component* is a maximal strongly connected subgraph of Δ.

Let R be an associative unital ring.

Recall that an element a of a ring R is a *left (right) zero divisor*, if there is a nonzero element $b \in R$ such that $ab = 0$ (respectively, $ba = 0$). An element that is both left and right zero divisor is called a *two-sided zero divisor*. We call a ring a *domain*, if it has no zero divisors except 0, i.e., $ab = 0$ implies that either $a = 0$ or $b = 0$.

Notation 2.11. *For an element $r \in R$ we denote $O_R\,(r) = \{s \in R : rs = sr = 0\}$. By $O_R\,(\mathcal{X})$, where $\mathcal{X}$ is a subset of R, we denote the set of elements in the ring R orthogonal to each element from $\mathcal{X}$.*

Remark 2.12. The zero element $0 \in R$ is orthogonal to every element of the ring. On the contrary, if an element $r \in R$ is either not a left or not a right zero divisor, then there does not exist a nonzero element $x \in R$ such that $xr = rx = 0$, i.e., there are no nonzero elements of R orthogonal to r. In view of this, we a priori remove 0 and elements that do not divide zero on at least one side from the vertices of the orthogonality graph.

Definition 2.13 ([9, Definition 2.15]). For each ring R define an undirected *orthogonality graph* $O\,(R)$ such that its vertices are all the two-sided zero divisors of $R \setminus \{0\}$, and there is an edge between a pair of vertices if and only if they are orthogonal.

If $\mathcal{X}$ is a subset of R, then $O(\mathcal{X})$ denotes the induced subgraph of $O(R)$ on the intersection of $\mathcal{X}$ and the set of the two-sided zero divisors of $R \setminus \{0\}$.

Lemma 2.14 ([9, Lemma 2.17]). *$O(R)$ is empty if and only if R is a domain.*

To establish further relationships between the ω-commuting and orthogonality graphs, using Definition 2.3, we introduce

Definition 2.15. The *directed orthogonality graph* $DO(R)$ is such that its vertices are all the two-sided zero divisors of $R \setminus \{0\}$, and there are arcs (u,v) and (v,u) between pair of vertices if and only if u and v are orthogonal.

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The digraph $DO(R)$ is a symmetric digraph. For a subset $\mathcal{X}$ of the matrix algebra $M_n(\mathbb{F})$ and a coefficient $\omega \in \mathbb{F}$, $\omega \neq 0, \pm 1$, the digraph $DO(\mathcal{X})$ is the maximal symmetric subgraph of the ω-commuting digraph $\Delta_\omega(\mathcal{X})$.

2.2. ω-commuting matrices.

Definition 2.16. Given $\omega \in \mathbb{F}$ and a subset $\mathcal{X} \subseteq M_n(\mathbb{F})$, consider a *generalized centralizer*

$$\mathcal{C}^\omega(\mathcal{X}) = \bigcap_{A \in \mathcal{X}} \mathcal{C}^\omega(A),$$

where

$$\mathcal{C}^\omega(A) = \{B \in M_n(\mathbb{F}) : AB = \omega BA\}.$$

Proposition 2.17 ([14, Proposition 7]). *Let $\mathbb{F}$ be an arbitrary field, $A \in M_n(\mathbb{F})$ be a Jordan matrix of the form $A = A_1 \oplus \ldots \oplus A_m$, where $A_i = J_{n_i}(\lambda_i)$ is a Jordan block (some of λ_i may be repeated). Set $\omega \in \mathbb{F}$, $\omega \neq 0, 1$. Consider a matrix $X \in \mathcal{C}^\omega(A)$. Partition X into m^2 blocks X_{kl} of sizes $n_k \times n_l$ according to the block partition of A. Then,*

1. *$X_{rs} = 0$ if $r = s$ and $\lambda_r \neq 0$, or if $r \neq s$ and $\lambda_r \neq \omega\lambda_s$;*
2. *if $\lambda_r = 0$,*

$$X_{rr} = \begin{bmatrix} y_{r,r;1} & y_{r,r;2} & y_{r,r;3} & \cdots & y_{r,r;n_r} \\ 0 & \omega y_{r,r;1} & \omega y_{r,r;2} & \cdots & \omega y_{r,r;n_r-1} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & & \omega^{n_r-2} y_{r,r;1} & \omega^{n_r-2} y_{r,r;2} \\ 0 & 0 & 0 & \cdots & \omega^{n_r-1} y_{r,r;1} \end{bmatrix};$$

3. *if $\lambda_r = \omega\lambda_s$,*

$$X_{rs} = \begin{bmatrix} 0 & \cdots & y_{r,s;1} & y_{r,s;2} & \cdots & y_{r,s;n_r} \\ 0 & 0 & 0 & \omega y_{r,s;1} & \cdots & \omega y_{r,s;n_r-1} \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega^{n_r-1} y_{r,s;1} \end{bmatrix}, \; n_r < n_s.$$

$$X_{rs} = \begin{bmatrix} y_{r,s;1} & y_{r,s;2} & \cdots & y_{r,s;n_s} \\ 0 & \omega y_{r,s;1} & \ddots & \omega y_{r,s;n_s-1} \\ 0 & 0 & \ddots & \ddots \\ 0 & 0 & & \omega^{n_s-1} y_{r,s;1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \; n_r \geq n_s.$$

3. The ω-commuting digraph for the upper-triangular matrix algebra.

In accordance with Definition 1.4, for a field $\mathbb{F}$, $\omega \in \mathbb{F}$, and $n \geq 2$, the *ω-commuting digraph* of the upper-triangular matrix algebra $T_n(\mathbb{F})$ denoted by $\Delta_\omega(T_n(\mathbb{F}))$ is the induced directed subgraph of $\Delta_\omega(M_n(\mathbb{F}))$ on the intersection of $T_n(\mathbb{F})$ and the vertex set of $\Delta_\omega(M_n(\mathbb{F}))$.

3.1. Vertex set structure in the digraph $\Delta_\omega(T_n(\mathbb{F}))$. By definition, the vertex set of $\Delta_\omega(T_n(\mathbb{F}))$ consists of all singular nonzero triangular matrices and those nonsingular matrices $A \in T_n(\mathbb{F})$ for which A and ωA have a common eigenvalue. Note that all singular nonzero triangular matrices are also the vertices of the orthogonality graphs $O(T_n(\mathbb{F}))$ and $DO(T_n(\mathbb{F}))$.

Let us further examine those non-singular matrices that are the vertices of the ω-commuting digraph $\Delta_\omega(T_n(\mathbb{F}))$.

Notation 3.1. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ that belongs to the vertex set of $\Delta_\omega(M_n(\mathbb{F}))$, that is, for some indices $p, q = 1, \dots, n$ $a_{pp} = \omega a_{qq}$, or equivalently, $\mathcal{C}^\omega(A) \neq \{O\}$. Based on the relations between possible values of p, q, let us divide such matrices into the following three disjoint sets.*
I. The set $\mathcal{T}$ contains such matrices A for which

- *there exists a pair of indices $1 \leq i, j \leq n$ such that $i < j$ and $a_{ii} = \omega a_{jj}$,*
- *for all $1 \leq r, s \leq n$ it holds that if $s > r$ then $a_{ss} \neq \omega a_{rr}$.*

II. The set $\mathcal{B}$ contains such matrices A for which there exists two pairs of indices i, j and r, s from $\{1, \dots, n\}$, such that $i < j$, $r < s$, $a_{ii} = \omega a_{jj}$ and $a_{ss} = \omega a_{rr}$.
III. The set $\mathcal{H}$ contains such matrices A for which

- *there exists a pair of indices $1 \leq r, s \leq n$ such that $r < s$ and $a_{ss} = \omega a_{rr}$,*
- *for all $1 \leq i, j \leq n$ it holds that if $i < j$, then $a_{ii} \neq \omega a_{jj}$.*

Proposition 3.2. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Then, the sets of matrices $\mathcal{H}$ and $\mathcal{T}$ in Notation 3.1 are nonempty. The set $\mathcal{B}$ exists and is nonempty for all $n \geq 3$.*

Proof. A diagonal matrix $D_1 = \operatorname{diag}(\omega, 1, \dots, 1, 1) \in T_n(\mathbb{F})$ belongs to the set $\mathcal{T}$. Indeed $d_{11} = \omega d_{22}$ and for all $1 \leq r, s \leq n$ it holds that if $s > r$ then $d_{ss} \neq \omega d_{rr}$ since $\omega \neq \pm 1$. Similarly, a diagonal matrix $D_3 = \operatorname{diag}(1, 1, \dots, 1, \omega) \in T_n(\mathbb{F})$ belongs to the set $\mathcal{H}$.

For $n = 2$ if $a_{11} = \omega a_{22}$, then $a_{22} \neq \omega a_{11}$, or if $a_{22} = \omega a_{11}$, then $a_{11} \neq \omega a_{22}$, since $\omega \neq \pm 1$, so the set $\mathcal{B}$ is empty. If $n \geq 3$, there exists a diagonal matrix $D_2 = \operatorname{diag}(\omega, 1, \dots, 1, \omega) \in T_n(\mathbb{F})$, which belongs to the set $\mathcal{B}$. ∎

Proposition 3.3. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ that belongs to the vertex set of $\Delta_\omega(M_n(\mathbb{F}))$. Then, any matrix from $\mathcal{C}^\omega(A) \cap T_n(\mathbb{F})$ is nil-triangular.*

Proof. Consider an arbitrary matrix $X \in \mathcal{C}^\omega(A) \cap T_n(\mathbb{F})$. Then, the diagonal entries of X satisfy ω-commutativity equations $a_{ii} x_{ii} = \omega a_{ii} x_{ii}$. Since $\omega \neq 1$, the invertibility of A implies that $x_{ii} = 0$ for all $i = 1, \dots, n$, that is, the matrix X is nil-triangular. ∎

Proposition 3.4. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ from the set $\mathcal{H}$ in Notation 3.1. Then, $\mathcal{C}^\omega(A) \cap T_n(\mathbb{F}) = \{O\}$.*

Proof. Consider an arbitrary matrix $X \in \mathcal{C}^\omega(A) \cap T_n(\mathbb{F})$. By Proposition 3.3, the matrix X is nil-triangular.

By condition for all $1 \leq i, j \leq n$ it holds that if $i < j$, then $a_{ii} \neq \omega a_{jj}$. We use induction on $l = 1, \dots, n-1$ to prove that the element $x_{k,k+l} = 0$ for all $k = 1, \dots, n - l$.

The induction base: $l = 1$. For all $k = 1, \dots, n - 1$, consider the $(k, k + 1)$-th entry of the matrix $AX - \omega XA$. We have

$$a_{kk}x_{k,k+1} - \omega x_{k,k+1}a_{k+1,k+1} = 0,$$

or equivalently

$$(a_{kk} - \omega a_{k+1,k+1})x_{k,k+1} = 0.$$

Since $a_{kk} \neq \omega a_{k+1,k+1}$ the equation above implies that $x_{k,k+1} = 0$ for all $k = 1, \ldots, n-1$.

The induction step: assume the statement holds for all indices $r < l$ and let us prove it for l. For all $k = 1, \ldots, n-l$ consider the $(k, k+l)$-th entry of the matrix $AX - \omega XA$. By induction hypothesis, we have $x_{rs} = 0$ for all $r < s + l$, therefore

$$a_{kk}x_{k,k+l} - \omega x_{k,k+l}a_{k+l,k+l} = 0,$$

or equivalently,

$$(a_{kk} - \omega a_{k+l,k+l})x_{k,k+l} = 0.$$

Since $a_{kk} \neq \omega a_{k+l,k+l}$, the equation above implies that $x_{k,k+l} = 0$ for all $k = 1, \ldots, n-l$.

Consequently, $X = O$ and $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) = \{O\}$. ∎

Proposition 3.5. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ from one of the sets $\mathcal{T}$ or $\mathcal{B}$ in Notation 3.1. Then, $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) \neq \{O\}$.*

Proof. By condition, there exists a pair of indices $1 \leq i, j \leq n$ such that $i < j$ and $a_{ii} = \omega a_{jj}$. First among all such pairs of indices we choose i, j with the minimal possible i. Next for the chosen i take the maximal possible j.

Consider a nilpotent matrix $X \in T_n(\mathbb{F})$ of the following block form

$$X = \begin{bmatrix} O_{i,j-1} & X' \\ O & O_{n-i,n-j+1} \end{bmatrix},$$

that is, the element x_{ij} is located in the lower left corner of the matrix X'. Consider the matrix equation $AX - \omega XA = O$ as a system of linear equations in the elements of X. Notice that the matrix $AX - \omega XA$ has the same block structure as the matrix X, the nontrivial equations arise for the elements located on the intersection of the first i rows with the last $n - j + 1$ columns. We claim that this system has nonzero solutions for which the element x_{ij} acts as a free parameter. To prove the claim, let us solve the equations moving to the right starting with the j-th column and going upward in each column as in the Gaussian elimination. Formally, we use the induction on the number of the column $t = j, \ldots, n$.

The induction base: $t = j$. We consider i equations, moving upward from the last (i-th) to the first. The last equation $a_{ii}x_{ij} - \omega a_{jj}x_{ij} = 0$ is trivial, since $a_{ii} = \omega a_{jj}$. Here, we use the induction on $d = 1, \ldots, i-1$ to demonstrate that $x_{i-d,j}$ can be expressed linearly in x_{ij}.

- *The induction base:* $d = 1$. The equation for the $(i-1)$-st row is

  $$a_{i-1,i-1}x_{i-1,j} + a_{i-1,i}x_{ij} - \omega x_{i-1,j}a_{jj} = 0.$$

  The coefficient $a_{i-1,i-1} - \omega a_{jj} \neq 0$ by the minimality of i, so from this equation, we express $x_{i-1,j}$ linearly in x_{ij}: $x_{i-1,j} = -a_{i-1,i}(a_{i-1,i-1} - \omega a_{jj})^{-1}x_{ij}$.
- *The induction step:* in $(i-d)$-th row the equation is

  $$a_{i-d,i-d}x_{i-d,j} + a_{i-d,i-d+1}x_{i-d+1,j} + \ldots + a_{i-d,i}x_{ij} - \omega x_{i-d,j}a_{jj} = 0.$$

The coefficient $a_{i-d,i-d} - \omega a_{jj} \neq 0$ by the minimality of i, thus from this equation we express $x_{i-d,j}$ linearly in $x_{i-d+1,j}, \ldots, x_{i-1,j}, x_{ij}$. By induction hypothesis each of these terms is expressed linearly in x_{ij}, so is $x_{i-d,j}$.

The induction step: suppose the statement holds for all $j \leq l < t$, let us prove it for $l = t$. Again here we use the induction on $d = 0, \ldots, i-1$ to demonstrate that $x_{i-d,t}$ can be expressed linearly in x_{ij}.

- *The induction base:* $d = 0$. The equation for the i-th row is
$$a_{ii}x_{it} - \omega(x_{ij}a_{jt} + x_{i,j+1}a_{j+1,t} + \ldots + x_{it}a_{tt}) = 0.$$
The coefficient $a_{ii} - \omega a_{tt} \neq 0$ by the maximality of j, so from this equation we express $x_{i,t}$ linearly in $x_{ij}, x_{i,j+1}, \ldots, x_{i,t-1}$. By induction hypothesis for t each of these terms is expressed linearly in x_{ij}, so is $x_{i,t}$.
- *The induction step:* in $(i-d)$-th row the equation is
$$a_{i-d,i-d}x_{i-d,t} + a_{i-d,i-d+1}x_{i-d+1,t} + \ldots + a_{i-d,i}x_{it} - \omega(x_{i-d,j}a_{jt} + x_{i-d,j+1}a_{j+1,t} + \ldots + x_{i-d,t}a_{tt}) = 0.$$
The coefficient $a_{i-d,i-d} - \omega a_{tt} \neq 0$ by the minimality of i, thus from this equation we express $x_{i-d,t}$ linearly in $x_{i-d+1,t}, \ldots, x_{i-1,t}, x_{it}$ and $x_{i-d,j}, x_{i-d,j+1}, \ldots, x_{i-d,t-1}$. By induction hypotheses each of these terms is expressed linearly in x_{ij}, so is $x_{i-d,t}$.

Therefore, we have constructed a matrix $X \in \mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F})$ with elements parameterized by x_{ij}, so for each $x_{ij} \neq 0$ there exists a nonzero matrix in $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F})$. Consequently, $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) \neq \{O\}$. □

Remark 3.6. Notice that if we consider ω^{-1} instead of ω, then the set $\mathcal{B}$ in Notation 3.1 remains the same, for the new set $\mathcal{T}$ the condition is $a_{ii} = \omega^{-1}a_{jj}$, or $a_{jj} = \omega a_{ii}$, that is, it is the set $\mathcal{H}$ for the coefficient ω, and viceversa.

Thus, replacing ω with ω^{-1} in Propositions 3.3–3.5, we conclude that

Corollary 3.7. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ that belongs to the vertex set of $\Delta_{\omega}(M_n(\mathbb{F}))$. Then,*
1. for a matrix A in the set $\mathcal{T}$ in Notation 3.1, it holds that $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) \neq \{O\}$ and $\mathcal{C}^{\omega^{-1}}(A) \cap T_n(\mathbb{F}) = \{O\}$;
2. for a matrix A in the set $\mathcal{B}$, it holds that $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) \neq \{O\}$ and $\mathcal{C}^{\omega^{-1}}(A) \cap T_n(\mathbb{F}) \neq \{O\}$;
3. for a matrix A in the set $\mathcal{H}$, it holds that $\mathcal{C}^{\omega}(A) \cap T_n(\mathbb{F}) = \{O\}$ and $\mathcal{C}^{\omega^{-1}}(A) \cap T_n(\mathbb{F}) \neq \{O\}$.

In terms of the ω-commutativity digraph, this result implies

Proposition 3.8. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$ and $n \geq 2$. Consider an invertible matrix $A \in T_n(\mathbb{F})$ that belongs to the vertex set of $\Delta_{\omega}(M_n(\mathbb{F}))$. Then,*
1. each matrix A in the set $\mathcal{T}$ in Notation 3.1 is a vertex of $\Delta_{\omega}(T_n(\mathbb{F}))$ with a zero indegree and a positive outdegree, so this vertex acts as a tail of certain arcs, but is not a head;
2. each matrix A in the set $\mathcal{B}$ is a vertex of $\Delta_{\omega}(T_n(\mathbb{F}))$ with positive indegree and outdegree, so this vertex acts both as a tail and as a head of some arcs;
3. each matrix A in the set $\mathcal{H}$ is a vertex of $\Delta_{\omega}(T_n(\mathbb{F}))$ with a zero outdegree and a positive indegree, so this vertex acts as a head of certain arcs, but is not a tail.

Note that the sets $\mathcal{T}$ and $\mathcal{H}$ are nonempty for all $n \geq 2$ by Proposition 3.2, and previous statement implies, in particular, that there are no arcs starting from $u \in \mathcal{H}$. Consequently, there exists an ordered pair of vertices (u, v), $u \in \mathcal{H}$, $v \in \mathcal{T}$, which is not strongly connected. In terms of the digraph $\Delta_\omega(T_n(\mathbb{F}))$, this fact can be reformulated as follows.

Theorem 3.9. *Let $\mathbb{F}$ be an arbitrary field. For any $n \geq 2$ and $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$, the digraph $\Delta_\omega(T_n(\mathbb{F}))$ is not strongly connected.*

Further we will describe connected components and evaluate their diameters $n = 2$ and $n \geq 3$ separately.

3.2. Matrices of order 2.

It is a usual situation that the relation graph of the 2×2 matrix algebra has a different structure and connectivity properties than the graph in the case of greater matrix sizes, see e.g. [24, Theorem 2.16, Remark 2.18, Corollary 2.19], [9, Lemma 4.1], [10, Lemma 2.2]. We demonstrate that for the ω-commutativity digraph of the upper-triangular matrix algebra the case of 2×2 matrices also differs from the general situation.

Several sets of vertices that consist of singular matrices that are inherited from the orthogonality graph, so recall the corresponding result from [10].

Lemma 3.10 ([10, Lemma 2.2]). *Let $\mathbb{F}$ be an arbitrary field. Then, the graph $O(T_2(\mathbb{F}))$ is disconnected and is a union of its subgraphs with the following sets of vertices:*

1. *the set*
$$V_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \,\middle|\, 0 \neq a \in \mathbb{F} \right\} \bigcup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \,\middle|\, 0 \neq b \in \mathbb{F} \right\};$$
2. *the set*
$$V_3 = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \,\middle|\, 0 \neq a \in \mathbb{F} \right\};$$
3. *for each $0 \neq \alpha \in \mathbb{F}$, the set*
$$V_{4,\alpha} = \left\{ \begin{bmatrix} c & c\alpha \\ 0 & 0 \end{bmatrix} \,\middle|\, 0 \neq c \in \mathbb{F} \right\} \bigcup \left\{ \begin{bmatrix} 0 & d \\ 0 & -d/\alpha \end{bmatrix} \,\middle|\, 0 \neq d \in \mathbb{F} \right\}.$$

The diameters of the connected components corresponding to all of the vertex sets V_1,$V_{4,\alpha}$ equal 1, if $\mathbb{F} = \mathbb{Z}_2$, and 2, if $|\mathbb{F}| > 2$. The connected component corresponding to the vertex set V_3 is a one-vertex graph of diameter 0, if $\mathbb{F} = \mathbb{Z}_2$; the diameter of this component equals 1, otherwise.

Lemma 3.11. *Let $\mathbb{F}$ be an arbitrary field and $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$. Using the Notation 3.1, consider the sets*

$$\mathcal{H} = \bigcup_{0 \neq \alpha \in \mathbb{F}} \left\{ \begin{bmatrix} \omega\alpha & b \\ 0 & \alpha \end{bmatrix} \,\middle|\, b \in \mathbb{F} \right\} \subset T_2(\mathbb{F}),$$

$$\mathcal{T} = \bigcup_{0 \neq \alpha \in \mathbb{F}} \left\{ \begin{bmatrix} \alpha & b \\ 0 & \omega\alpha \end{bmatrix} \,\middle|\, b \in \mathbb{F} \right\} \subset T_2(\mathbb{F})$$

and

$$V = \mathcal{H} \cup \mathcal{T} \cup V_3 \subset T_2(\mathbb{F}).$$

Then,
1. the directed subgraph $\Delta_\omega(V)$ of $\Delta_\omega(T_2(\mathbb{F}))$ is weakly connected with the underlying graph $U\Delta_\omega(V)$ of diameter 2;

2. the digraph $\Delta_\omega(V)$ contains the following 3 types of arcs: (T,U),(U,H), (U_1,U_2), where $H\in\mathcal{H}$, $T\in\mathcal{T}$, $U,U_1,U_2\in V_3$ are arbitrary;
3. the maximal distance between reachable vertices in $\Delta_\omega(V)$ is equal to 2;
4. the digraph $\Delta_\omega(V)$ consists of the following connected components: one-vertex components corresponding to each matrix in $\mathcal{H}\cup\mathcal{T}$ and the component V_3 of diameter 1.

Proof. By Definition 1.4 and Notation 3.1, the set V is a subset of the vertex set in the graph $\Delta_\omega(T_2(\mathbb{F}))$. Note that V_3 contains all nonzero nil-triangular matrices from $T_2(\mathbb{F})$ and these matrices are scalar multiples of the matrix $E_{12}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$.

Let us prove Item 2. By Propositions 3.3 and 3.8, there are no arcs with heads from the set $\mathcal{T}$ and for each vertex $T\in\mathcal{T}$ there exists an arc (T,N_T), where N_T is a nonzero nil-triangular matrix in $T_2(\mathbb{F})$. As shown above, $N_T=\nu E_{12}$, therefore any matrix from V_3 is a scalar multiple of N_T, hence there exist arcs (T,U) for all $U\in V_3$. Similarly, there are no arcs with tails from $\mathcal{H}$ and for each vertex $H\in\mathcal{H}$ there exists an arc (N_H,H), where N_H is a nonzero nil-triangular matrix in $T_2(\mathbb{F})$. Consequently, there exist arcs (U,H) for all $U\in V_3$. Since $E_{12}^2=O$, any two matrices $U_1,U_2\in V_3$ are orthogonal, hence they ω-commute and there is an arc (U_1,U_2).

Items 1 and 3 follow from Item 2. Indeed, consider a pair of vertices $A,B\in V$, $A\neq B$. If $A,B\in V_3$, then there are arcs $A\to B$, $B\to A$, the vertices are strongly connected and $\mathrm{d}\,(A,B)=1$. If $A\in\mathcal{T},B\in\mathcal{H}$, then there exists a directed path $A\to E_{11}\to B$, but $AB\neq\omega BA$, thus $\mathrm{d}\,(A,B)=2$. If both $A,B\in\mathcal{T}$, then two arcs $A\to E_{12}$ and $B\to E_{12}$ provide a path from A to B in $U\Delta_\omega(V)$. Similarly, if both $A,B\in\mathcal{H}$, then two arcs $E_{12}\to A$ and $E_{12}\to B$ provide a path from A to B in $U\Delta_\omega(V)$.

Let us prove Item 4. By Item 2, the subgraph $\Delta_\omega(V_3)$ is strongly connected of diameter at most 1. The condition on the coefficient ω implies that $|\mathbb{F}|\geq 4$; hence, there exist at least two distinct matrices in V_3 and $\operatorname{diam}\Delta_\omega(V_3)=1$. Since there are no arcs with heads from the set $\mathcal{T}$, thus there are no directed paths ending in $T\in\mathcal{T}$, and a vertex $T\in\mathcal{T}$ cannot be contained in a connected component, which contains any other vertex. Analogously, there are no arcs with tails from the set $\mathcal{H}$; thus, there are no directed paths starting in $H\in\mathcal{H}$, and a vertex $H\in\mathcal{H}$ cannot be contained in a connected component, which contains any other vertex. ☐

THEOREM 3.12. *Let $\mathbb{F}$ be an arbitrary field. Let $\omega\in\mathbb{F}\setminus\{0,\pm1\}$. Then,*
1. the underlying graph $U\Delta_\omega\left(T_2\left(\mathbb{F}\right)\right)$ is disconnected and is a union of its connected subgraphs with the following sets of vertices:

- *the set*

$$V_1=\left\{\begin{bmatrix}a&0\\0&0\end{bmatrix}\Big|\,0\neq a\in\mathbb{F}\right\}\bigcup\left\{\begin{bmatrix}0&0\\0&b\end{bmatrix}\Big|\,0\neq b\in\mathbb{F}\right\};$$

- *for each $0\neq\alpha\in\mathbb{F}$, the set*

$$V_{2,\alpha}=\left\{\begin{bmatrix}c&c\alpha\\0&0\end{bmatrix}\Big|\,0\neq c\in\mathbb{F}\right\}\bigcup\left\{\begin{bmatrix}0&d\\0&-d/\alpha\end{bmatrix}\Big|\,0\neq d\in\mathbb{F}\right\};$$

- *$V=\mathcal{H}\cup\mathcal{T}\cup V_3$, where*

$$\mathcal{H}=\bigcup_{0\neq\alpha\in\mathbb{F}}\left\{\begin{bmatrix}\omega\alpha&b\\0&\alpha\end{bmatrix}\Big|\,b\in\mathbb{F}\right\},$$

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$$\mathcal{T} = \bigcup_{0\neq\alpha\in\mathbb{F}} \left\{ \begin{bmatrix} \alpha & b \\ 0 & \omega\alpha \end{bmatrix} \middle| \, b \in \mathbb{F} \right\},$$

$$V_3 = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| \, 0 \neq a \in \mathbb{F} \right\}.$$

2. The diameters of the connected components of the graph $U\Delta_\omega(T_2(\mathbb{F}))$ *corresponding to all of the vertex sets* V_1,$V_{2,\alpha}$, V *are equal to* 2.
3. The digraph $\Delta_\omega(V)$ *consists of the following connected components: one-vertex components corresponding to each matrix in* $\mathcal{H}\cup\mathcal{T}$ *and the components* V_1,$V_{2,\alpha}$ *of diameter* 2 *and* V_3 *of diameter* 1.

Proof. 1. Notice that by definition the union of the sets V_1, $V_{2,\alpha}$, for all $0\neq\alpha\in\mathbb{F}$ and V_3 consists of all singular nonzero matrices from $T_2(\mathbb{F})$. Lemma 3.10 implies that they are disjoint. The sets $\mathcal{H}$ and $\mathcal{T}$ are disjoint by construction due to the condition $\omega\neq\pm1$ and contain non-singular matrices. All non-singular matrices that are vertices of $\Delta_\omega(T_2(\mathbb{F}))$ are contained in $\mathcal{H}\cup\mathcal{T}$ by Notation 3.1 and Proposition 3.2. Therefore, the disjoint union $V_1 \sqcup \bigcup_{0\neq\alpha\in\mathbb{F}} V_{2,\alpha} \sqcup V_3 \sqcup \mathcal{H} \sqcup \mathcal{T}$ is exactly the vertex set of the digraph $\Delta_\omega(T_2(\mathbb{F}))$ and its underlying graph $U\Delta_\omega(T_2(\mathbb{F}))$.

The Jordan normal form of any matrix from V_1 and $V_{2,\alpha}, 0\neq\alpha\in\mathbb{F}$, is a diagonal matrix of the form $D = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$ for some $\delta\neq0$, $\delta\in\mathbb{F}$. Thus, by Proposition 2.17, we obtain that $\mathcal{C}^\omega(D) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix} \middle| \, \varepsilon\in\mathbb{F} \right\}$, $\dim\mathcal{C}^\omega(D) = 1$. In particular, it implies that $\mathcal{C}^\omega(D) = \mathcal{C}^{\omega^{-1}}(D) = O(D)$. Analogously, by Proposition 2.17 for any nonzero $\varepsilon\in\mathbb{F}$ one has $\mathcal{C}^\omega\left(\begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix}\right) = \langle D\rangle$. Consequently, the directed subgraph $\Delta_\omega(V_1)$ coincides with the symmetric digraph $DO(V_1)$ and is strongly connected. The digraph $\Delta_\omega(T_2(\mathbb{F}))$ has no arcs with heads (or tails) in V_1 and tails (or heads) in $\bigcup_{0\neq\alpha\in\mathbb{F}} V_{2,\alpha} \sqcup V_3 \sqcup \mathcal{H} \sqcup \mathcal{T}$. Thus, the subgraph $U\Delta_\omega(V_1)$ is a connected component of the graph $U\Delta_\omega(T_2(\mathbb{F}))$. For each fixed $0\neq\alpha\in\mathbb{F}$, the conjugation by invertible matrix $P_\alpha = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \in T_2(\mathbb{F})$ maps $\begin{bmatrix} c & c\alpha \\ 0 & 0 \end{bmatrix} \in V_{2,\alpha}$ to $\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$, and maps $\begin{bmatrix} 0 & d \\ 0 & -d/\alpha \end{bmatrix}$ to $\begin{bmatrix} 0 & 0 \\ 0 & -d/\alpha \end{bmatrix}$, hence $P_\alpha^{-1} V_{2,\alpha} P_\alpha = V_1$. As an automorphism of the algebra $T_2(\mathbb{F})$ conjugation by P_α preserves orthogonality and ω-commutativity relations. Therefore, for any matrix $A\in V_{2,\alpha}$, it is also true that $\mathcal{C}^\omega(A) = \mathcal{C}^{\omega^{-1}}(A) = O(A)$. Therefore, all reasonings that worked for $\Delta_\omega(V_1)$ are also valid for $\Delta_\omega(V_{2,\alpha})$.

By Propositions 3.3–3.5 and Lemma 3.11, all arcs with tails in $\mathcal{T}$ have heads in V_3, and all arcs with heads in $\mathcal{H}$ have tails in V_3. Applying Proposition 2.17 to the Jordan matrix $E_{12}\in V_3$, we have

$$\mathcal{C}^\omega(E_{12}) = \left\{ \begin{bmatrix} u & v \\ 0 & \omega u \end{bmatrix} \middle| \, u,v\in\mathbb{F} \right\},$$

$$\mathcal{C}^{\omega^{-1}}(E_{12}) = \left\{ \begin{bmatrix} \omega u & v \\ 0 & u \end{bmatrix} \middle| \, u,v\in\mathbb{F} \right\},$$

and by linearity of the ω-commutativity relation $\mathcal{C}^\omega(\beta E_{12}) = \mathcal{C}^\omega(E_{12})$, $\mathcal{C}^{\omega^{-1}}(\beta E_{12}) = \mathcal{C}^{\omega^{-1}}(E_{12})$ for all $\beta\neq0$. Consequently, all nonzero matrices from $\mathcal{C}^\omega(N)$ are contained in $\mathcal{H}\sqcup V_3$, and all nonzero matrices from $\mathcal{C}^{\omega^{-1}}(N)$ are contained in $\mathcal{T}\sqcup V_3$ for all $N\in V_3$. The subgraph $U\Delta_\omega(V)$ is connected by Lemma 3.11. Therefore, the set V is a connected component of the graph $U\Delta_\omega(T_2(\mathbb{F}))$.

2. The condition on the coefficient ω implies that $|\mathbb{F}|\geq4$, hence $\operatorname{diam} DO(V_1) = \operatorname{diam} U\Delta_\omega(V_1) = \operatorname{diam} DO(V_{2,\alpha}) = \operatorname{diam} U\Delta_\omega(V_{2,\alpha}) = 2$. The equation $\operatorname{diam} U\Delta_\omega(V) = 2$ holds by Lemma 3.11.

Arguments from the proof of Items 1 and 2 and Lemma 3.11 imply Item 3. ◻

3.3. General case: matrices of order $n \geq 3$. Let us recall some technique from [10] that is useful for the ω-commuting digraph. The main result is that the orthogonality graph $O(T_n(\mathbb{F}))$ is connected.

THEOREM 3.13 ([10, Theorem 2.8]). *Let $\mathbb{F}$ be an arbitrary field and $n \geq 3$. Then, the graph $O(T_n(\mathbb{F}))$ is connected and has diameter 4.*

COROLLARY 3.14. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F}$, $\omega \neq 0$ and $n \geq 3$. Then, the directed subgraph $DO(T_n(\mathbb{F}))$ of the digraph $\Delta_\omega(T_n(\mathbb{F}))$ is strongly connected and has diameter 4.*

DEFINITION 3.15 ([10, Definition 2.4]). Let $\mathbb{F}$ be an arbitrary field and $n \geq 3$. A singular matrix $A \in T_n(\mathbb{F})$ is said to be *bad* if has only one zero diagonal entry, which is located either at position $(1,1)$ or (n,n). Otherwise, a singular matrix $A \in T_n(\mathbb{F})$ is said to be *good*.

COROLLARY 3.16. *Let $\mathbb{F}$ be an arbitrary field and $n \geq 3$. If $A, B \in T_n(\mathbb{F})$, A is a nonzero nilpotent matrix, B is a good matrix, then $\mathrm{d}\,(A,B) \leq 3$ in $O(T_n(\mathbb{F}))$.*

Proof. Follows from Item 1 of the proof [10, Theorem 2.8] since a nilpotent matrix is also good. ◻

LEMMA 3.17 ([10, Lemma 2.3]). *Let $\mathbb{F}$ be an arbitrary field and $n \geq 3$. Consider bad matrices $A, B \in T_n(\mathbb{F})$ in the following block forms:*

$$A = \begin{bmatrix} 0 & \bar{a} \\ O_{(n-1)\times 1} & A_1 \end{bmatrix}, B = \begin{bmatrix} B_1 & \bar{b} \\ O_{1\times(n-1)} & 0 \end{bmatrix},$$

where $A_1, B_1 \in T_{n-1}(\mathbb{F})$ are invertible matrices, $\bar{a}$ is a row, $\bar{b}$ is a column over $\mathbb{F}$ of length $n-1$. Then,

$$O_{M_n(\mathbb{F})}(A) \cap T_n(\mathbb{F}) = \left\{ \begin{bmatrix} c_0 & \bar{c} \\ O_{(n-1)\times 1} & O_{n-1} \end{bmatrix} \;\middle|\; c_0 \in \mathbb{F}, \bar{c} = -c_0 \bar{a} A_1^{-1} \right\},$$

$$O_{M_n(\mathbb{F})}(B) \cap T_n(\mathbb{F}) = \left\{ \begin{bmatrix} O_{n-1} & \bar{d} \\ O_{1\times(n-1)} & d_0 \end{bmatrix} \;\middle|\; d_0 \in \mathbb{F}, \bar{d} = -d_0 B_1^{-1} \bar{b} \right\}.$$

PROPOSITION 3.18 ([9, Corollary 5.10]). *Let $\mathbb{F}$ be a field and $n \geq 3$. If NT_n is the subalgebra of nil-triangular matrices in $M_n(\mathbb{F})$, then* $\operatorname{diam} O(NT_n) = 2$.

COROLLARY 3.19. *Let $\mathbb{F}$ be an arbitrary field, $\omega \in \mathbb{F}$, $\omega \neq 0$ and $n \geq 3$. Then, the directed subgraph $DO(NT_n)$ of the digraph $\Delta_\omega(T_n(\mathbb{F}))$ is strongly connected and has diameter 2.*

THEOREM 3.20. *Let $\mathbb{F}$ be an arbitrary field, $n \geq 3$ and $\omega \in \mathbb{F} \setminus \{0, \pm 1\}$. Consider the ω-commuting digraph $\Delta_\omega(T_n(\mathbb{F}))$ and the disjoint partition of its vertex set V, $V = \mathcal{H} \sqcup \mathcal{B} \sqcup \mathcal{T} \sqcup \mathcal{S}$, where the sets $\mathcal{H}, \mathcal{B}, \mathcal{T}$ defined in Notation 3.1 contain invertible matrices, the set $\mathcal{S}$ contains all nonzero singular matrices. Then,*
1. the digraph $\Delta_\omega(T_n(\mathbb{F}))$ is weakly connected with the underlying undirected graph having diameter 4;
2. the directed subgraph $\Delta_\omega(\mathcal{B} \cup \mathcal{S})$ is a strongly connected digraph of diameter 4;
3. the digraph $\Delta_\omega(T_n(\mathbb{F}))$ consists of the following connected components: one-vertex components corresponding to each matrix in $\mathcal{H} \cup \mathcal{T}$ and the component $\Delta_\omega(\mathcal{B} \cup \mathcal{S})$.

Proof. 1. Let us first demonstrate that any two matrices A, B are connected in the underlying graph $U\Delta_\omega(T_n(\mathbb{F}))$ by a path of length at most 4.

i. If both matrices A and B are singular, they are also vertices in the directed orthogonality graph $DO(T_n(\mathbb{F})) \subset \Delta_\omega(T_n(\mathbb{F}))$ and the result follows from Corollary 3.14. In this case, the vertices A and B are also strongly connected in the digraph $\Delta_\omega(T_n(\mathbb{F}))$.

ii. Suppose $A \in \mathcal{T} \cup \mathcal{B}$, $B \in \mathcal{B} \cup \mathcal{H}$. In this case, by Proposition 3.3 and Corollary 3.7, there exist nilpotent matrices $M, N \in \mathcal{S}$ for which there are arcs (A, M) and (N, B). By Corollary 3.19 $\mathrm{d}\,(M, N) \leq 2$ in the directed subgraph $DO(T_n(\mathbb{F}))$, with a concrete directed path $M \to E_{1n} \to N$. Therefore, one has a directed path $A \to M \to E_{1n} \to N \to B$ of length 4 and $\mathrm{d}\,(A, B) \leq 4$ in $\Delta_\omega(T_n(\mathbb{F}))$.

iii. Suppose $A, B \in \mathcal{T}$. In this case by Propositions 3.3 and 3.5, there exist nilpotent matrices $M, N \in \mathcal{S}$ for which there are arcs (A, M) and (B, N). By Corollary 3.19 $\mathrm{d}\,(M, N) \leq 2$ in the directed subgraph $DO(T_n(\mathbb{F}))$, with a concrete directed path $M \to E_{1n} \to N$. Therefore, after deleting arrows in a directed path $A \to M \to E_{1n} \to N$ and in an arc $B \to N$, we obtain a path $A - M - E_{1n} - N - B$ of length 4 in $U\Delta_\omega(T_n(\mathbb{F}))$.

iv. Similarly suppose $A, B \in \mathcal{H}$. In this case by Proposition 3.3 and Corollary 3.7, there exist nilpotent matrices $M, N \in \mathcal{S}$ for which there are arcs (M, A) and (N, B). By Corollary 3.19 $\mathrm{d}\,(M, N) \leq 2$ in the directed subgraph $DO(T_n(\mathbb{F}))$, with a concrete directed path $M \to E_{1n} \to N$. Therefore after deleting arrows in a directed path $M \to E_{1n} \to N \to B$ and in an arc $M \to A$, we obtain a path $A - M - E_{1n} - N - B$ of length 4 in $U\Delta_\omega(T_n(\mathbb{F}))$.

v. Suppose $A \in \mathcal{T} \cup \mathcal{B}$, $B \in \mathcal{S}$. In this case by Propositions 3.3 and 3.5, there exists a nilpotent matrix $M \in \mathcal{S}$ for which there is an arc (A, M). The matrices M and B are strongly connected, so A and B are connected in $\Delta_\omega(T_n(\mathbb{F}))$. It remains to evaluate the distance $\mathrm{d}\,(A, B)$. If B is a good matrix, then $\mathrm{d}\,(M, B) \leq 3$ in $DO(T_n(\mathbb{F}))$ by Corollary 3.16. Therefore, $\mathrm{d}\,(A, B) \leq 1 + \mathrm{d}\,(M, B) \leq 4$. Assume further that B is a bad matrix. Following the proof of Proposition 3.5, it is possible to choose the matrix M with a block structure $M = \begin{bmatrix} O_{i,j-1} & M' \\ O & O_{n-i,n-j+1} \end{bmatrix}$, for some $1 \leq i < j \leq n$, $M' \neq O$. Clearly, the matrix M' here cannot be an invertible $(n-1) \times (n-1)$ matrix since for $i = n-1$ the only possibility for j is n and in this case M' has one column. Therefore, the matrix M admits the following block form $M = \begin{bmatrix} O_{(n-1)\times 1} & M'' \\ 0 & O_{1\times(n-1)} \end{bmatrix}$, where M'' is a singular matrix in $M_{n-1}(\mathbb{F})$.

- Let $b_{11} = 0$, that is, $B = \begin{bmatrix} 0 & \bar{b} \\ O_{(n-1)\times 1} & B_1 \end{bmatrix}$. By Lemma 3.17, there exists a matrix $C = \begin{bmatrix} 1 & \bar{c} \\ O_{(n-1)\times 1} & O_{n-1} \end{bmatrix}$, $\bar{c} = -\bar{b}B_1^{-1}$ orthogonal to B. Then adopting the reasoning from the proof of [10, Theorem 2.8], consider a nonzero column $\bar{x} \in \mathbb{F}^{n-1}$ such that $M''\bar{x} = 0$ and define $X = \begin{bmatrix} O_{1,n-1} & x' \\ O_{n-1} & \bar{x} \end{bmatrix} \in T_n(\mathbb{F})$, where $x' = -\bar{c}\bar{x}$. Then by construction $CX = XC = O$ and $MX = XM = O$, therefore we have a path $A \to M \to X \to C \to B$ of length 4 in $\Delta_\omega(T_n(\mathbb{F}))$.
- Let $b_{nn} = 0$, that is $B = \begin{bmatrix} B_1 & \bar{b} \\ O_{1\times(n-1)} & 0 \end{bmatrix}$. By Lemma 3.17, there exists a matrix $C = \begin{bmatrix} O_{n-1} & \bar{c} \\ O_{1\times(n-1)} & 1 \end{bmatrix}$, $\bar{c} = -B_1^{-1}\bar{b}$ orthogonal to B. Since M'' is singular, there exists a nonzero row $\bar{y} \in \mathbb{F}^{n-1}$ such that $\bar{y}M'' = 0$ and define $Y = \begin{bmatrix} \bar{y} & y' \\ O_{n-1} & O_{(n-1)\times 1} \end{bmatrix}$, where $y' = -\bar{y}\bar{c}$. Then by construction $CY = YC = O$ and $MY = YM = O$; therefore, we have a path $A \to M \to Y \to C \to B$ of length 4 in $\Delta_\omega(T_n(\mathbb{F}))$.

vi. Suppose $A \in \mathcal{S}$, $B \in \mathcal{B} \cup \mathcal{H}$. In this case by Propositions 3.3 and 3.5, there exists a nilpotent matrix $N \in \mathcal{S}$ for which there is an arc (N, B). The matrices A and N are strongly connected, so A and B are connected in $\Delta_\omega(T_n(\mathbb{F}))$. It remains to evaluate the distance $\mathrm{d}\,(A, B)$. If A is a good matrix, then $\mathrm{d}\,(A, N) \leq 3$ in $DO(T_n(\mathbb{F}))$ by Corollary 3.16. Therefore, $\mathrm{d}\,(A, B) \leq 4$. Assume further that A is a bad matrix. Following the proof of Proposition 3.5 for the coefficient ω^{-1}, it is possible to choose the matrix N with a block structure $M = \begin{bmatrix} O_{i,j-1} & N' \\ O & O_{n-i,n-j+1} \end{bmatrix}$, for some $1 \leq i < j \leq n$, $N' \neq O$. As in the previous item, the matrix N' here cannot be an invertible $(n-1) \times (n-1)$ matrix. Therefore arguing for the matrices N, A as above, $\mathrm{d}\,(N, A) \leq 3$ in the directed orthogonality graph $DO(T_n(\mathbb{F}))$. Consequently, $\mathrm{d}\,(A, B) \leq \mathrm{d}\,(A, N) + 1 \leq 4$ in $\Delta_\omega(T_n(\mathbb{F}))$.

It remains to provide a pair of vertices $A, B \in V$ for which $\mathrm{d}\,(A, B) = 4$ in $U\Delta_\omega(T_n(\mathbb{F}))$. We give an example with two matrices $A, B \in \mathcal{S}$, so that it also works for Item 2. Take $A = \operatorname{diag}(0, 1, \ldots, 1)$, $B = J_n(0)$. By Proposition 2.17, we have $\mathcal{C}^\omega(A) = \mathcal{C}^{\omega^{-1}}(A) = \{\alpha E_{11} | \alpha \in \mathbb{F}\}$,

$$\mathcal{C}^\omega(B) = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ 0 & \omega x_1 & \omega x_2 & \ldots & \omega x_{n-1} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & & \omega^{n-2} x_1 & \omega^{n-2} x_2 \\ 0 & 0 & 0 & \ldots & \omega^{n-1} x_1 \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{F} \right\},$$

$$\mathcal{C}^{\omega^{-1}}(B) = \left\{ \begin{bmatrix} y_1 & y_2 & y_3 & \ldots & y_n \\ 0 & \omega^{-1} y_1 & \omega^{-1} y_2 & \ldots & \omega^{-1} y_{n-1} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & & \omega^{2-n} y_1 & \omega^{2-n} y_2 \\ 0 & 0 & 0 & \ldots & \omega^{1-n} y_1 \end{bmatrix} : y_1, \ldots, y_n \in \mathbb{F} \right\}.$$

If $Y \in \mathcal{C}^{\omega^{-1}}(B)$, then

$$E_{11} Y = \begin{bmatrix} y_1 & y_2 & y_3 & \ldots & y_n \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$Y E_{11} = y_1 E_{11}$. Consequently, the equation $E_{11} Y = \omega^{\pm 1} Y E_{11}$ holds if and only if $Y = O$. Similarly, for $X \in \mathcal{C}^\omega(B)$, the equation $E_{11} X = \omega^{\pm 1} X E_{11}$ holds if and only if $X = O$. That is, there is no directed path of the form $A \to \alpha E_{11} \to Y \to B$ and $\mathrm{d}\,(A, B) > 3$. Combined with Item i, this implies that $\mathrm{d}\,(A, B) = 4$ in $\Delta_\omega(T_n(\mathbb{F}))$. Moreover, this relations imply that there is no path of length 3 in $U\Delta_\omega(T_n(\mathbb{F}))$ obtained by removing arrows from the arcs of $\Delta_\omega(T_n(\mathbb{F}))$.

2. Follows from Item 1 (Items (i)–(ii), (v)–(vi) for the strong connectivity and upper bound of the diameter, example $A = \operatorname{diag}(0, 1, \ldots, 1)$, $B = J_n(0)$ for the sharpness of the diameter bound).

3. Follows from Items 1–2, since for vertices from $\mathcal{T} \cup \mathcal{H}$ the reasoning from proof of Lemma 3.11 is applicable. ☐

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