Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

# BOUNDING THE SOLUTIONS OF SYLVESTER-LIKE ABSOLUTE VALUE EQUATION*

MARZIEH DEHGHANI-MADISEH† AND MILAN HLADÍK‡

**Abstract.** We develop several methods (including two direct methods and an iterative method) for computing an enclosure of the solutions of the so-called Sylvester-like absolute value equations (AVEs). The proposed direct methods are modifications of the Bauer–Skeel and Hansen–Bliek–Rohn bounds, which were introduced for outer approximation of the solutions of the standard and generalized AVEs. These approaches, while requiring diagonalizability of certain nonnegative matrices, have the advantage of considerably reducing computational costs, in contrast to simple Kroneckerization, i.e., the direct application of the aforementioned bounds to the Kronecker form of the Sylvester-like AVEs. We also propose an iterative approach, which refines some initial bounds and produces highly efficient enclosures for the solutions. Moreover, the iterative method can be terminated at any time and provides a numerically guaranteed distance to the unique solution.

**Key words.** Interval arithmetic, Absolute value equation, Matrix equations, Solution set.

**AMS subject classifications.** 65G40, 15A24.

## 1. Introduction.

*The problem.* In this paper, we consider the matrix equation

$$AXB + C|X|D = E, \tag{1.1}$$

where $A, C \in \mathbb{R}^{m\times m}$, $B, D \in \mathbb{R}^{n\times n}$, and $E \in \mathbb{R}^{m\times n}$ are input matrices and $X \in \mathbb{R}^{m\times n}$ is the unknown rectangular matrix. The matrix equation (1.1) is a nonlinear equation, which is called the Sylvester-like absolute value equation (Sylvester-like AVE).

The goal of this paper is to present efficient bounds for the solutions of (1.1). To this end, we first briefly review the vector and matrix AVEs and introduce the notation and some basic techniques on which we will build.

*Standard AVE and GAVE.* For the case $n = 1$, the Sylvester-like AVE (1.1) reduces to the following generalized absolute value equation (GAVE)

$$Ax + B|x| = e, \tag{1.2}$$

where $A, B \in \mathbb{R}^{m\times m}$ and $e \in \mathbb{R}^m$. More specifically, for $B = I$, the GAVE (1.2) reduces to the following AVE

$$Ax + |x| = e. \tag{1.3}$$

---

[*] Received by the editors on August 14, 2025. Accepted for publication on January 27, 2026. Handling Editor: Melina Freitag. Corresponding Author: Milan Hladík.

[†] Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran (m.dehghani@scu.ac.ir). Supported by the Shahid Chamran University of Ahvaz under the grant number SCU.MM1403.33518.

[‡] Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 11800, Prague, Czech Republic (hladik@kam.mff.cuni.cz). Supported by the Czech Science Foundation Grant 25-15714S.

The equations (1.2) and (1.3), which are special forms of the Sylvester-like AVE (1.1), have interesting connections to various branches of science. The GAVE (1.2) is equivalent to the linear complementarity problem and so has important applications in optimization field such as linear programming, convex quadratic programming, interval analysis, and mixed integer programming; see [14, 19, 23, 28, 29] and the survey paper [12].

One of the essential studies in the area of AVEs (1.2) and (1.3) is checking their solvability and the existence or nonexistence of solutions for them. Characterization of unique solvability was reviewed by Kumar et al. [17]. Due to its intractability, various sufficient conditions have been derived [11, 23, 22, 24, 30, 35]. As it has been shown in Mangasarian [19], solving the AVEs of types (1.2) and (1.3) is an NP-hard problem. Despite that (or motivated by that), various iterative methods have been proposed for solving or estimating AVE or GAVE; see, for instance, [4, 5, 15, 16, 19, 20, 21, 26, 27, 30, 33, 36]. There are also classes of problems that can be solved in polynomial time [10, 23, 36].

*Sylvester-like AVE.* Clearly, solving the Sylvester-like AVE (1.1) exactly is an NP-hard problem, too. Certain sufficient or necessary conditions for unique solvability of (1.1) were proposed in [6, 34] and surveyed in Kumar et al. [17]. To be concrete, (1.1) has a unique solution for each $E \in \mathbb{R}^{m\times n}$ if any of the following two conditions holds:

$$\rho(|A^{-1}C|)\rho(|DB^{-1}|) < 1,$$

or

$$\|A^{-1}C\| \cdot \|DB^{-1}\| < 1.$$

Regarding the algorithms, there are considerably less methods discussed than for the standard AVE or GAVE; a particular example of the approaches investigated is Picard-type iterations [25, 32].

Other types of the matrix AVE have also been investigated; for instance, the form $AXB + |CXD| = E$ was addressed by Li [18]. Note that some variants of the matrix equation (1.1) appear in characterizing the solution set of a system of matrix equations with interval coefficients [1, 2, 3, 7, 8].

*Vectorization.* Using the Kronecker product "⊗" and the vectorization operator "vec," Hashemi [6] showed that the Sylvester-like AVE (1.1) can be transformed into the following GAVE

$$Sx + G|x| = e, \tag{1.4}$$

in which

$$S = B^\top \otimes A, \quad G = D^\top \otimes C, \quad x = \operatorname{vec}(X), \quad \text{and} \quad e = \operatorname{vec}(E).$$

For a given matrix $E \in \mathbb{R}^{m\times n}$, $\operatorname{vec}(E)$ is an $mn$-dimensional vector, which is obtained by stacking the columns of $E$. In this way, we can solve the matrix AVE (1.1) by transforming it into the standard AVE (1.3). However, this would cause the increase of the size and computational complexity. Thus, we will use the form (1.4) only to derive certain formulas, while we will compute with the original matrices.

If $S$ is invertible, then premultiplying the GAVE (1.4) by $S^{-1}$ yields the following AVE

$$x - h = T|x|, \tag{1.5}$$

where

$$h = S^{-1}e, \quad T = -S^{-1}G. \tag{1.6}$$

Obviously, the AVE (1.5) is an equivalent reformulation of (1.1). Thus, a direct approach for handling the Sylvester-like AVE (1.1) employs the existing methods for evaluating the solutions of (1.5). Hladík [9] adapted the Bauer–Skeel and Hansen–Bliek–Rohn bounds for enclosing the solutions of an AVE of the form (1.5), but straightforward applying the same approaches on (1.5) requires $\mathcal{O}((mn)^3)$ arithmetic operations, which makes the problem very prohibitive and, from a computational point of view, is not efficient. We want to propose a modification of the Bauer–Skeel and Hansen–Bliek–Rohn bounds and an iterative technique for enclosing the solutions of (1.1), in such a way that they reduce the computational complexity to the cubic order.

*Notation.* In this paper, for a square matrix $A$, $\rho(A)$ stands for the spectral radius of $A$, which is the maximum of the absolute values of its eigenvalues. The absolute value of $A$ is denoted by $|A|$ and is understood componentwise. Interval quantities are our essential tools for providing some enclosures to the solutions of (1.1). These types of quantities are shown in boldface letters. An interval number $\mathbf{x}$ is a closed interval of the form $\mathbf{x} = [\underline{x}, \overline{x}]$ and is defined as the set

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}.$$

The midpoint and radius of $\mathbf{x}$ are defined as $\operatorname{mid}(\mathbf{x}) = \frac{1}{2}(\overline{x} + \underline{x})$ and $\operatorname{rad}(\mathbf{x}) = \frac{1}{2}(\overline{x} - \underline{x})$, respectively. An interval matrix $\mathbf{B}$ is determined by two matrices $\underline{B}, \overline{B} \in \mathbb{R}^{m\times n}$ as its bounds is defined similarly as

$$\mathbf{B} = [\underline{B}, \overline{B}] = \{\tilde{B} \in \mathbb{R}^{m\times n} : \underline{B} \leq \tilde{B} \leq \overline{B}\}.$$

The concept of the midpoint and radius of interval matrices is understood componentwise. For a vector $a = (a_1, \dots, a_n)$, $\operatorname{Diag}(a)$ is the diagonal matrix having entries $a_1, \dots, a_n$ on the diagonal. The notation ./ stands for the Hadamard (i.e., componentwise) division.

*Structure of the paper.* In Section 2, first the Bauer–Skeel and Hansen–Bliek–Rohn bounds, which have been proposed by Hladík [9], are recalled. We then develop their modification of them together to serve for enclosing the solutions of the Sylvester-like AVE (1.1) in such a way that the computational costs are significantly reduced. Next, we derive an iterative technique to reduce the overestimation of the bounding methods. Moreover, under certain assumption, we can prove convergence to the unique solution. Numerical examples are provided in Section 3 to compare the proposed approaches with respect to the execution times and quality of the obtained results. Finally, the paper is concluded in Section 4.

**2. Enclosing the solutions of** (1.1)**.** In the following lemma, we recall some basic properties of the Kronecker product and the spectral radius, which will be used in the subsequent subsections. Items (3), (5), and (6) are straightforward, while the others are proved, for example, in Horn and Johnson [13].

Lemma 2.1. *For matrices $A$, $B$, $C$, and $D$ of compatible sizes, we have*

*(1)* $\operatorname{vec}(ABC) = (C^\top \otimes A)\operatorname{vec}(B)$;
*(2)* $(A \otimes B)(C \otimes D) = (AC \otimes BD)$;
*(3)* $\operatorname{Diag}(\operatorname{vec}(A))^{-1}\operatorname{vec}(B) = \operatorname{vec}(B./A)$;
*(4) for nonsingular square matrices $A$ and $B$, the matrix $A \otimes B$ is also nonsingular, and* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
*(5)* $|A \otimes B| = |A| \otimes |B|$;
*(6)* $\rho(A^\top) = \rho(A)$;
*(7) for square matrices $A$ and $B$, we have* $\rho(A \otimes B) = \rho(A)\rho(B)$.

*Preliminaries.* Recall from (1.5) that as long as the matrix $S$ in (1.4) is nonsingular, the Sylvester-like AVE (1.1) can be reformulated as the AVE

$$x - h = T|x|, \tag{2.7}$$

in which $h$ and $T$ are defined by (1.6). So, from here on, we assume that $A$ and $B$ in (1.1) are nonsingular, which by part (4) of Lemma 2.1 guarantees the nonsingularity of $S = B^\top \otimes A$. Now, define $T_1 = |B^{-\top} D^\top|$ and $T_2 = |A^{-1}C|$. By parts (2), (4), and (5) of Lemma 2.1, we obtain

$$\begin{aligned} |T| &= |S^{-1}G| = |(B^\top \otimes A)^{-1}(D^\top \otimes C)| = |(B^{-\top}D^\top) \otimes A^{-1}C| \\ &= |B^{-\top}D^\top| \otimes |A^{-1}C| = T_1 \otimes T_2. \end{aligned} \tag{2.8}$$

Suppose that $T_1$ and $T_2$ are diagonalizable, so there exist nonsingular matrices $S_{T_1}$ and $S_{T_2}$ and diagonal matrices $D_{T_1}$ and $D_{T_2}$ such that

$$T_1 = S_{T_1} D_{T_1} S_{T_1}^{-1}, \quad T_2 = S_{T_2} D_{T_2} S_{T_2}^{-1}. \tag{2.9}$$

In fact, the diagonal entries of $D_{T_1}$ and $D_{T_2}$ are the eigenvalues of $T_1$ and $T_2$, respectively. Also, the columns of $S_{T_1}$ and $S_{T_2}$, respectively, form their corresponding eigenvectors.

Using (2.8), (2.9), and parts (2) and (4) of Lemma 2.1, we can write

$$\begin{aligned} |T| = T_1 \otimes T_2 &= (S_{T_1} D_{T_1} S_{T_1}^{-1}) \otimes (S_{T_2} D_{T_2} S_{T_2}^{-1}) \\ &= (S_{T_1} \otimes S_{T_2})(D_{T_1} \otimes D_{T_2})(S_{T_1}^{-1} \otimes S_{T_2}^{-1}) \\ &= (S_{T_1} \otimes S_{T_2})(D_{T_1} \otimes D_{T_2})(S_{T_1} \otimes S_{T_2})^{-1}. \end{aligned}$$

The above relation yields

$$|T| = S_{|T|} D_{|T|} S_{|T|}^{-1},$$

in which $S_{|T|} = S_{T_1} \otimes S_{T_2}$ and $D_{|T|} = D_{T_1} \otimes D_{T_2}$. It is obvious that $D_{|T|}$ has a diagonal structure. If we define $\Lambda = I_{mn} - D_{|T|}$, then $\Lambda$ is diagonal, too. Let the $m$-by-$n$ matrix $R$ be such that

$$\Lambda = \mathrm{Diag}(\mathrm{vec}(R)). \tag{2.10}$$

By the Neumann series Theorem, we know that if $\rho(|T|) < 1$, then $(I_{mn} - |T|)$ is invertible with nonnegative inverse, which by (2.10) can be written as

$$(I_{mn} - |T|)^{-1} = S_{|T|}(I_{mn} - D_{|T|})^{-1} S_{|T|}^{-1} = S_{|T|}\, \mathrm{Diag}(\mathrm{vec}(R))^{-1} S_{|T|}^{-1}. \tag{2.11}$$

## 2.1. A modification of the Bauer–Skeel bounds.

Consider the absolute value equation of the following form

$$x - b = B|x|, \tag{2.12}$$

where $B$ and $b$ are known matrix and vector, respectively, and $x$ is the unknown vector. Hladík [9] proposed the following Bauer–Skeel type bounds for the solutions of an AVE of type (2.12).

Lemma 2.2. *If $\rho(|B|) < 1$, then the unique solution $x$ of AVE (2.12) satisfies*

$$|x - b| \le (I - |B|)^{-1}|B||b|. \tag{2.13}$$

By Lemma 2.2, we conclude that the solutions of (2.12) lie within the interval vector

$$\mathbf{x}^{BS} = \big[b - (I - |B|)^{-1}|B||b|,\, b + (I - |B|)^{-1}|B||b|\big]. \tag{2.14}$$

For enclosing the solutions of our problem, i.e., the Sylvester-like AVE (1.1), we can apply Lemma 2.2 on the AVE (2.7), which is a reformulation of (1.1) and also it follows the form (2.12). But this naive idea needs a prohibitive computational cost. We want to modify Lemma 2.2 for matrix equation (1.1) so that the cost is significantly reduced. Recall from (1.6) that matrix $T$ is defined as $T = -S^{-1}G$.

Lemma 2.3. *Consider the Sylvester-like AVE (1.1). If*

$$\rho(|A^{-1}C|)\rho(|DB^{-1}|) < 1, \tag{2.15}$$

*then $\rho(|T|) < 1$.*

*Proof.* Using (2.8), we can write

$$|T| = |B^{-\top}D^{\top}| \otimes |A^{-1}C|,$$

which by parts (6) and (7) of Lemma 2.1 yields

$$\begin{aligned}\rho(|T|) &= \rho\big(|B^{-\top}D^{\top}| \otimes |A^{-1}C|\big) = \rho\big(|B^{-\top}D^{\top}|\big)\rho(|A^{-1}C|)\\ &= \rho(|DB^{-1}|)\rho(|A^{-1}C|),\end{aligned}$$

and the proof is completed. ◻

Theorem 2.4. *Consider the Sylvester-like AVE (1.1). Suppose that (2.15) is satisfied and matrices $|A^{-1}C|$ and $|B^{-\top}D^{\top}|$ are diagonalizable. Using the introduced notations above, define*

$$M = \big(S_{T_2}^{-1}|A^{-1}C||A^{-1}EB^{-1}||DB^{-1}|S_{T_1}^{-\top}\big)\, ./R. \tag{2.16}$$

*Then the interval matrix*

$$\mathbf{X}^{BS} = \left[A^{-1}EB^{-1} - S_{T_2}MS_{T_1}^{\top},\, A^{-1}EB^{-1} + S_{T_2}MS_{T_1}^{\top}\right], \tag{2.17}$$

*encloses the unique solution of (1.1).*

*Proof.* Let $X$ be the solution of (1.1), so $x = \mathrm{vec}(X)$ solves the AVE (2.7), which is a reformulation of (1.1). By condition (2.15) and Lemma 2.3, we obtain $\rho(|T|) < 1$. So by Lemma 2.2, we conclude that $x$ satisfies

$$|x - h| \le (I_{mn} - |T|)^{-1}|T||h|. \tag{2.18}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

Using (2.18), (2.11), and Lemma 2.1, we can write

$$
\begin{aligned}
|x-h| &\le (I_{mn}-|T|)^{-1}|T||h| \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} S_{|T|}^{-1}|T||(B^\top \otimes A)^{-1}\,\mathrm{vec}(E)| \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} S_{|T|}^{-1}|T||(B^{-\top} \otimes A^{-1})\,\mathrm{vec}(E)| \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} S_{|T|}^{-1}|(B^\top \otimes A)^{-1}(D^\top \otimes C)|\,\mathrm{vec}(|A^{-1}EB^{-1}|) \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} S_{|T|}^{-1}|(B^{-\top}D^\top) \otimes (A^{-1}C)|\,\mathrm{vec}(|A^{-1}EB^{-1}|) \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} S_{|T|}^{-1}(|B^{-\top}D^\top| \otimes |A^{-1}C|)\,\mathrm{vec}(|A^{-1}EB^{-1}|) \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1} (S_{T_1} \otimes S_{T_2})^{-1}\,\mathrm{vec}(|A^{-1}C||A^{-1}EB^{-1}||DB^{-1}|) \\
&= S_{|T|}\big(\mathrm{Diag}(\mathrm{vec}(R))\big)^{-1}\,\mathrm{vec}(S_{T_2}^{-1}|A^{-1}C||A^{-1}EB^{-1}||DB^{-1}|S_{T_1}^{-\top}) \\
&= S_{|T|}\,\mathrm{vec}\big([S_{T_2}^{-1}|A^{-1}C||A^{-1}EB^{-1}||DB^{-1}|S_{T_1}^{-\top}]./R\big) \\
&= (S_{T_1} \otimes S_{T_2})\,\mathrm{vec}(M) \\
&= \mathrm{vec}(S_{T_2}MS_{T_1}^\top). \qquad (2.19)
\end{aligned}
$$

Further, by Lemma 2.1, we have

$$
\begin{aligned}
|x-h| &= |\,\mathrm{vec}(X) - S^{-1}e| = |\,\mathrm{vec}(X) - (B^\top \otimes A)^{-1}\,\mathrm{vec}(E)| \\
&= |\,\mathrm{vec}(X) - \mathrm{vec}(A^{-1}EB^{-1})| = \mathrm{vec}(|X - A^{-1}EB^{-1}|). \qquad (2.20)
\end{aligned}
$$

So by relations (2.19) and (2.20), we obtain

$$\mathrm{vec}(|X - A^{-1}EB^{-1}|) \le \mathrm{vec}(S_{T_2}MS_{T_1}^\top),$$

which yields

$$|X - A^{-1}EB^{-1}| \le S_{T_2}MS_{T_1}^\top,$$

or equivalently

$$X \in \big[A^{-1}EB^{-1} - S_{T_2}MS_{T_1}^\top,\ A^{-1}EB^{-1} + S_{T_2}MS_{T_1}^\top\big].$$

This means that the interval matrix $\mathbf{X}^{BS}$ defined by (2.17) is an enclosure for the solution of the Sylvester-like AVE (1.1). ☐

### 2.2. A modification of the Hansen–Bliek–Rohn bounds.

The following Hansen–Bliek–Rohn type bounds have also been derived by Hladík [9] for the unique solution of an AVE of type (2.12).

Lemma 2.5. *If $\rho(|B|) < 1$, then each solution $x$ of AVE (2.12) satisfies $x \in \mathbf{x}^{HBR}$, in which $\mathbf{x}^{HBR}$ is defined componentwisely as*

$$\mathbf{x}_i^{HBR} = \frac{b_i + (u_i/d_i - |b|_i)[-1,1]}{1 + (1 - 1/d_i)[-1,1]}, \quad \forall i, \qquad (2.21)$$

*where*

$$C = I - |B|, \quad u = C^{-1}|b|, \quad d_i = (C^{-1})_{ii}. \qquad (2.22)$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

REMARK 2.6. *If* $X \in \mathbb{R}^{m\times n}$*, then* $x = \operatorname{vec}(X) \in \mathbb{R}^{mn}$ *and for* $k = 1, \ldots, mn$*, we have*

$$x_k = X_{ij},$$

*where*

$$i = (\lfloor k-1 \rfloor \bmod m) + 1, \quad j = \left\lfloor \frac{k-1}{m} \right\rfloor + 1. \tag{2.23}$$

Now we are ready to present a modification of Lemma 2.5 to provide an enclosure for the solution of the Sylvester-like AVE (1.1) with low computational cost.

THEOREM 2.7. *Consider the Sylvester-like AVE* (1.1)*. Suppose that* (2.15) *is satisfied and matrices* $|A^{-1}C|$ *and* $|B^{-\top}D^{\top}|$ *are diagonalizable. Using the introduced notations above, define*

$$N = \left(S_{T_2}^{-1}|A^{-1}EB^{-1}|S_{T_1}^{-\top}\right)./R$$

*and*

$$W^{(k)} = \left[S_{T_2}^{-1}Z^{(k)}S_{T_1}^{-\top}\right]./R, \quad k = 1, \ldots, mn,$$

*in which* $Z^{(k)} \in \mathbb{R}^{m\times n}$ *is a matrix whose all elements are zero except for the* $(i,j)$*-th element, which is defined by* (2.23) *and has the value of one. Then, the interval matrix* $\mathbf{X}^{HBR}$ *defined by*

$$\mathbf{X}_{ij}^{HBR} = \frac{H_{ij} + (\tilde{U}_{ij}/\tilde{D}_{ij} - |H|_{ij})[-1,1]}{1 + (1 - 1/\tilde{D}_{ij})[-1,1]}, \tag{2.24}$$

*where*

$$H = A^{-1}EB^{-1}, \quad \tilde{U} = S_{T_2}NS_{T_1}^{\top}, \quad \tilde{D} = S_{T_2}W^{(k)}S_{T_1}^{\top},$$

*encloses the solutions of* (1.1).

*Proof.* Suppose $X$ is a solution of (1.1), so $x = \operatorname{vec}(X)$ solves the AVE (2.7). Condition (2.15) yields $\rho(|T|) < 1$, so by Lemma 2.5, $x \in \mathbf{x}^{HBR}$, where $\mathbf{x}^{HBR}$ is defined as

$$\mathbf{x}_k^{HBR} = \frac{h_k + (\tilde{u}_k/\tilde{d}_k - |h|_k)[-1,1]}{1 + (1 - 1/\tilde{d}_k)[-1,1]}, \quad k = 1, \ldots, mn, \tag{2.25}$$

in which

$$\tilde{C} = I_{mn} - |T|, \quad \tilde{u} = \tilde{C}^{-1}|h|, \quad \tilde{d}_k = (\tilde{C}^{-1})_{kk}.$$

For a fixed integer $1 \le k \le mn$, let $(i,j) = ((\lfloor k-1 \rfloor \bmod m) + 1, \lfloor \frac{k-1}{m} \rfloor + 1)$. By the above notations and Lemma 2.1, we can write

$$h = (B^{\top} \otimes A)^{-1}\operatorname{vec}(E) = \operatorname{vec}(A^{-1}EB^{-1}) = \operatorname{vec}(H),$$

so

$$h_k = H_{ij}. \tag{2.26}$$

By relation (2.11), we have

$$\tilde{C}^{-1} = (I_{mn} - |T|)^{-1} = S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} S_{|T|}^{-1}.$$

Thus,

$$\begin{aligned}
\tilde{u} = \tilde{C}^{-1}|h| &= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} S_{|T|}^{-1} \operatorname{vec}(|A^{-1}EB^{-1}|)\\
&= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} (S_{T_1} \otimes S_{T_2})^{-1} \operatorname{vec}(|A^{-1}EB^{-1}|)\\
&= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} \operatorname{vec}\big(S_{T_2}^{-1}|A^{-1}EB^{-1}|S_{T_1}^{-\top}\big)\\
&= S_{|T|} \operatorname{vec}\big([S_{T_2}^{-1}|A^{-1}EB^{-1}|S_{T_1}^{-\top}]./R\big)\\
&= (S_{T_1} \otimes S_{T_2}) \operatorname{vec}(N)\\
&= \operatorname{vec}(S_{T_2} N S_{T_1}^{\top}) = \operatorname{vec}(\tilde{U}),
\end{aligned}$$

which yields

$$\tilde{u}_k = \tilde{U}_{ij}. \tag{2.27}$$

For $\tilde{d}_k$, by a simple calculation, we see that

$$\tilde{d}_k = (\tilde{C}^{-1})_{kk} = (\tilde{C}^{-1} \operatorname{vec}(Z^{(k)}))_k.$$

Further, we have

$$\begin{aligned}
\tilde{C}^{-1} \operatorname{vec}(Z^{(k)}) &= (I_{mn} - |T|)^{-1} \operatorname{vec}(Z^{(k)})\\
&= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} S_{|T|}^{-1} \operatorname{vec}(Z^{(k)})\\
&= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} (S_{T_1} \otimes S_{T_2})^{-1} \operatorname{vec}(Z^{(k)})\\
&= S_{|T|}\big(\operatorname{Diag}(\operatorname{vec}(R))\big)^{-1} \operatorname{vec}\big(S_{T_2}^{-1} Z^{(k)} S_{T_1}^{-\top}\big)\\
&= S_{|T|} \operatorname{vec}\big([S_{T_2}^{-1} Z^{(k)} S_{T_1}^{-\top}]./R\big)\\
&= (S_{T_1} \otimes S_{T_2}) \operatorname{vec}(W^{(k)})\\
&= \operatorname{vec}(S_{T_2} W^{(k)} S_{T_1}^{\top}) = \operatorname{vec}(\tilde{D}),
\end{aligned}$$

from which we obtain

$$\tilde{d}_k = \tilde{D}_{ij}. \tag{2.28}$$

Now in relation (2.25), suppose $\mathbf{X}^{HBR}$ is an $m$-by-$n$ interval matrix such that $\mathbf{x}^{HBR} = \operatorname{vec}(\mathbf{X}^{HBR})$, so

$$\mathbf{x}_k^{HBR} = \mathbf{X}_{ij}^{HBR}, \tag{2.29}$$

and by relations (2.25)-(2.29), we conclude that

$$\mathbf{X}_{ij}^{HBR} = \frac{H_{ij} + (\tilde{U}_{ij}/\tilde{D}_{ij} - |H|_{ij})[-1,1]}{1 + (1 - 1/\tilde{D}_{ij})[-1,1]}. \qquad \square$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

Theorem 2.8. *The methods for computing the enclosures of the solutions of the Sylvester-like AVE (1.1) based on Theorems 2.4 and 2.7 have complexity $\mathcal{O}(m^3+n^3)$.*

*Proof.* The computational cost of the spectral decompositions in (2.9) needs a complexity that is cubic in terms of the dimension of the involved matrices and adding together yields the computational complexity $\mathcal{O}(m^3+n^3)$. The cost for evaluating the inverse of $m$-by-$m$ and $n$-by-$n$ matrices is also cubic in terms of the size of input matrices, which together is $\mathcal{O}(m^3+n^3)$. The rest of the calculations in Theorems 2.4 and 2.7 depend mainly on addition, multiplication, or point-wise divisions of $m$-by-$m$, $m$-by-$n$, or $n$-by-$n$ matrices, which result in complexity $\mathcal{O}(m^3+n^3)$. The computational cost of the remaining parts is negligible, so the proof is completed. ∎

It is worth mentioning that if we directly employ the Bauer–Skeel or Hansen–Bliek–Rohn methods introduced in [9] for bounding the solutions of (1.1), then it requires $\mathcal{O}((mn)^3)$ arithmetic operations, which makes the problem much more demanding.

**2.3. An iterative refinement approach.** In this subsection, we propose an iterative technique to provide some outer estimations for the solutions of the Sylvester-like AVE (1.1).

Let an initial interval matrix $\mathbf{X}$, which is an enclosure for the solutions of (1.1), be given and suppose that $X$ is a solution of (1.1). Since

$$|\mathbf{X}| \subseteq [|\operatorname{mid}(\mathbf{X})| - \operatorname{rad}(\mathbf{X}), |\operatorname{mid}(\mathbf{X})| + \operatorname{rad}(\mathbf{X})],$$

we have

$$\begin{aligned} X &= A^{-1}EB^{-1} - (A^{-1}C)|X|(DB^{-1}) \\ &\in A^{-1}EB^{-1} - (A^{-1}C)|\mathbf{X}|(DB^{-1}) \\ &\subseteq A^{-1}EB^{-1} - (A^{-1}C)|\operatorname{mid}(\mathbf{X})|(DB^{-1}) + [-1,1]|A^{-1}C|\operatorname{rad}(\mathbf{X})|DB^{-1}| \\ &\equiv \mathcal{F}(\mathbf{X}). \end{aligned}$$

Thus, we can start with $\mathbf{X}^0 = \mathbf{X}$ and consider the iterations

$$\mathbf{X}^k \coloneqq \mathcal{F}(\mathbf{X}^{k-1}) \cap \mathbf{X}^{k-1}, \quad k = 1, \dots .$$

Under common assumption, we can prove convergence even for the simplified version

$$\mathbf{X}^k \coloneqq \mathcal{F}(\mathbf{X}^{k-1}), \quad k = 1, \dots . \tag{2.30}$$

Theorem 2.9. *Suppose that $\rho(|DB^{-1}|)\rho(|A^{-1}C|) < 1$. Then, the sequence (2.30) converges linearly to the unique solution of the Sylvester-like AVE (1.1).*

*Proof.* The radius of the iterations (2.30) changes as follows. Denote $\mathbf{Y} \coloneqq \mathcal{F}(\mathbf{X})$. Then,

$$\operatorname{rad}(\mathbf{X}) \mapsto \operatorname{rad}(\mathbf{Y}) = |A^{-1}C|\operatorname{rad}(\mathbf{X})|DB^{-1}|.$$

In the vectorized form, we can write it as

$$\begin{aligned} \operatorname{rad}(\mathbf{y}) &= \operatorname{vec}(\operatorname{rad}(\mathbf{Y})) = \operatorname{vec}\left(|A^{-1}C|\operatorname{rad}(\mathbf{X})|DB^{-1}|\right) \\ &= \left(|DB^{-1}|^\top \otimes |A^{-1}C|\right)\operatorname{rad}(\mathbf{x}). \end{aligned}$$

Note that $\rho(|DB^{-1}|^\top \otimes |A^{-1}C|) = \rho(|DB^{-1}|)\rho(|A^{-1}C|) < 1$. Thus, there exists a norm such that

$$\|\operatorname{rad}(\mathbf{x})\| \mapsto \|\operatorname{rad}(\mathbf{y})\| = \||DB^{-1}|^\top \otimes |A^{-1}C|\| \cdot \|\operatorname{rad}(\mathbf{x})\| = \alpha \|\operatorname{rad}(\mathbf{x})\|,$$

where $\alpha < 1$. Therefore, the sequence (2.30) converges linearly. ∎

Iterations (2.30) can be used in different ways. First, we can apply a few of iterations to improve the bounds, possibly also to determine the sign of the solutions and get rid of the absolute values. Second, we can iterate to determine the solution with a given accuracy. The accuracy is provided by the radius of $\mathbf{X}^k$, and it is a guaranteed bound since $\mathbf{X}^k$ provably contains the solution. Thus, in contrast to the standard point-to-point based iterations, we control the accuracy and have numerically rigorous distance to the solution for free.

THEOREM 2.10. *The iterative method* (2.30) *has complexity* $\mathcal{O}(m^3 + n^3)$ *arithmetic operations per iteration.*

*Proof.* The cost for evaluating the inverse of matrices $A$ and $B$ is cubic in terms of the dimension of the involved matrices, which together needs $\mathcal{O}(m^3 + n^3)$ arithmetic operations. Other parts of calculating the members of sequence (2.30) mainly depend on multiplication of $m$-by-$m$, $m$-by-$n$, or $n$-by-$n$ matrices, which lead to $\mathcal{O}(m^3 + n^3)$ arithmetic operations. Since the cost of the remaining parts of the iterative formula (2.30) is negligible, the proof is completed. ∎

## 3. Numerical examples.

We carried out several numerical tests in order to demonstrate the effectiveness of the methods presented in Section 2. We compare the computing times and quality of the enclosures computed by the following methods:

- Bauer–Skeel bounds (BS): The approach introduced in Lemma 2.2 and applied to the AVE (2.7);
- Modification of the Bauer–Skeel bounds (MBS): The new approach (2.17) proposed in Theorem 2.4 and applied to the matrix AVE (1.1);
- Hansen–Bliek–Rohn bounds (HBR): The approach introduced in Lemma 2.5 applied to the AVE (2.7);
- Modification of the Hansen–Bliek–Rohn bounds (MHBR): The new approach (2.24) proposed in Theorem 2.7 and applied to the matrix AVE (1.1);
- The iterative refinement approach (ITR): The new iterative method (2.30) applied to the matrix AVE (1.1).

Note that both BS and MBS methods yield the same enclosure; only the running times are different since the methods achieve the enclosure by different ways. That is why we do not display the enclosure $\mathbf{X}^{BS}$. For analogous reason, we also do not display the enclosure $\mathbf{X}^{HBR}$.

In the displayed tables, the execution times are recorded in seconds. We write "MO" for the failure of a particular method because of memory overflow. For comparing quality of the numerical results, we display the relative sums of radii with respect to the enclosure computed by the MBS method, i.e., for the received enclosure $\mathbf{X}$ and the enclosure $\mathbf{Y}$ computed by the MBS method, we display

$$Ratio = \frac{\sum_i \sum_j \operatorname{rad}(\mathbf{X}_{ij})}{\sum_i \sum_j \operatorname{rad}(\mathbf{Y}_{ij})},$$

In all examples, the stopping criterion for the ITR method is $\operatorname{rad}(\mathbf{X}^k) < 10^{-6}$. To work with intervals, we have utilized the Intlab package [31]. The computations were done in MATLAB R2017a on a machine with Intel(R) Core(TM) 2.20 GHz Dual CPU and with 4 GB RAM.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

Example 3.1. *Consider the Sylvester-like AVE*

$$AXB + C|X|D = E, \tag{3.31}$$

*in which*

$$A = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ -1 & 1 & 2 & -1 \\ 2 & -1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 1.5 \\ -2.5 & 1 & 5 & 0 \\ -1 & 0 & -2.5 & -0.5 \\ 10 & 0.5 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -0.3 & -0.1 & 0.1 \\ 0.5 & -0.1 & 0.3 & 0 \\ 0.1 & 0 & -0.1 & 0.2 \\ 0.1 & 0 & 0 & 0.1 \end{pmatrix}, \quad D = \begin{pmatrix} 0.01 & 0 & -0.02 & -0.03 \\ -0.01 & -0.03 & 0 & -0.02 \\ 0 & -0.01 & 0.02 & -0.03 \\ 0.03 & -0.02 & -0.01 & 0.02 \end{pmatrix},$$

$$E = \begin{pmatrix} 137.479 & 2.045 & -47.493 & 18.513 \\ -163.867 & -4.116 & 19.951 & -25.022 \\ 90.997 & 9.477 & 10.020 & 16.456 \\ 179.518 & -4.529 & 5.004 & 27.973 \end{pmatrix}.$$

*To verify the assumptions of our methods, we calculate*

$$\rho(|A^{-1}C|) = 6.5811, \quad \rho(|DB^{-1}|) = 0.0532 \quad \rho(|A^{-1}C|)\rho(|DB^{-1}|) = 0.3502.$$

*Now, the MBS method is applicable and yields the following enclosure for* (3.31)

$$\mathbf{X}^{MBS} = \begin{pmatrix} [-1.6280, 2.7670] & [-4.1967, -1.0867] & [-4.7140, 2.3447] & [1.9784, 3.5525] \\ [-1.5069, 0.9902] & [1.7390, 3.5036] & [-0.7056, 3.3011] & [-1.5867, -0.6944] \\ [2.5784, 4.9532] & [-1.1661, 0.4991] & [2.4863, 6.2752] & [8.4513, 9.3014] \\ [0.3731, 3.1046] & [-0.3514, 1.5772] & [5.0957, 9.4756] & [2.3693, 3.3485] \end{pmatrix},$$

*and the MHBR method gives the following enclosure*

$$\mathbf{X}^{MHBR} = \begin{pmatrix} [-1.5294, 2.7670] & [-4.1967, -1.4325] & [-4.7140, 0.3142] & [2.2624, 3.5553] \\ [-1.5069, 0.9532] & [1.9062, 3.5036] & [0.3160, 3.3011] & [-1.5867, -0.8078] \\ [2.7491, 4.9532] & [-1.1661, 0.4081] & [3.0713, 7.0560] & [6.9504, 11.4537] \\ [0.6279, 3.1046] & [-0.1246, 1.5772] & [4.1991, 14.2506] & [2.3859, 3.5227] \end{pmatrix}.$$

*Using the initial interval matrix*

$$\mathbf{X}^0 = \begin{pmatrix} [-1000, 1000] & [-1000, 1000] & [-1000, 1000] & [-1000, 1000] \\ [-1000, 1000] & [-1000, 1000] & [-1000, 1000] & [-1000, 1000] \\ [-1000, 1000] & [-1000, 1000] & [-1000, 1000] & [-1000, 1000] \\ [-1000, 1000] & [-1000, 1000] & [-1000, 1000] & [-1000, 1000] \end{pmatrix},$$

*the ITR method after $k = 15$ iterations obtains the following enclosure for the solutions of* (3.31)

$$\mathbf{X}^{ITR} = \begin{pmatrix} [0.9999, 1.0001] & [-2.0001, -1.9999] & [-0.0001, 0.0001] & [2.9999, 3.0001] \\ [-0.0001, 0.0001] & [2.9999, 3.0001] & [1.9999, 2.0001] & [-1.0001, -0.9999] \\ [3.9999, 4.0001] & [-0.0001, 0.0001] & [4.9999, 5.0001] & [8.9999, 9.0001] \\ [1.9999, 2.0001] & [0.9999, 1.0001] & [7.9999, 8.0001] & [2.9999, 3.0001] \end{pmatrix}.$$

*As one can see, the result of the ITR method is the best, and the result obtained by the MHBR method is slightly tighter than the one obtained by the MBS method. It is to be noted that the exact solution of the Sylvester-like AVE* (3.31) *is*

$$X = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 3 & 2 & -1 \\ 4 & 0 & 5 & 9 \\ 2 & 1 & 8 & 3 \end{pmatrix}.$$

Example 3.2. *Now consider the Sylvester-like AVE*

$$AXB + C|X|D = E, \tag{3.32}$$

*where*

$$A = \begin{pmatrix} -1 & 2 & -2 & 5 & 1 \\ 0 & 1 & 2 & -3 & 1 \\ 3 & 4 & -2 & 1 & 0 \\ 3 & -1 & 1 & 2 & 4 \\ 2 & -1 & 2 & -4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -4 & 0.5 & 0.2 & 0 \\ 1 & 0.2 & 2 & 0 & -0.3 \\ 0.4 & 0.5 & -0.2 & -1 & 0 \\ 0.2 & -3 & 0 & 0.5 & 0.1 \\ -2 & 0.2 & -0.1 & 0 & 0.2 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.02 & 0 & 0.03 & -0.01 & 0.03 \\ 0.04 & 0.03 & 0.02 & -0.01 & 0.01 \\ 0.05 & 0.02 & -0.01 & 0.03 & 0 \\ 0.01 & 0 & 0.03 & -0.01 & 0.02 \\ 0.01 & -0.03 & 0 & 0.01 & 0.02 \end{pmatrix}, \quad D = \begin{pmatrix} 0.08 & -0.01 & 0.01 & 0.02 & -0.02 \\ 0.02 & -0.03 & 0 & -0.02 & -0.03 \\ -0.01 & 0.02 & -0.03 & 0.04 & -0.02 \\ 0 & -0.01 & 0.02 & 0.05 & 0.03 \\ 0.05 & 0.03 & -0.02 & -0.01 & 0.02 \end{pmatrix},$$

$$E = \begin{pmatrix} -28.7946 & -34.3007 & -24.8029 & -7.3951 & 7.6966 \\ 42.4488 & 1.5960 & 20.7011 & 10.0162 & -5.9073 \\ 107.4475 & -121.7019 & -24.0977 & -6.6804 & 3.2983 \\ 119.0225 & -215.8017 & 44.3989 & 21.2112 & -0.0051 \\ 86.0087 & -54.5998 & 36.7997 & 13.9069 & -7.3026 \end{pmatrix}.$$

*We calculate*

$$\rho(|A^{-1}C|) = 0.1391, \quad \rho(|DB^{-1}|) = 0.6172, \quad \rho(|A^{-1}C|)\rho(|DB^{-1}|) = 0.0859,$$

*so that the assumptions are satisfied. The MBS method yields the following enclosure for the solutions of* (3.32)

$$\mathbf{X}^{MBS} =$$

$$\begin{pmatrix} [7.8482, 8.1889] & [-1.0445, -0.9628] & [0.9071, 1.0723] & [1.7403, 2.2089] & [-2.1929, -1.7770] \\ [1.8900, 2.1243] & [-3.0312, -2.9734] & [-0.0675, 0.0474] & [-2.1728, -1.8494] & [-3.1485, -2.8618] \\ [-1.5078, -0.4114] & [1.8588, 2.1258] & [-3.2934, -2.7537] & [3.1885, 4.7014] & [-2.6451, -1.3133] \\ [-0.2341, 0.2756] & [-1.0658, -0.9417] & [1.8644, 2.1160] & [4.6205, 5.3239] & [2.7064, 3.3236] \\ [4.5911, 5.3361] & [2.9165, 3.0960] & [-2.1673, -1.8034] & [-1.4648, -0.4384] & [1.5137, 2.4194] \end{pmatrix},$$

*and the MHBR method gives*

$$\mathbf{X}^{MHBR} =$$

$$\begin{pmatrix} [7.8398, 8.1974] & [-1.0374, -0.9699] & [0.9118, 1.0678] & [1.7443, 2.2201] & [-2.1855, -1.8052] \\ [1.8834, 2.1310] & [-3.0250, -2.9796] & [-0.0641, 0.0440] & [-2.1812, -1.8526] & [-3.2271, -2.8026] \\ [-1.5351, -0.3849] & [1.8836, 2.1012] & [-3.2802, -2.7756] & [3.0794, 5.1438] & [-2.6256, -1.5227] \\ [-0.2462, 0.2877] & [-1.0547, -0.9528] & [1.8714, 2.1103] & [4.2717, 5.7859] & [2.6264, 3.4812] \\ [4.5771, 5.3542] & [2.9323, 3.0804] & [-2.1585, -1.8153] & [-1.4878, -0.4654] & [1.6342, 2.4049] \end{pmatrix}.$$

Table 1
*Assumptions for Example 3.3*

| $m$ | $\rho(\lvert A^{-1}C\rvert)$ | $\rho(\lvert DB^{-1}\rvert)$ | $\rho(\lvert A^{-1}C\rvert)\rho(\lvert DB^{-1}\rvert)$ |
|---|---|---|---|
| 10 | 0.1000 | 0.0985 | 0.0098 |
| 20 | 0.1000 | 0.1810 | 0.0181 |
| 30 | 0.1000 | 0.2719 | 0.0272 |
| 40 | 0.1000 | 0.3647 | 0.0365 |
| 50 | 0.1000 | 0.4581 | 0.0458 |
| 60 | 0.1000 | 0.5517 | 0.0552 |
| 70 | 0.1000 | 0.6456 | 0.0646 |
| 80 | 0.1000 | 0.7395 | 0.0740 |
| 90 | 0.1000 | 0.8335 | 0.0834 |
| 100 | 0.1000 | 0.9276 | 0.0928 |
| 110 | 0.1000 | 1.0217 | 0.1022 |
| 120 | 0.1000 | 1.1158 | 0.1116 |
| 130 | 0.1000 | 1.2099 | 0.1210 |
| 140 | 0.1000 | 1.3041 | 0.1304 |
| 150 | 0.1000 | 1.3982 | 0.1398 |
| 160 | 0.1000 | 1.4923 | 0.1492 |
| 170 | 0.1000 | 1.5865 | 0.1586 |
| 180 | 0.1000 | 1.6807 | 0.1681 |
| 190 | 0.1000 | 1.7748 | 0.1775 |
| 200 | 0.1000 | 1.8690 | 0.1869 |

*If we apply the enclosure* $\mathbf{X}^{MBS}$ *obtained by the MBS method as an initial point for the sequence* (2.30), *then after* $k = 6$ *iterations we obtain*

$$\mathbf{X}^{ITR} =$$

$$\begin{pmatrix} [7.9999, 8.0001] & [-1.0001, -0.9999] & [0.9999, 1.0001] & [1.9999, 2.0001] & [-2.0001, -1.9999] \\ [1.9999, 2.0001] & [-3.0001, -2.9999] & [-0.0001, 0.0001] & [-2.0001, -1.9999] & [-3.0001, -2.9999] \\ [-1.0001, -0.9999] & [1.9999, 2.0001] & [-3.0001, -2.9999] & [3.9999, 4.0001] & [-2.0001, -1.9999] \\ [-0.0001, 0.0001] & [-1.0001, -0.9999] & [1.9999, 2.0001] & [4.9999, 5.0001] & [2.9999, 3.0001] \\ [4.9999, 5.0001] & [2.9999, 3.0001] & [-2.0001, -1.9999] & [-1.0001, -0.9999] & [1.9999, 2.0001] \end{pmatrix}.$$

*As one can see, the ITR method yields a tighter enclosure than the other ones. In fact, this method improves the enclosure obtained by any method. The exact solution of the Sylvester-like AVE* (3.32) *is*

$$X = \begin{pmatrix} 8 & -1 & 1 & 2 & -2 \\ 2 & -3 & 0 & -2 & -3 \\ -1 & 2 & -3 & 4 & -2 \\ 0 & -1 & 2 & 5 & 3 \\ 5 & 3 & -2 & -1 & 2 \end{pmatrix}.$$

Example 3.3. *Let us consider the Sylvester-like AVE*

$$AXB + C|X|D = E, \tag{3.33}$$

Table 2
*Results for Example 3.3*

| | Time | | | | Ratio | |
|---|---|---|---|---|---|---|
| $m$ | MBS | BS | MHBR | HBR | MBS | MHBR |
| 10 | 0.0026 | 0.0916 | 0.0115 | 0.4422 | 1 | 0.9609 |
| 20 | 0.0066 | 0.4634 | 0.0260 | 0.9763 | 1 | 0.9726 |
| 30 | 0.0070 | 1.6534 | 0.0399 | 3.3245 | 1 | 0.9966 |
| 40 | 0.0146 | 9.2573 | 0.1194 | 12.135 | 1 | 1.0075 |
| 50 | 0.0107 | 34.875 | 0.3163 | 29.268 | 1 | 1.0135 |
| 60 | 0.0215 | 111.16 | 0.5180 | 83.442 | 1 | 1.0176 |
| 70 | 0.0291 | 301.61 | 0.9940 | 202.51 | 1 | 1.0215 |
| 80 | 0.0284 | 650.39 | 1.4492 | 402.42 | 1 | 1.0228 |
| 90 | 0.0339 | 1432.1 | 2.9156 | 832.23 | 1 | 1.0224 |
| 100 | 0.0465 | 2843.2 | 4.6642 | 1723.4 | 1 | 1.0251 |
| 110 | 0.1180 | MO | 5.8970 | MO | 1 | 1.0247 |
| 120 | 0.1840 | MO | 7.1882 | MO | 1 | 1.0252 |
| 130 | 0.2034 | MO | 13.2465 | MO | 1 | 1.0262 |
| 140 | 0.2697 | MO | 16.5070 | MO | 1 | 1.0302 |
| 150 | 0.2654 | MO | 13.5688 | MO | 1 | 1.0284 |
| 160 | 0.1688 | MO | 28.8961 | MO | 1 | 1.0106 |
| 170 | 0.1790 | MO | 32.8483 | MO | 1 | 1.0099 |
| 180 | 0.2770 | MO | 29.0497 | MO | 1 | 1.0114 |
| 190 | 0.1916 | MO | 34.5732 | MO | 1 | 1.0109 |
| 200 | 0.2349 | MO | 47.8639 | MO | 1 | 1.0138 |

*where $A$, $B$, $C$, $D$, and $E$ are generated by the following Matlab functions*

```
alpha=10^-1; A=gallery('parter',m);
B=gallery('parter',m)-ones(m); C=alpha*gallery('parter',m);
D=alpha*gallery('parter',m); E=ones(m);
```

*We assess enclosures for the matrix equation (3.33) by different numerical methods and various values of $m$. The assumptions of our methods were satisfied; see Table 1. The computed numerical results are displayed in Table 2.*

*We see from Table 2 that our proposed methods MBS and MHBR are very much faster than their corresponding approaches, i.e., the BS and HBR methods when they are applied on transformed form of equation (3.33). This result is a confirmation on what is stated in Theorem 2.8. The MBS method needs lower running time than the MHBR method, while the MHBR method yields tighter enclosures than the MBS method. The BS and HBR methods fail from dimension $m = 110$ onward due to memory overflow.*

Example 3.4. *Consider the Sylvester-like AVE*

$$AXB + C|X|D = E, \tag{3.34}$$

TABLE 3
*Assumptions for Example 3.4*

| $m$ | $\rho(\lvert A^{-1}C\rvert)$ | $\rho(\lvert DB^{-1}\rvert)$ | $\rho(\lvert A^{-1}C\rvert)\rho(\lvert DB^{-1}\rvert)$ |
|---|---|---|---|
| 10 | 0.2631 | $2.2577 * 10^{-4}$ | $5.9393 * 10^{-5}$ |
| 20 | 0.3248 | $2.2696 * 10^{-4}$ | $7.3723 * 10^{-5}$ |
| 30 | 0.3618 | $2.2652 * 10^{-4}$ | $8.1952 * 10^{-5}$ |
| 40 | 0.3887 | $2.2600 * 10^{-4}$ | $8.7843 * 10^{-5}$ |
| 50 | 0.4100 | $2.2554 * 10^{-4}$ | $9.2476 * 10^{-5}$ |
| 60 | 0.4278 | $2.2514 * 10^{-4}$ | $9.6314 * 10^{-5}$ |
| 70 | 0.4431 | $2.2479 * 10^{-4}$ | $9.9604 * 10^{-5}$ |
| 80 | 0.4566 | $2.2449 * 10^{-4}$ | $1.0249 * 10^{-4}$ |
| 90 | 0.4686 | $2.2421 * 10^{-4}$ | $1.0507 * 10^{-4}$ |
| 100 | 0.4686 | $2.2421 * 10^{-4}$ | $1.0507 * 10^{-4}$ |
| 110 | 0.4895 | $2.2375 * 10^{-4}$ | $1.0953 * 10^{-4}$ |
| 120 | 0.4987 | $2.2355 * 10^{-4}$ | $1.1150 * 10^{-4}$ |
| 130 | 0.5073 | $2.2337 * 10^{-4}$ | $1.1332 * 10^{-4}$ |
| 140 | 0.5153 | $2.2320 * 10^{-4}$ | $1.1503 * 10^{-4}$ |
| 150 | 0.5229 | $2.2304 * 10^{-4}$ | $1.1663 * 10^{-4}$ |
| 160 | 0.5300 | $2.2290 * 10^{-4}$ | $1.1813 * 10^{-4}$ |
| 170 | 0.5367 | $2.2276 * 10^{-4}$ | $1.1956 * 10^{-4}$ |
| 180 | 0.5431 | $2.2264 * 10^{-4}$ | $1.2092 * 10^{-4}$ |
| 190 | 0.5492 | $2.2252 * 10^{-4}$ | $1.2221 * 10^{-4}$ |
| 200 | 0.5551 | $2.2240 * 10^{-4}$ | $1.2345 * 10^{-4}$ |

*in which $A$, $B$, $C$, $D$, and $E$ are produced as follows*

```
alpha=10^-4; A=10*gallery('lehmer',m)-ones(m);
B=gallery('lehmer',m)-ones(m); C=gallery('lehmer',m)-ones(m);
D=alpha*gallery('lehmer',m); E=5*ones(m);
```

*The enclosures for the Sylvester-like AVE (3.34) were computed by applying the BS, MBS, HBR, and MHBR methods for different values of $m$. The assumptions of our methods were satisfied, as observed in Table 3. The obtained results including the execution times and relative sums of radii with respect to the MBS method are displayed in Table 4.*

*The numbers presented in Table 4 show that the MBS and MHBR methods are much faster than the BS and HBR methods. Indeed, applying BS and HBR methods on the transformed form of the Sylvester-like AVE (3.34) needs $\mathcal{O}(m^6)$ arithmetic operations, which makes the problem prohibitive. The MHBR method gives slightly tighter enclosures than the MBS method; this is while it performs slower than the MBS method. The BS and HBR methods fail from $m = 110$ onward due to memory overflow.*

Now, we apply the ITR method to provide some enclosures for the solutions of the Sylvester-like AVEs (3.33) and (3.34), respectively, examined in Examples 3.3 and 3.4. For this purpose, we consider the enclosure obtained by the MBS method in each case as the initial point for the iterative technique introduced by formula (2.30). The results obtained together with the number of iterations "Iter" by executing the ITR method for

Table 4
*Results for Example 3.4*

| | Time | | | | Ratio | |
|---|---|---|---|---|---|---|
| $m$ | MBS | BS | MHBR | HBR | MBS | MHBR |
| 10 | 0.0030 | 0.0893 | 0.0073 | 0.5431 | 1 | 0.9999 |
| 20 | 0.0046 | 0.5364 | 0.0171 | 1.5481 | 1 | 0.9999 |
| 30 | 0.0045 | 1.9635 | 0.0205 | 5.2391 | 1 | 0.9999 |
| 40 | 0.0072 | 10.660 | 0.0463 | 15.314 | 1 | 0.9999 |
| 50 | 0.0124 | 37.726 | 0.0818 | 32.324 | 1 | 0.9999 |
| 60 | 0.0142 | 120.47 | 0.1671 | 91.314 | 1 | 0.9999 |
| 70 | 0.0208 | 314.88 | 0.3446 | 234.41 | 1 | 0.9999 |
| 80 | 0.0267 | 661.77 | 0.6544 | 441.82 | 1 | 0.9999 |
| 90 | 0.0345 | 1531.2 | 0.7648 | 876.34 | 1 | 0.9999 |
| 100 | 0.0436 | 2812.4 | 1.1004 | 1713.4 | 1 | 0.9999 |
| 110 | 0.1318 | MO | 1.4310 | MO | 1 | 0.9999 |
| 120 | 0.1365 | MO | 2.0773 | MO | 1 | 0.9999 |
| 130 | 0.1398 | MO | 2.6732 | MO | 1 | 0.9999 |
| 140 | 0.1655 | MO | 3.0248 | MO | 1 | 0.9999 |
| 150 | 0.1504 | MO | 3.7551 | MO | 1 | 0.9999 |
| 160 | 0.1939 | MO | 5.3897 | MO | 1 | 0.9999 |
| 170 | 0.1989 | MO | 5.9321 | MO | 1 | 0.9999 |
| 180 | 0.2245 | MO | 7.5383 | MO | 1 | 0.9999 |
| 190 | 0.2337 | MO | 9.7953 | MO | 1 | 0.9999 |
| 200 | 0.2732 | MO | 10.8863 | MO | 1 | 0.9999 |

different values of $m$ are displayed in Table 5. The running times also involve the times of the MBS method used for the initial enclosures. Figure 1 shows the log–log plot of the radius of the obtained enclosure by executing the ITR method for Examples 3.3 and 3.4 (for dimension $m = 50$) versus iteration $k$, respectively.

From the numbers displayed in Table 5, we see that the ITR method has greatly improved the enclosures obtained by the MBS method with very few repetitions in each case. Overall, the MBS and ITR methods are similar in terms of the execution times, and the MHBR method is slower among the other two methods. As it is shown in Fig. 1, increasing the number of repetitions, the radius of the obtained enclosures in both examples also significantly decreases.

**4. Concluding remarks.** In this paper, we investigated a Sylvester-like absolute value equation, a nonlinear matrix equation that contains several important classes of AVEs as special cases. By utilizing the transformed form of this equation and properties of the Kronecker product and vectorization operator, we proposed a modification to the Bauer–Skeel and Hansen–Bliek–Rohn bounds, significantly reducing the computational costs of enclosing solutions to the Sylvester-like AVE. This reduction in computational costs was illustrated through numerical examples. Furthermore, we proposed an iterative technique for enclosing the solutions, achieving significant improvement at the starting point with minimal computational cost. We proved convergence to the system's unique solution under a widely used unique solvability assumption.

Table 5
*Results of the ITR method for Examples 3.3 and 3.4*

| | Example 3.3 | | | Example 3.4 | | |
|---|---|---|---|---|---|---|
| $m$ | Time | Ratio | Iter | Time | Ratio | Iter |
| 10 | 0.0089 | $9.4745 * 10^{-5}$ | 5 | 0.0104 | $5.8914 * 10^{-5}$ | 2 |
| 20 | 0.0146 | $5.8664 * 10^{-6}$ | 4 | 0.0121 | $7.3233 * 10^{-5}$ | 2 |
| 30 | 0.0172 | $2.0215 * 10^{-5}$ | 4 | 0.0078 | $8.1484 * 10^{-5}$ | 2 |
| 40 | 0.0373 | $1.7896 * 10^{-6}$ | 5 | 0.0119 | $8.7395 * 10^{-5}$ | 2 |
| 50 | 0.0389 | $4.4703 * 10^{-6}$ | 5 | 0.0186 | $9.2044 * 10^{-5}$ | 2 |
| 60 | 0.0551 | $9.4278 * 10^{-6}$ | 5 | 0.0273 | $9.5896 * 10^{-5}$ | 2 |
| 70 | 0.0894 | $1.7690 * 10^{-5}$ | 5 | 0.0401 | $9.9197 * 10^{-5}$ | 2 |
| 80 | 0.2832 | $2.2546 * 10^{-6}$ | 6 | 0.0386 | $1.0209 * 10^{-4}$ | 2 |
| 90 | 0.2729 | $4.1039 * 10^{-6}$ | 6 | 0.0868 | $1.0468 * 10^{-4}$ | 2 |
| 100 | 0.1988 | $7.0069 * 10^{-6}$ | 6 | 0.0841 | $1.0702 * 10^{-4}$ | 2 |
| 110 | 0.2728 | $1.1360 * 10^{-5}$ | 6 | 0.1666 | $1.0915 * 10^{-4}$ | 2 |
| 120 | 0.3694 | $1.9694 * 10^{-6}$ | 7 | 0.1560 | $1.1113 * 10^{-4}$ | 2 |
| 130 | 0.3831 | $3.2017 * 10^{-6}$ | 7 | 0.2044 | $1.1295 * 10^{-4}$ | 2 |
| 140 | 0.4589 | $5.0193 * 10^{-6}$ | 7 | 0.2138 | $1.1466 * 10^{-4}$ | 2 |
| 150 | 0.4857 | $7.6254 * 10^{-6}$ | 7 | 0.1945 | $1.1627 * 10^{-4}$ | 2 |
| 160 | 0.4772 | $1.6830 * 10^{-6}$ | 8 | 0.2433 | $1.1778 * 10^{-4}$ | 2 |
| 170 | 0.5263 | $2.5825 * 10^{-6}$ | 8 | 0.2659 | $1.1921 * 10^{-4}$ | 2 |
| 180 | 0.6961 | $3.8663 * 10^{-6}$ | 8 | 0.3002 | $1.2057 * 10^{-4}$ | 2 |
| 190 | 0.6527 | $5.6622 * 10^{-6}$ | 8 | 0.3098 | $1.2187 * 10^{-4}$ | 2 |
| 200 | 0.8241 | $1.5196 * 10^{-6}$ | 9 | 0.3577 | $1.2311 * 10^{-4}$ | 2 |

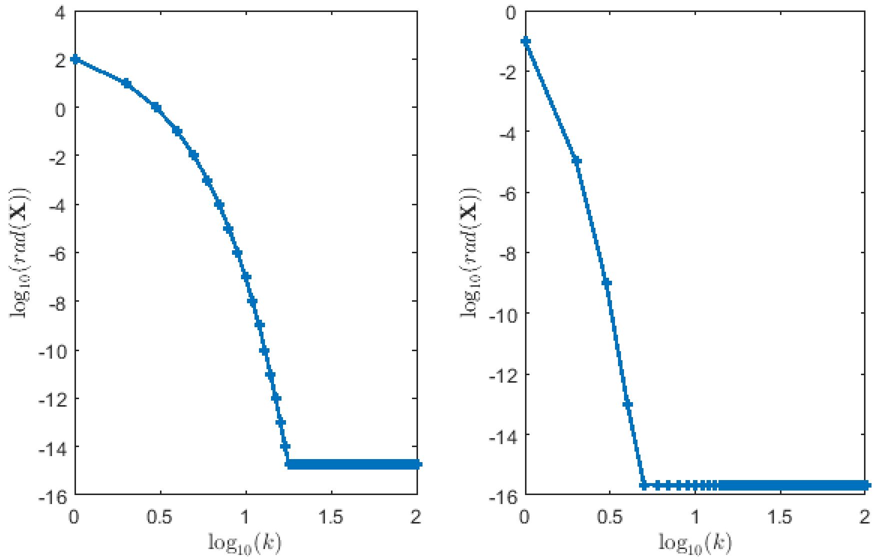

Figure 1. *Log–log scale of the radius of the enclosure obtained by the ITR method versus iteration $k$; on the left-hand side is Example 3.3, and on the right-hand side is Example 3.4.*

## REFERENCES

[1] M. Dehghani-Madiseh and M. Hladík. Efficient approaches for enclosing the united solution set of the interval generalized Sylvester matrix equation. *Appl. Numer. Math.*, 126:18–33, 2018.

[2] M. Dehghani-Madiseh and M. Hladík. Enclosing the solution set of the parametric generalised Sylvester matrix equation $A(p)XB(p) + C(p)XD(p) = F(p)$. *Int. J. Syst. Sci.*, 50(11):2153–2167, 2019.

[3] M. Dehghani-Madiseh. An efficient approach for enclosing the solution set of the interval coupled Sylvester matrix equations. *Linear Multilinear Algebra*, 72(13):2149–2168, 2024.

[4] H. Esmaeili, E. Mahmoodabadi, and M. Ahmadi. A uniform approximation method to solve absolute value equation. *Bull. Iran. Math. Soc.*, 41(5):1259–1269, 2015.

[5] F.K. Haghani. On generalized traubs method for absolute value equations. *J. Optim. Theory Appl.*, 166(2):619–625, 2015.

[6] B. Hashemi. Sufficient conditions for the solvability of a Sylvester-like absolute value matrix equation. *Appl. Math. Lett.*, 112:106818, 2021.

[7] B. Hashemi and M. Dehghan. Results concerning interval linear systems with multiple righthand sides and the interval matrix equation AX = B. *J. Comput. Appl. Math.*, 235(9):2969–2978, 2011.

[8] B. Hashemi and M. Dehghan. The interval Lyapunov matrix equation: Analytical results and an efficient numerical technique for outer estimation of the united solution set. *Math. Comput. Model.*, 55(3–4):622–633, 2012.

[9] M. Hladík. Bounds for the solutions of absolute value equations. *Comput. Optim. Appl.*, 69(1):243–266, 2018.

[10] M. Hladík. Properties of the solution set of absolute value equations and the related matrix classes. *SIAM J. Matrix Anal. Appl.*, 44(1):175–195, 2023.

[11] M. Hladík and H. Moosaei. Some notes on the solvability conditions for absolute value equations. *Optim. Lett.*, 17(1):211–218, January 2023.

[12] M. Hladík, H. Moosaei, F. Hashemi, S. Ketabchi, and P.M. Pardalos. An overview of absolute value equations: From theory to solution methods and challenges. *Comput. Optim. Appl.*, 2025. in press, DOI: 10.1007/s10589-025-00717-5.

[13] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[14] S. Hu and Z. Huang. A note on absolute value equations. *Optim. Lett.*, 4(3):417–424, 2010.

[15] X. Jiang and Y. Zhang. A smoothing-type algorithm for absolute value equations. *J. Ind. Manag. Optim.*, 9(4):789–798, 2013.

[16] S. Ketabchi and H. Moosaei. An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side. *Comput. Math. Appl.*, 64(6):1882–1885, 2012.

[17] S. Kumar, Deepmala, M. Hladík, and H. Moosaei. Characterization of unique solvability of absolute value equations: An overview, extensions, and future directions. *Optim. Lett.*, 18(4):889–907, 2024.

[18] C. Li. Sufficient conditions for the unique solution of a new class of Sylvester-like absolute value equations. *J. Optim. Theory Appl.*, 195(2):676–683, 2022.

[19] O.L. Mangasarian. Absolute value programming. *Comput. Optim. Appl.*, 36(1):43–53, 2007.

[20] O.L. Mangasarian. A generalized newton method for absolute value equations. *Optim. Lett.*, 3(1):101–108, 2009.

[21] O.L. Mangasarian. Absolute value equation solution via linear programming. *J. Optim. Theory Appl.*, 161(3):870–876, 2014.

[22] O.L. Mangasarian. Sufficient conditions for the unsolvability and solvability of the absolute value equation. *Optim. Lett.*, 11(7):1469–1475, 2017.

[23] O.L. Mangasarian and R.R. Meyer. Absolute value equations. *Linear Algebra Its Appl.*, 419(2–3):359–367, 2006.

[24] F. Mezzadri. On the solution of general absolute value equations. *Appl. Math. Lett.*, 107:106462, 2020.

[25] S. Mollahasani and F.P.A. Beik. Absolute value equations with tensor product structure: Unique solvability and numerical solution. *Appl. Math.*, 67(5):657–674, 2022.

[26] H. Moosaei, S. Ketabchi, M.A. Noor, J. Iqbal, and V. Hooshyarbakhsh. Some techniques for solving absolute value equations. *Appl. Math. Comput.*, 268:696–705, 2015.

[27] M.A. Noor, J. Iqbal, K.I. Noor, and E.A. Al-Said. On an iterative method for solving absolute value equations. *Optim. Lett.*, 6(5):1027–1033, 2012.

[28] O. Prokopyev. On equivalent reformulations for absolute value equations. *Comput. Optim. Appl.*, 44(3):363–372, 2009.

[29] J. Rohn. Forty necessary and sufficient conditions for regularity of interval matrices: A survey. *Electron. J. Linear Algebra*, 18:500–512, 2009.

[30] J. Rohn, V. Hooshyarbakhsh, and R. Farhadsefat. An iterative method for solving absolute value equations and sufficient conditions for unique solvability. *Optim. Lett.*, 8(1):35–44, 2014.

[31] S.M. Rump. INTLAB – INTerval LABoratory. In: T. Csendes (editor), *Developments in Reliable Computing.* Kluwer Academic Publishers, Dordrecht, 77–104, 1999. https://www.tuhh.de/ti3/rump/intlab/.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 39-57, February 2026.

[32] W.-L. Tang and S.-X. Miao. On the solvability and Picard-type method for absolute value matrix equation. *Comp. Appl. Math.*, 41:78, 2022.

[33] A. Wang, H. Wang, and Y. Deng. Interval algorithm for absolute value equations. *Cent. Eur. J. Math.*, 9(5):1171–1184, 2011.

[34] L.M. Wang and C.X. Li. New sufficient conditions for the unique solution of a square Sylvester-like absolute value equation. *Appl. Math. Lett.*, 116:106966, 2021.

[35] S. Wu and S. Shen. On the unique solution of the generalized absolute value equation. *Optim. Lett.*, 15(1):1–8, 2021.

[36] M. Zamani and M. Hladík. A new concave minimization algorithm for the absolute value equation solution. *Optim. Lett.*, 15(6):2241–2254, 2021.