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SOME ADDITIVE RESULTS FOR THE GENERALIZED DRAZIN INVERSE IN A BANACH ALGEBRA*

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Abstract. In this note, additive results are presented for the generalized Drazin inverse in Banach algebra. Necessary and sufficient conditions are given for the generalized Drazin invertibility of the sum of two commuting generalized Drazin invertible elements. These results are a generalization of the results from the paper [C.Y. Deng and Y. Wei. New additive results for the generalized Drazin inverse. *J. Math. Anal. Appl.*, 370:313–321, 2010.] to the Banach algebra case.

Key words. Banach algebra, Generalized Drazin inverse, Additive properties.

AMS subject classifications. 15A09.

1. Introduction. Let $\mathcal{A}$ be a complex Banach algebra with the unit 1. By $\mathcal{A}^{-1}$, $\mathcal{A}^{\mathsf{nil}}$ and $\mathcal{A}^{\mathsf{qnil}}$, we denote the sets of all invertible, nilpotent, and quasi-nilpotent elements in $\mathcal{A}$, respectively. By $\sigma(a)$ we denote the spectrum of the element $a \in \mathcal{A}$ and by $\mathrm{acc}(\sigma(\mathrm{a}))$ we denote the set of all accumulation points of $\sigma(a)$. If $\mathcal{B}$ is a sub-algebra of $\mathcal{A}$ and $a \in \mathcal{B}$, then by $\sigma_{\mathcal{B}}(a)$ we denote the spectrum of the element a in the sub-algebra $\mathcal{A}$. Let us recall that the Drazin inverse of $a \in \mathcal{A}$ [4], is the element $x \in \mathcal{A}$ (denoted by a^{D}) which satisfies

$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k, \tag{1.1}$$

for some nonnegative integer k. The least such k is the index of a, denoted by $ind(a)$. When $ind(a) = 1$, then the Drazin inverse a^{D} is called the group inverse and is denoted by a^g or $a^{\#}$.

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The conditions from (1.1) are equivalent to

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathsf{nil}}. \tag{1.2}$$

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [8]. The condition $a - a^2x \in \mathcal{A}^{\mathsf{nil}}$ from (1.2) was replaced by the condition $a - a^2x \in \mathcal{A}^{\mathsf{qnil}}$. Hence, the generalized Drazin inverse of a is the element $x \in \mathcal{A}$ (written as a^{d}) which satisfies

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathsf{qnil}}.$$

The set $\mathcal{A}^{\mathsf{d}}$ consists of all $a \in \mathcal{A}$ such that a^{d} exists. The Drazin index of a, ind(a), is the nilpotency index of $a - a^2b$ if $a - a^2b \in \mathcal{A}^{\mathsf{nil}}$ and ind(a) $= \infty$, otherwise. It is well-known that for $a \in \mathcal{A}$, a^{d} exists if and only if $0 \notin \mathrm{acc}(\sigma(\mathrm{a}))$ and in that case a^{d} is unique [8].

An alternative definition of the generalized Drazin inverse in a normed algebra and a ring is also given in [5, 6, 7]. For interesting properties of the generalized Drazin inverse see [8].

Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be an idempotent ($p = p^2$). Then we write

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p),$$

and use the notations

$$a_{11} = pap, \quad a_{12} = pa(1-p), \quad a_{21} = (1-p)ap, \quad a_{22} = (1-p)a(1-p).$$

Every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p.$$

Let a^π be the spectral idempotent of a corresponding to $\{0\}$. It is well-known that $a \in \mathcal{A}^{\mathsf{d}}$ can be represented in the following matrix form,

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p, \tag{1.3}$$

relative to $p = aa^{\mathsf{d}} = 1 - a^\pi$, where a_{11} is invertible in the algebra $p\mathcal{A}p$ and a_{22} is quasinilpotent element of the algebra $(1-p)\mathcal{A}(1-p)$. Using that representation, the Drazin inverse of a is presented by

$$a^{\mathsf{d}} = \begin{bmatrix} (a_{11})^{-1}_{p\mathcal{A}p} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

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where $(a_{11})^{-1}_{p\mathcal{A}p}$ is the inverse of a_{11} in sub-algebra $p\mathcal{A}p$.

In this note, we generalize the results from [2] to the Banach algebra case.

First we state the following result which is proved in [9] for matrices, and extended in [3] for bounded linear operators and in [1] for arbitrary elements of a Banach algebra.

THEOREM 1.1. *Let $x, y \in \mathcal{A}$ and*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{(1-p)}$$

relative to the idempotent $p \in \mathcal{A}$.

(1) *If $a \in (p\mathcal{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathcal{A}(1-p))^{\mathsf{d}}$, then x and y are generalized Drazin invertible and*

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{bmatrix}_p, \quad y^{\mathsf{d}} = \begin{bmatrix} b^{\mathsf{d}} & 0 \\ u & a^{\mathsf{d}} \end{bmatrix}_{(1-p)} \tag{1.4}$$

where $u = \sum\limits_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} c b^n b^{\pi} + \sum\limits_{n=0}^{\infty} a^{\pi} a^n c (b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}} c b^{\mathsf{d}}$.

(2) *If $x \in \mathcal{A}^{\mathsf{d}}$ and $a \in (p\mathcal{A}p)^{\mathsf{d}}$, then $b \in ((1-p)\mathcal{A}(1-p))^{\mathsf{d}}$ and $x^{\mathsf{d}}, y^{\mathsf{d}}$ are given by (1.4).*

2. Results. In this section, we present our main results.

THEOREM 2.1. *Let $a, b \in \mathcal{A}^{\mathsf{d}}$ and $ab = ba$. Then $a + b \in \mathcal{A}^{\mathsf{d}}$ if and only if $1 + a^{\mathsf{d}} b \in \mathcal{A}^{\mathsf{d}}$. In this case, we have*

$$\begin{aligned}(a+b)^{\mathsf{d}} = {} & a^{\mathsf{d}}(1 + a^{\mathsf{d}} b)^{\mathsf{d}} b b^{\mathsf{d}} + (1 - b b^{\mathsf{d}}) \left[\sum_{n=0}^{\infty} (-b)^n (a^{\mathsf{d}})^n \right] a^{\mathsf{d}} \\ & + b^{\mathsf{d}} \left[\sum_{n=0}^{\infty} (b^{\mathsf{d}})^n (-a)^n \right] a^{\pi},\end{aligned}$$

$$\begin{aligned}\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| \le {} & \|bb^{\mathsf{d}}\| \|a^{\mathsf{d}}\| \left[\|(1 + a^{\mathsf{d}} b)^{\mathsf{d}}\| + 1 \right] + \|(1 - bb^{\mathsf{d}})\| \left[\sum_{n=1}^{\infty} \|(-b)^n (a^{\mathsf{d}})^n\| \right] \|a^{\mathsf{d}}\| \\ & + \|b^{\mathsf{d}}\| \left[\sum_{n=0}^{\infty} \|(b^{\mathsf{d}})^n (-a)^n\| \right] \|a^{\pi}\|,\end{aligned}$$

and

$$(a+b)(a+b)^{\mathsf{d}} = (aa^{\mathsf{d}} + ba^{\mathsf{d}})(1 + a^{\mathsf{d}} b)^{\mathsf{d}} b b^{\mathsf{d}} + (1 - bb^{\mathsf{d}}) a a^{\mathsf{d}} + b b^{\mathsf{d}} a^{\pi},$$

$$\|(a+b)(a+b)^{\mathsf{d}} - aa^{\mathsf{d}}\| \le \left[\|aa^{\mathsf{d}} + ba^{\mathsf{d}}\| \|(1 + a^{\mathsf{d}} b)^{\mathsf{d}}\| + \|1 - 2aa^{\mathsf{d}}\| \right] \|bb^{\mathsf{d}}\|.$$

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Proof. Since a is generalized Drazin invertible, we will suppose that a is given by (1.3). Let $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p$.

From $ab = ba$, we get that $b_{12} = (a_{11})^{-1}_{p\mathcal{A}p} b_{12} a_{22}$ which implies that $b_{12} = (a_{11})^{-n}_{p\mathcal{A}p} b_{12} a_{22}^n$, for arbitrary $n \in \mathbb{N}$. Since a_{22} is a quasi-nilpotent, we get that $b_{12} = 0$. Similarly, by $ab = ba$ it follows that $b_{21} = a_{22} b_{21} (a_{11})^{-1}_{p\mathcal{A}p}$, i.e., $b_{21} = 0$. Also, $a_{11} b_{11} = b_{11} a_{11}$ and $a_{22} b_{22} = b_{22} a_{22}$.

Since, $b \in \mathcal{A}^{\mathsf{d}}$ and $\sigma(b) = \sigma(b_{11})_{p\mathcal{A}p} \cup \sigma(b_{22})_{(1-p)\mathcal{A}(1-p)}$, using the Theorem 4.2 from the famous paper of Koliha [8], we get that $b_{11} \in p\mathcal{A}p$ and $b_{22} \in (1-p)\mathcal{A}(1-p)$, so $b_{11}, b_{22} \in \mathcal{A}^{\mathsf{d}}$ and we can represent b_{11} and b_{22} as

$$b_{11} = \begin{bmatrix} b'_{11} & 0 \\ 0 & b'_{22} \end{bmatrix}_{p_1} \quad \text{and} \quad b_{22} = \begin{bmatrix} b''_{11} & 0 \\ 0 & b''_{22} \end{bmatrix}_{p_2},$$

where $p_1 = b_{11} b_{11}^{\mathsf{d}}$, $p_2 = b_{22} b_{22}^{\mathsf{d}}$, b'_{11}, b''_{11} are invertible in the algebras $p_1\mathcal{A}p_1$ and $p_2\mathcal{A}p_2$ respectively, and b'_{22}, b''_{22} are quasi-nilpotent. Since b_{11} commutes with an invertible a_{11} and b_{22} commutes with quasi-nilpotent a_{22}, as before we prove that

$$a_{11} = \begin{bmatrix} a'_{11} & 0 \\ 0 & a'_{22} \end{bmatrix}_{p_1} \quad \text{and} \quad a_{22} = \begin{bmatrix} a''_{11} & 0 \\ 0 & a''_{22} \end{bmatrix}_{p_2}.$$

Since $p_1 p = p p_1 = p_1$, from the fact that a_{11} is invertible in the subalgebra $p\mathcal{A}p$ we get that a'_{11} and a'_{22} are invertible in the algebras $p_1\mathcal{A}p_1$ and $(p-p_1)\mathcal{A}(p-p_1)$ respectively. Also, a''_{11} and a''_{22} are quasi-nilpotent and we have that a'_{ii} commutes with b'_{ii} and a''_{ii} commutes with b''_{ii}, for $i = 1, 2$.

Since a'_{22} is invertible and b'_{22} is quasi-nilpotent and they commute, we have that $(a'_{22})^{-1}_{(p-p_1)\mathcal{A}(p-p_1)} b'_{22}$ is quasi-nilpotent, so $(p-p_1) + (a'_{22})^{-1}_{(p-p_1)\mathcal{A}(p-p_1)} b'_{22}$ is invertible in $(p-p_1)\mathcal{A}(p-p_1)$ and $a'_{22} + b'_{22} \in \mathcal{A}^{\mathsf{d}}$.

Similarly, we conclude that $a''_{11} + b''_{11} \in \mathcal{A}^{\mathsf{d}}$.

Also, $a''_{22} + b''_{22}$ is generalized Drazin invertible.

Now, we obtain that

$$a + b = a'_{11} + b'_{11} + a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22}.$$

Since, $a'_{11} + b'_{11} \in p_1\mathcal{A}p_1$ and $b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22} \in (p-p_1)\mathcal{A}(p-p_1)$ we have that

$$a + b \in \mathcal{A}^{\mathsf{d}} \Leftrightarrow \left(a'_{11} + b'_{11} \in \mathcal{A}^{\mathsf{d}} \ \text{ and } \ a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22} \in \mathcal{A}^{\mathsf{d}} \right).$$

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Firstly, we will consider the generalized Drazin invertibility of $y = a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22}$. From $p_2 y p_2 = a''_{11} + b''_{11}$ and $(1-p_2)y(1-p_2) = a'_{22} + b'_{22} + a''_{22} + b''_{22}$, we conclude that

$$y \in \mathcal{A}^{\mathsf{d}} \Leftrightarrow \left(a''_{11} + b''_{11} \in \mathcal{A}^{\mathsf{d}} \ \text{ and } \ a'_{22} + b'_{22} + a''_{22} + b''_{22} \in \mathcal{A}^{\mathsf{d}} \right).$$

Before, we show that $a''_{11} + b''_{11} \in \mathcal{A}^{\mathsf{d}}$, so $y \in \mathcal{A}^{\mathsf{d}}$ if and only if $z = a'_{22} + b'_{22} + a''_{22} + b''_{22} \in \mathcal{A}^{\mathsf{d}}$. Notice that $z = pzp + (1-p)z(1-p)$, where $pzp = a'_{22} + b'_{22} \in \mathcal{A}^{\mathsf{d}}$ and $(1-p)z(1-p) = a''_{22} + b''_{22} \in \mathcal{A}^{\mathsf{d}}$, so $z \in \mathcal{A}^{\mathsf{d}}$. Hence, $y \in \mathcal{A}^{\mathsf{d}}$ and we get that $a+b \in \mathcal{A}^{\mathsf{d}}$ if and only if $a'_{11} + b'_{11} \in \mathcal{A}^{\mathsf{d}}$.

Now,

$$(a'_{11} + b'_{11})^{\mathsf{d}} = a'_{11}(p_1 + (a'_{11})^{-1}_{p_1\mathcal{A}p_1} b'_{11})^{\mathsf{d}} = p_1 p a^{\mathsf{d}}(1 + a^{\mathsf{d}}b)^{\mathsf{d}} b b^{\mathsf{d}} p p_1.$$

By the first equation, we easily obtain

$$\begin{aligned} (a+b)^{\mathsf{d}} - a^{\mathsf{d}} &= a^{\mathsf{d}}(1 + a^{\mathsf{d}}b)^{\mathsf{d}} b b^{\mathsf{d}} + (1 - bb^{\mathsf{d}}) \left[\sum_{n=0}^{\infty} (-b)^n (a^{\mathsf{d}})^n \right] a^{\mathsf{d}} \\ &\quad + b^{\mathsf{d}} \left[\sum_{n=0}^{\infty} (b^{\mathsf{d}})^n (-a)^n \right] a^{\pi} - a^{\mathsf{d}} \\ &= a^{\mathsf{d}}(1 + a^{\mathsf{d}}b)^{\mathsf{d}} b b^{\mathsf{d}} - bb^{\mathsf{d}} a^{\mathsf{d}} + (1 - bb^{\mathsf{d}}) \left[\sum_{n=1}^{\infty} (-b)^n (a^{\mathsf{d}})^n \right] a^{\mathsf{d}} \\ &\quad + b^{\mathsf{d}} \left[\sum_{n=0}^{\infty} (b^{\mathsf{d}})^n (-a)^n \right] a^{\pi}. \end{aligned}$$

From above, we can estimate

$$\begin{aligned} \|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| &\le \|bb^{\mathsf{d}}\| \|a^{\mathsf{d}}\| \left[\|(1 + a^{\mathsf{d}}b)^{\mathsf{d}}\| + 1 \right] + \|(1 - bb^{\mathsf{d}})\| \left[\sum_{n=1}^{\infty} \|(-b)^n (a^{\mathsf{d}})^n\| \right] \|a^{\mathsf{d}}\| \\ &\quad + \|b^{\mathsf{d}}\| \left[\sum_{n=0}^{\infty} \|(b^{\mathsf{d}})^n (-a)^n\| \right] \|a^{\pi}\|, \end{aligned}$$

and

$$\begin{aligned} \|(a+b)(a+b)^{\mathsf{d}} - aa^{\mathsf{d}}\| &= \|(aa^{\mathsf{d}} + ba^{\mathsf{d}})(1 + a^{\mathsf{d}}b)^{\mathsf{d}} bb^{\mathsf{d}} - bb^{\mathsf{d}} aa^{\mathsf{d}} + bb^{\mathsf{d}} a^{\pi}\| \\ &\le \left[\|aa^{\mathsf{d}} + ba^{\mathsf{d}}\| \|(1 + a^{\mathsf{d}}b)^{\mathsf{d}}\| + \|1 - 2aa^{\mathsf{d}}\| \right] \|bb^{\mathsf{d}}\|. \quad \square \end{aligned}$$

Corollary 2.2. *Let $a, b \in \mathcal{A}^{\mathsf{d}}$ be such that $ab = ba$ and $1 + a^{\mathsf{d}}b \in \mathcal{A}^{\mathsf{d}}$.*

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(1) *If b is quasi-nilpotent, then*

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(a^{\mathsf{d}})^{n+1}(-b)^n = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}.$$

(2) *If $b^k = 0$, then $(a+b)^{\mathsf{d}} = \sum_{n=0}^{k-1}(a^{\mathsf{d}})^{n+1}(-b)^n = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}$.*

(3) *If $b^k = b$ $(k \geq 3)$, then $b^{\mathsf{d}} = b^{k-2}$ and*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b^{k-1} + (1-b^{k-1})a^{\mathsf{d}} \\ &\quad + b^{k-2}\left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^n(-a)^n\right]a^{\pi} \\ &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b^{k-1} + (1-b^{k-1})a^{\mathsf{d}} \\ &\quad + b^{k-2}(1+ab^{k-2})^{\mathsf{d}}a^{\pi}.\end{aligned}$$

(4) *If $b^2 = b$, then $b^{\mathsf{d}} = b$ and*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b + (1-b)a^{\mathsf{d}} + b\left[\sum_{n=0}^{\infty}(-a)^n\right]a^{\pi} \\ &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b + (1-b)a^{\mathsf{d}} + b(1+a)^{\mathsf{d}}a^{\pi}.\end{aligned}$$

(5) *If $a^2 = a$ and $b^2 = b$, then $1+ab$ is invertible and $a(1+ab)^{-1}b = \frac{1}{2}ab$. In this case,*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a(1+ab)^{-1}b + b(1-a) + (1-b)a \\ &= a+b-\tfrac{3}{2}ab.\end{aligned}$$

THEOREM 2.3. *Let $a \in \mathcal{A}^{\mathsf{d}}$ and $b \in \mathcal{A}$ be such that $\|ba^{\mathsf{d}}\| < 1, a^{\pi}ba^{\pi} = a^{\pi}b$ and $a^{\pi}ab = a^{\pi}ba$. If $a^{\pi}b \in \mathcal{A}^{\mathsf{d}}$, then $a+b \in \mathcal{A}^{\mathsf{d}}$. In this case,*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}} + (1+a^{\mathsf{d}}b)^{-1}a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n \\ &\quad + \left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2}ba^{\pi}(a+b)^n\right]a^{\pi} \\ &\quad \times \left[1-(a+b)a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right],\end{aligned}$$

and

$$\begin{aligned}\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| &\leq \frac{\|a^{\mathsf{d}}\|\|ba^{\mathsf{d}}\|}{1-\|ba^{\mathsf{d}}\|} + \|(1+a^{\mathsf{d}}b)^{-1}a^{\pi}\|\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|(-a)\|^n \\ &\quad + \left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|ba^{\mathsf{d}}\|}{1-\|ba^{\mathsf{d}}\|}\right)^{n+2}\|b\|\|a+b\|^n\right]\|a^{\pi}\|^2 \\ &\quad + \|a^{\pi}\|^2\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\right)^{n+2}\|b\|\|a+b\|^{n+1}\right]\left[\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|a\|^n\right].\end{aligned}$$

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Proof. Since $a \in \mathcal{A}^{\mathsf{d}}$ and $a^\pi b(1-a^\pi) = 0$, we have that for $p = 1 - a^\pi$

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} b_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p$$

where a_1 is invertible in the algebra $p\mathcal{A}p$ and a_2 is quasi-nilpotent element of the algebra $(1-p)\mathcal{A}(1-p)$. Also by $a^\pi ab = a^\pi ba$ and the fact that $a^\pi b \in \mathcal{A}^{\mathsf{d}}$, we conclude that $a_2b_2 = b_2a_2$ and $b_2 \in \mathcal{A}^{\mathsf{d}}$. Since $\sigma(ba^{\mathsf{d}}) \cup \{0\} = \sigma(a^{\mathsf{d}}b) \cup \{0\}$, it follows from $\|ba^{\mathsf{d}}\| < 1$ that $1 + a^{\mathsf{d}}b$ is invertible. Now, from Theorem 2.1, we have that

$$(a_2+b_2)^{\mathsf{d}} = \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n.$$

Using Theorem 1.1, we get that

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} (a_1+b_1)^{-1} & S \\ 0 & \sum\limits_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n \end{bmatrix}_p,$$

where

$$S = \left[\sum_{n=0}^{\infty}(a_1+b_1)^{-n-2}b_3(a_2+b_2)^n\right]\left[1-p-(a_2+b_2)\sum_{n=0}^{\infty}(b_2^{\mathsf{d}})^{n+1}(-a_2)^n\right]$$
$$-(a_1+b_1)^{-1}b_3\sum_{n=0}^{\infty}(b_2^{\mathsf{d}})^{n+1}(-a_2)^n.$$

We know that

$$\begin{bmatrix} (a_1+b_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & \sum\limits_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n \end{bmatrix}_p = a^\pi \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n.$$

Also, by computation we get

$$\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}_p = \left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2} ba^\pi(a+b)^n\right]a^\pi$$
$$\times\left[1-(a+b)a^\pi\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right]$$
$$-(1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}ba^\pi\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n.$$

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Hence,

$$
\begin{aligned}
(a+b)^{\mathsf{d}} &= (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}} + (1+a^{\mathsf{d}}b)^{-1}a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n \\
&\quad + \left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2} ba^{\pi}(a+b)^n\right] a^{\pi} \\
&\quad \times \left[1-(a+b)\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right].
\end{aligned}
$$

Since

$$(a_1+b_1)^{-1}\oplus 0 = (1+a_1^{-1}b_1)^{-1}a_1^{-1}\oplus 0 = a_1^{-1}(1+b_1a_1^{-1})^{-1}$$

It follows that

$$(1+a^{\mathsf{d}}b)^{\mathsf{d}}a^{\mathsf{d}}\oplus 0 = a^{\mathsf{d}}(1+ba^{\mathsf{d}})^{\mathsf{d}}$$

By the above facts and the condition $\|ba^{\mathsf{d}}\|<1$, we obtain

$$
\begin{aligned}
\|(a+b)^{\mathsf{d}}-a^{\mathsf{d}}\| &= \| \sum_{n=1}^{\infty}a^{\mathsf{d}}(ba^{\mathsf{d}})^n + (1+a^{\mathsf{d}}b)^{-1}a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n \\
&\quad + \left[\sum_{n=0}^{\infty}\left(\sum_{n=0}^{\infty}a^{\mathsf{d}}(ba^{\mathsf{d}})^n\right)^{n+2} ba^{\pi})(a+b)^n\right] a^{\pi} \\
&\quad \times \left[1-(a+b)a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right] \| \\
&\leq \frac{\|a^{\mathsf{d}}\|\|ba^{\mathsf{d}}\|}{1-\|ba^{\mathsf{d}}\|} + \|(1+a^{\mathsf{d}}b)^{-1}a^{\pi}\|\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|(-a)\|^n \\
&\quad + \left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|ba^{\mathsf{d}}\|}{1-\|ba^{\mathsf{d}}\|}\right)^{n+2}\|b\|\|a+b\|^n\right]\|a^{\pi}\|^2 \\
&\quad + \|a^{\pi}\|^2\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|ba^{\mathsf{d}}\|}{1-\|ba^{\mathsf{d}}\|}\right)^{n+2}\|b\|\|a+b\|^{n+1}\right]\left[\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|a\|^n\right]. \qquad \square
\end{aligned}
$$

Corollary 2.4. *Let $a\in\mathcal{A}^{\mathsf{d}}$ and $b\in\mathcal{A}$ be such that $\|ba^{\mathsf{d}}\|<1, a^{\pi}b(1-a^{\pi})=0$ and $a^{\pi}ab=a^{\pi}ba$,*

(1) *If $baa^{\mathsf{d}}=0$ and b is quasi-nilpotent, then $a+b\in\mathcal{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(a^{\mathsf{d}})^{n+2}b(a+b)^n + a^{\mathsf{d}}.$$

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(2) *If* $a^\pi b = ba^\pi$, $\sigma(a^\pi b) = 0$, *then* $a + b \in \mathcal{A}^{\mathsf{d}}$ *and*

$$(a + b)^{\mathsf{d}} = (1 + a^{\mathsf{d}} b)^{-1} a^{\mathsf{d}} = a^{\mathsf{d}} (1 + ba^{\mathsf{d}})^{-1}.$$

The following theorem is a generalization of Theorem 2.3 and a generalization of Theorem 6 from [2].

THEOREM 2.5. *Let* $a, b \in \mathcal{A}^{\mathsf{d}}$ *and let* q *be an idempotent such that* $aq = qa$, $(1 - q)bq = 0$, $(ab - ba)q = 0$, *and* $(1 - q)(ab - ba) = 0$. *If* $(a + b)q$ *and* $(1 - q)(a + b)$ *are generalized Drazin invertible, then* $a + b \in \mathcal{A}^{\mathsf{d}}$ *and*

$$\begin{aligned}(a + b)^{\mathsf{d}} = &\sum_{n=0}^{\infty} S^{n+2} qb(1 - q)(a + b)^n (1 - q)\Big[1 - (a + b)S\Big] \\ &+ \Big[1 - (a + b)S\Big] q \sum_{n=0}^{\infty} (a + b)^n qb(1 - q)S^{n+2} \\ &+ (1 - Sqb)\,(1 - q)S + Sq,\end{aligned}$$

where

$$\begin{aligned} S &= a^{\mathsf{d}}(1 + a^{\mathsf{d}} b)^{\mathsf{d}} bb^{\mathsf{d}} + (1 - bb^{\mathsf{d}}) \left[\sum_{n=0}^{\infty} (-b)^n (a^{\mathsf{d}})^{n+1}\right] \\ &+ \left[\sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+1} (-a)^n\right] a^\pi. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 2.3. □

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