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COMPUTING STANDARD CANONICAL FORMS OF REGULAR LINEAR TIME VARYING DAES VIA A PRELIMINARY STAGE*

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Abstract. For regular linear time-invariant differential-algebraic equations (DAEs), the corresponding matrix pencil is regular, and the computation of a standard canonical form (SCF) is well understood. Although the investigation of linear DAEs with time-varying coefficients is more complex, it is analogously related to the examination of pairs of time-dependent matrix functions. We show how the computation of an SCF for linear time-varying DAEs becomes possible if a suitable block structure of the pair of matrix functions is found in a preliminary stage. Starting from this preliminary stage, an iterative process delivers an SCF. This iteration terminates in finitely many steps due to the nilpotent structure of the involved matrix functions. The corresponding transformation matrix functions can be provided systematically, which leads also to new representations of the canonical subspaces and projectors related to the original DAE. We demonstrate how the structured canonical forms resulting in the tractability and strangeness frameworks and some DAEs in Hessenberg form from applications can be transformed into this preliminary stage and discuss some examples.

Key words. Pairs of time-varying matrix functions, Differential-algebraic equation, Equivalence transformations, Regularity, Nilpotency, Higher index, Canonical form, Block structures, Hessenberg form.

AMS subject classifications. 15A99, 34A09, 34A12, 34A30.

1. Introduction.

We consider linear differential-algebraic equations (DAEs) in the standard form,

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{1.1}$$

given by the ordered pair $\{E, F\}$ of sufficiently smooth, at least continuous, matrix functions $E, F : \mathcal{I} \to \mathbb{R}^{m\times m}$, $\mathcal{I} \subseteq \mathbb{R}$ an interval, and suppose that $E(t)$ is singular for all $t \in \mathcal{I}$. Solutions are continuously differentiable functions $x(\cdot)$ that satisfy the DAE pointwise on the specified interval. We focus on equivalence transformations of the DAE (1.1) and the associated ordered pair $\{E, F\}$ into several canonical forms revealing explicitly inherent structures of (1.1) to some extent.

The question of particularly favorable equivalent representations, that is, the transformability of DAEs into canonical forms, plays its role from the beginning of the DAE theory.

The most popular form is the so-called *standard canonical form* (SCF), cf. [6], [4]: A pair has SCF if

$$\left\{\begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix}\right\}, \quad \Omega : \mathcal{I} \to \mathbb{R}^{d\times d}, \quad N : \mathcal{I} \to \mathbb{R}^{a\times a}, \quad d \geq 0, \quad m = d + a, \tag{1.2}$$

where N is strictly upper (or lower) triangular. If in addition N is constant, then the system is in *strong standard canonical form* (SSCF). In this article, we use the upper triangular formulation only.

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In the SCF (1.2), the upper part may be absent ($d = 0$). In the lower part, the matrix function N is pointwise nilpotent and may undergo rank changes.

Recently, in [17], the mostly used DAE approaches are investigated and compared with focus on all their individual constant-rank conditions supporting the constructions. As common ground, by the Equivalence Theorem [17, Theorem 8.1], it is pointed out that regularity of (1.1) on the given arbitrary interval $\mathcal{I} \subseteq \mathbb{R}$ formally means the existence of an index $\mu \in \mathbb{N}$ and constant so-called canonical characteristics

$$\begin{aligned} &r < m,\ \theta_0 = 0, && \text{for } \mu = 1,\\ &r < m,\ \theta_0 \geq \cdots \geq \theta_{\mu-2} > \theta_{\mu-1} = 0, && \text{for } \mu > 1, \end{aligned} \tag{1.3}$$

that can be expressed in terms of each of the considered approaches. In particular, $r := \operatorname{rank} E$ is constant and for an SCF the nilpotent matrix N as well as the powers of N may not undergo rank changes. Moreover, in [18], it is succeeded to verify the [17, Conjecture 8.2]. This implies: The DAE (1.1) is regular with the characteristics (1.3) if and only if it is equivalently transformable into SSCF with the same characteristics.

In the end, with no restriction of generality, we may suppose that the constant matrix N of the SSCF has a certain elementary block structure that is fully determined by the canonical characteristics only [18]. This insight forms the basis of the algorithm presented here.

The paper is organized as follows:

In Section 2, we recompile and reinterpret many results from literature that are fundamental for our approach, in particular a definition of regularity based on transformability into SSCF and an according definition of the canonical subspaces related to DAEs. Moreover, we discuss the differences between our approach and some approaches from literature to compute SCFs.

In Section 3, we introduce particular block structured upper triangular matrix functions as well as specific constant elementary nilpotent matrices with a block structural pattern that depends on the canonical characteristics only.

Based on this block structure, in Section 4, we introduce a new canonical form quite like an SCF, the so-called *Quasi*SCF, for later considerations. A *Quasi*SCF is equivalent to an SCF in case of regularity and will be an intermediate step in our computations.

These previously mentioned forms allow us to describe the particular assumptions for the block structure we use in Section 5, where an appropriate starting pair for the iterative main SCF-procedure in the next section is provided. This starting pair can be seen as a preliminary stage of the SCF and is referred to as *Pre*SCF.

In Section 6, then a procedure is described that, starting from such a block-structured preliminary stage, computes an SSCF step-by-step via a *Quasi*SCF. Through a more detailed examination of the transformation matrices used in the steps, a description of the canonical subspaces and the canonical projector is presented in Section 7.

To emphasize the applicability of the procedure, first in Section 8, we show how the T-canonical and S-canonical forms that arise from the projector-based decoupling associated to the tractability framework and the reduction technique associated to the strangeness framework, respectively, can equivalently be transformed into the preliminary stage of an SCF. Second, in Section 9, we show how classes of DAEs in Hessenberg form that are relevant in applications can also be transformed into such a preliminary stage. Some technical details and challenging examples from literature are discussed in the appendix.

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The computations for all examples were carried out using the Python library SymPy for symbolic mathematics. The used prototype can be found at https://github.com/d-estevez/Compute_QuasiSCF_DAE.

List of symbols and abbreviations.

ODE	ordinary differential equation
IVP	initial value problem
DAE	differential-algebraic equation
SCF	standard canonical form
SSCF	strong SCF, i.e., SCF with constant nilpotent matrix N
QuasiSCF	particular block structure with nilpotent matrix $N^{(E_{c/r})}$ and $R \in BUT_{nonsingular}$
PreSCF	preliminary stage of a QuasiSCF (and therefore SCF and SSCF)
I_n	identity matrix of size n
$\mathcal{I} \subseteq \mathbb{R}$	interval
$m \in \mathbb{N}$	dimension of the DAE
$d \in \mathbb{N}$	degree of freedom, dimension of pure ODE part
$a = m - d \in \mathbb{N}$	dimension of pure DAE part
$\ker A$, $\operatorname{im} A$	kernel, image of a matrix valued function $A : \mathcal{I} \to \mathbb{R}^{m\times m}$
$\{E, F\}$	ordered pair of quadratic matrix valued functions $E, F : \mathcal{I} \to \mathbb{R}^{m\times m}$
$r \in \mathbb{N}$	rank of E
BUT	set of all block upper triangular $B : \mathcal{I} \to \mathbb{R}^{a\times a}$
$BUT_{nonsingular}$	set of all nonsingular $R \in BUT$
$N : \mathcal{I} \to \mathbb{R}^{a\times a}$	nilpotent matrix valued function
SUT	set of all block strictly upper triangular $N : \mathcal{I} \to \mathbb{R}^{a\times a}$
SUT_{column}	set of all $N \in SUT$ with full column rank blocks $N_{i,i+1}$
SUT_{row}	set of all $N \in SUT$ with full row rank blocks $N_{i,i+1}$
$N^{(C)} \in \mathbb{R}^{a\times a}$	constant strictly upper (and therefore nilpotent) matrix
$N^{(E_c)} \in \mathbb{R}^{a\times a}$	elementary constant nilpotent matrix from SUT_{column}
$N^{(E_r)} \in \mathbb{R}^{a\times a}$	elementary constant nilpotent matrix from SUT_{row}
$N^{(E_{c/r})} \in \mathbb{R}^{a\times a}$	either $N^{(E_c)}$ or $N^{(E_r)}$
S_{can}	canonical flow-subspace of the DAE
N_{can}	canonical complement to the flow-subspace
Π_{can}	projector onto S_{can} and along N_{can}

2. Fundamentals. Since we build on many concepts and results from literature, we begin with a compilation of the necessary background, interpreted for our purposes here.

2.1. Equivalence transformations. Let us first recall what equivalence means, see, e.g., [4], [14], [13], among others. The DAE (1.1) and the DAE

$$\bar{E}\bar{x}' + \bar{F}\bar{x} = \bar{q}, \tag{2.4}$$

and correspondingly the two pairs of matrix functions $\{E, F\}$ and $\{\bar{E}, \bar{F}\}$ are called *equivalent*, if there exist pointwise nonsingular matrix functions $L \in \mathcal{C}(\mathcal{I}, \mathbb{R}^{m\times m}), K \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{m\times m})$, such that

$$\bar{E} = LEK, \quad \bar{F} = LFK + LEK'. \tag{2.5}$$

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The second summand of $\bar{F}$, which is absent for constant K, makes the computation of an SCF in the general time-varying case considerably more challenging than for matrix pencils.

For equivalent pairs of matrix functions, we use the notation $\{E,F\} \sim \{\bar{E},\bar{F}\}$ (or $\{E,F\} \overset{L,K}{\sim} \{\bar{E},\bar{F}\}$ if we want to emphasize the used nonsingular transformation matrix functions), i.e., by definition it holds

$$\{E,F\} \quad \sim \quad \{LEK\ ,\ LFK+LEK'\}.$$

Recall that these transformations are reflexive, symmetric, and transitive. Moreover, for the DAE, they correspond to

- the premultiplication of (1.1) by L and
- and the transformation of the unknown function $x=K\bar{x}$

leading to

$$LE(K\bar{x})'+LF(K\bar{x})=Lq=:\bar{q},$$

that with (2.5) corresponds to the DAE (2.4).

2.2. Regularity. According to Equivalence Theorem [17, Theorem 8.1], regularity of (1.1) on the given arbitrary interval $\mathcal{I}\subseteq\mathbb{R}$ means the existence of an constant index $\mu\in\mathbb{N}$ and constant so-called canonical characteristics (1.3) that can be computed within the framework of various index concepts like differentiation, geometric, strangeness, tractability, among others. These canonical characteristics do not change under equivalence transformations.

Thereby, $r=\operatorname{rank} E(t), t\in\mathcal{I}$, that is, the leading coefficient has constant rank. The other characteristics $\theta_i\in\mathbb{N}$ stand for constant ranks of individual matrix functions associated with different approaches. Here is no place to go into detail since each of the approaches requires technical effort[1].

In terms of transferability into SCF, according to Equivalence Theorem [17, Theorem 8.1], the meaning of regularity is easier to formulate since the canonical characteristics (1.3) persist under equivalent transformations:

The DAE (1.1) is regular with the characteristics (1.3) if and only if it is transformable into SCF (2.4) in which all powers of N feature constant ranks, and

$$\begin{aligned}
&\operatorname{rank} N(t)^i=\sum_{k=i-1}^{\mu-2}\theta_k, \quad i=2,\dots,\mu, \quad && \textit{if } \mu\geq 2,\\
&\operatorname{rank} N(t)=0, && \textit{if } \mu=1.
\end{aligned}$$

Moreover in [18], it is succeeded to verify the [17, Conjecture 8.2]. This implies:

The DAE (1.1) is regular with the characteristics (1.3) if and only if it is transformable into SSCF with the same characteristics.

We use this result to establish formally the definition of regularity that we use in this article, although we emphasize once again that there are several equivalent definitions, see [17, Theorem 8.1], [18].

[1] We refer to [17, Section 8] for all the detailed formal definition as well as the phenomenological background

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Definition 2.1. *A pair $\{E, F\}$ is regular on an interval $\mathcal{I}$ iff there exist constant index μ and canonical characteristics*

$$r < m,\ \theta_0 \geq \cdots \geq \theta_{\mu-2} > \theta_{\mu-1} = 0,\ d = r - \sum_{i=0}^{\mu-2} \theta_i, \tag{2.6}$$

such that on $\mathcal{I}$ the pair $\{E, F\}$ is equivalent to an SSCF with a constant, nilpotent matrix $N^{(C)}$ of index μ and

$$\theta_i = \operatorname{rank} \left(N^{(C)}\right)^{i+1} - \operatorname{rank} \left(N^{(C)}\right)^{i+2}, \quad i = 0, \ldots, \mu - 2, \quad d = r - \sum_{i=0}^{\mu-2} \theta_i,$$

that is

$$\operatorname{rank} \left(N^{(C)}\right) = r - d, \quad \operatorname{rank} \left(N^{(C)}\right)^i = r - d - \sum_{k=0}^{i-2} \theta_k = \sum_{k=i-1}^{\mu-2} \theta_k, \quad i = 2, \ldots, \mu.$$

According to [18], this constant matrix $N^{(C)}$ can be assumed to have a certain elementary block-structure that is determined by the canonical characteristics only. We discuss this in Section 3 in more detail.

Remark 2.2. *It should be noted that different concepts of regularity are used in the DAE literature. In this regard, the comprehensive analysis [17] can be seen as an attempt to organize this terminology. We refer in particular to the arguments in [17, Sections 8.2 and 8.4] and the examples in [17, Chapter 7]. Strictly speaking, the above introduced regularity notion is restricted to open intervals*[2]*. If a DAE is regular on an interval $\mathcal{I}$, then this interval consists of so-called regular points only. If regularity is not given, the index and the characteristics may be different on different subintervals. In our opinion, any change in the index μ, the rank r, or a characteristic value θ_i indicates a critical point, which on the one hand may be associated with a singular flow, but on the other hand may merely be a so-called harmless critical point, see, e.g., [17], [16].*

If a DAE can be transformed into SCF over the entire interval, and there is then a change in rank in an $N(t)^i$, the points of change are only harmless critical points. In the terminology of [17], DAEs that can be transformed into SCF belong to the class of almost regular DAEs.

2.3. The canonical subspaces of a regular DAE. As pointed out, e.g., in [11], [12], there are two continuously time-varying canonical subspaces related to a DAE (1.1). The first is the so-called flow subspace

$$S_{can}(\bar{t}) = \Big\{ \bar{x} \in \mathbb{R}^m : \text{there are } \delta > 0 \text{ and a solution } x : (\bar{t} - \delta, \bar{t} + \delta) \cap \mathcal{I} \to \mathbb{R}^m \text{ of the homogeneous DAE such that } x(\bar{t}) = \bar{x} \Big\},$$

which is the set of consistent values for the homogeneous DAE. Here, we assume continuously differentiable solutions that satisfy the DAE pointwise.

Due to constraints, $\dim S_{can}(\bar{t}) \leq r < m$. For regular DAEs, the flow subspace S_{can} is varying in $\mathbb{R}^m$ depending on t and has constant dimension d. The second subspace $N_{can}(\bar{t})$ is its so-called canonical complement in $\mathbb{R}^m$, which plays its role in the formulation of accurate initial conditions. The initial condition

$$x(\bar{t}) - \bar{x} \in N_{can}(\bar{t}),$$

[2] If the interval is not open, for border points additional considerations with one-sided limits have to be taken into account.

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ensures uniquely solvable IVPs for (1.1). N_{can} may also vary in $\mathbb{R}^m$ depending on t and has constant dimension a. In contrast to DAEs, an explicit ODE would yield $S_{can} = \mathbb{R}^m$, $N_{can} = \{0\}$ for $t \in \mathcal{I}$. Conversely, so-called pure DAEs yield $S_{can} = \{0\}$, $N_{can} = \mathbb{R}^m$ for $t \in \mathcal{I}$.

Below, we describe both subspaces by means of SSCFs. Notice that if $\{E, F\}$ is already in SSCF, then obviously

$$S_{can} = \operatorname{im} \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \quad N_{can} = \operatorname{im} \begin{bmatrix} 0 \\ I_a \end{bmatrix}.$$

As mentioned before, transformability into SSCF is an equivalent criterion of regularity. Let us for a moment suppose that L, K are nonsingular matrix functions such that

$$\{E, F\} \overset{L,K}{\sim} \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(C)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\}, \tag{2.7}$$

with a constant strictly upper triangular (and therefore nilpotent) matrix $N^{(C)} \in \mathbb{R}^{a\times a}$ and a matrix function $\Omega : \mathcal{I} \to \mathbb{R}^{d\times d}$. For

$$\begin{bmatrix} \bar{q}_1(t) \\ \bar{q}_2(t) \end{bmatrix} := L(t)q(t), \quad \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} := K^{-1}(t)x(t),$$

we see that the transformed DAE decouples into two parts.

- The first part,
$$u' + \Omega u = \bar{q}_1(t), \tag{2.8}$$
is an explicit ODE living in $\mathbb{R}^d$.
If at $t_0 \in \mathcal{I}$ an initial value $u_0 \in \mathbb{R}^d$ is given, $u(\cdot)$ can be computed as the unique solution of the IVP.
- The second part,
$$N^{(C)} v' + v = \bar{q}_2(t),$$
is a so-called pure DAE, and v is uniquely determined by
$$v(t) = \sum_{i=0}^{\mu-1} \left(-N^{(C)}\right)^i (\bar{q}_2)^{(i)}(t), \quad t \in \mathcal{I}.$$

Consequently, all solutions of the original DAE (1.1) are given by

$$x(t) = K(t) \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad t \in \mathcal{I},$$

and, in particular,

$$x(t_0) = K(t_0) \begin{bmatrix} u_0 \\ 0 \end{bmatrix} + K(t_0) \begin{bmatrix} 0 \\ v(t_0) \end{bmatrix} = K(t_0) \begin{bmatrix} u_0 \\ v(t_0) \end{bmatrix}, \quad u_0 \in \mathbb{R}^d.$$

Using the fundamental matrix $U(t, t_0)$ of the explicit pure ODE (2.8) normalized at $t_0 \in \mathcal{I}$, we arrive at the solution representation of the original DAE

$$x(t) = K(t) \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = K(t) \begin{bmatrix} U(t, t_0)(c + \int\limits_{t_0}^{t} U(s, t_0)\bar{q}_1(s)ds) \\ \sum\limits_{i=0}^{\mu-1} \left(-N^{(C)}\right)^i \bar{q}_2^{(i)}(t) \end{bmatrix},$$

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in which $c \in \mathbb{R}^d$ is the free integration constant. Obviously, if $q \equiv 0$, then $x(t) \in \operatorname{im} K(t) \begin{bmatrix} I_d \\ 0 \end{bmatrix} = S_{can}(t)$. In contrast, for arbitrary given $x_0 \in \mathbb{R}^m$, the initial condition

$$x(t_0) = x_0, \tag{2.9}$$

implies the following restrictions concerning x_0 and q:

$$K(t_0) \begin{bmatrix} c \\ \sum_{i=0}^{\mu-1} ((-N^{(C)})^i \bar{q}_2^{(i)}(t_0)) \end{bmatrix} = x_0, \quad \text{i.e.,} \quad K^{-1}(t_0) x_0 = \begin{bmatrix} c \\ \sum_{i=0}^{\mu-1} ((-N^{(C)})^i \bar{q}_2^{(i)}(t_0)) \end{bmatrix}.$$

The integration constant $c \in \mathbb{R}^d$ can be chosen to satisfy the first d entries of $K^{-1}(t_0)x_0$. However, the other part requires an extra restriction concerning x_0 and q, otherwise solvability is lost.

An accurate initial condition should fix exactly the free constant $c \in \mathbb{R}^d$, that is

$$x(t_0) - x_0 \in \operatorname{im} K(t_0) \begin{bmatrix} 0 & \\ & I_a \end{bmatrix} = N_{can}(t_0) \quad \text{or, equivalently,} \quad K(t_0) \begin{bmatrix} I_d & \\ & 0 \end{bmatrix} K(t_0)^{-1}(x(t_0) - x_0) = 0.$$

The projector valued function $K \begin{bmatrix} I_d & \\ & 0 \end{bmatrix} K^{-1}$ is nothing else than the canonical projector introduced as completely decoupling projector function Π_{can} in the context of the projector-based analysis, see, e.g., [14, Definition 2.37].

The above considerations allow the following simpler definition that will be used in this article.

Definition 2.3. *Let the pair $\{E, F\}$ be regular on the interval $\mathcal{I}$ and let L, K be equivalence transformation matrix functions that provide an SSCF. Then, we define the canonical subspaces*

$$S_{can}(t) := \operatorname{im} K(t) \begin{bmatrix} I_d \\ 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & I_a \end{bmatrix} K^{-1}(t) \subset \mathbb{R}^m,$$

$$N_{can}(t) := \operatorname{im} K(t) \begin{bmatrix} 0 \\ I_a \end{bmatrix} = \ker \begin{bmatrix} I_d & 0 \end{bmatrix} K^{-1}(t) \subset \mathbb{R}^m,$$

with $S_{can}(t) \oplus N_{can}(t) = \mathbb{R}^m, \quad t \in \mathcal{I}$ and the time-varying canonical projector

$$\mathit{\Pi}_{can}(t) := K(t) \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} K^{-1}(t),$$

fulfilling $\ker \mathit{\Pi}_{can} = N_{can}, \quad \operatorname{im} \mathit{\Pi}_{can} = S_{can}$.

Remark 2.4. *Note that in general, if $\{E, F\}$ is regular and $\{E, F\} \overset{L,K}{\sim} \{\bar{E}, \bar{F}\}$, then*

$$S_{can} = K\bar{S}_{can}, \quad \text{and} \quad N_{can} = K\bar{N}_{can}.$$

2.4. About uniqueness. The uniqueness of the canonical subspaces S_{can} and N_{can}, and therefore also of the canonical projector $\mathit{\Pi}_{can}$, can be recognized using results for the SCF.

For all linear DAEs, even if they are not regular but transformable into SCF[3], the SCF is unique in the sense that the dimensions of the ODE and the pure DAE are unique, and that both parts are unique up to equivalence transformations.

[3]cf. Remark 2.2.

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THEOREM 2.5. *[2, Theorem 2.1]*[4] *[Uniqueness of SCF] Let* $d, a, \tilde{d}, \tilde{a} \in \mathbb{N}_0$, $m = d + a = \tilde{d} + \tilde{a}$, *and let*

$$\Omega \in C(\mathcal{I}, \mathbb{R}^{d\times d}), \quad \tilde{\Omega} \in C(\mathcal{I}, \mathbb{R}^{\tilde{d}\times\tilde{d}}), \quad \textit{and} \quad N \in C(\mathcal{I}, \mathbb{R}^{a\times a}), \quad \tilde{N} \in C(\mathcal{I}, \mathbb{R}^{\tilde{a}\times\tilde{a}}),$$

where both, N *and* $\tilde{N}$, *are pointwise strictly lower or strictly upper triangular matrix functions. If there exist nonsingular*

$$L \in C(\mathcal{I}, \mathbb{R}^{m\times m}), \quad K \in C^1(\mathcal{I}, \mathbb{R}^{m\times m}),$$

such that

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\} \overset{L,K}{\sim} \left\{ \begin{bmatrix} I_{\tilde{d}} & 0 \\ 0 & \tilde{N} \end{bmatrix}, \begin{bmatrix} \tilde{\Omega} & 0 \\ 0 & I_{\tilde{a}} \end{bmatrix} \right\},$$

then:

(i) $d = \tilde{d}$, $a = \tilde{a}$,

(ii) $L = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix}$, $K = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$, *with* $L_{11} = K_{11}^{-1}, \quad L_{22} = (K_{22} + NK_{22}')^{-1}$,

(iii) $\{I_d, \Omega\} \overset{K_{11}^{-1},K_{11}}{\sim} \left\{I_d, \tilde{\Omega}\right\}, \quad \{N, I_a\} \overset{L_{22},K_{22}}{\sim} \left\{\tilde{N}, I_a\right\}$.

Proof. A proof for strictly lower triangular N was given in [2]. For strictly upper triangular N, it follows with permutation matrices that invert the order of rows/columns. ∎

The canonical subspaces, which are unique, may be represented in different ways, i.e., using different bases. Furthermore, for a DAE, there is neither a unique ODE in $\mathbb{R}^d$ nor a unique pure DAE.

COROLLARY 2.6. *For* $m, a, d \in \mathbb{N}$, $m = d + a$, $E, F, L, \bar{L} \in C(\mathcal{I}, \mathbb{R}^{m\times m})$, $K, \bar{K} \in C^1(\mathcal{I}, \mathbb{R}^{m\times m})$,

$$\{E, F\} \overset{L,K}{\sim} \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\} \overset{\bar{L},\bar{K}}{\sim} \left\{ \begin{bmatrix} I_d & 0 \\ 0 & \tilde{N} \end{bmatrix}, \begin{bmatrix} \tilde{\Omega} & 0 \\ 0 & I_a \end{bmatrix} \right\},$$

in witch $\Omega, \bar{\Omega} \in C(\mathcal{I}, \mathbb{R}^{d\times d})$, $N, \bar{N} \in C(\mathcal{I}, \mathbb{R}^{a\times a})$,

$$\bar{L} = \begin{bmatrix} \bar{K}_{11}^{-1} & 0 \\ 0 & \bar{L}_{22} \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix},$$

and

$$\hat{K} := K \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix},$$

it holds

$$S_{can}(t) = \text{im } K(t) \begin{bmatrix} I_d \\ 0 \end{bmatrix} = \text{im } \hat{K}(t) \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \quad N_{can}(t) = \text{im } K(t) \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix} = \text{im } \hat{K}(t) \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix},$$

$$\mathit{\Pi}_{can}(t) = K(t) \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} K^{-1}(t) = \hat{K}(t) \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \hat{K}^{-1}(t).$$

Moreover, for

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} := K^{-1}(t)x(t), \quad \begin{bmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{bmatrix} := \hat{K}^{-1}(t)x(t) = \begin{bmatrix} \bar{K}_{11}^{-1} & 0 \\ 0 & \bar{K}_{22}^{-1} \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix},$$

[4]We slightly adapted this theorem from [2] were only strictly lower triangular matrices were considered.

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and the two ODEs

$$u' + \Omega u = 0, \quad \tilde{u}' + \tilde{\Omega}\tilde{u} = 0,$$

it holds

$$\tilde{\Omega} = \bar{K}_{11}^{-1}\Omega\bar{K}_{11} + \bar{K}_{11}^{-1}\bar{K}_{11}'. \tag{2.10}$$

Proof. The claim follows straight forward. ∎

Not surprisingly, since we are confronted with time-varying transformations, they may change the stability behavior.

EXAMPLE 2.7. *For the ODE*

$$u' + 2u = 0,$$

and the transformation with $K_{11}(t) = e^{-5t}$, we obtain

$$0 = \tilde{u}' + \left(e^{5t}2e^{-5t} + e^{5t}(-5)e^{-5t}\right)\tilde{u} = \tilde{u}' - 3\tilde{u}.$$

EXAMPLE 2.8. *(cf. Example C.1) For the ODE*

$$u' + \lambda u = 0,$$

and the transformation with $K_{11}(t) = \sqrt{(\eta t - 1)^2 + 1} =: \gamma(t)$, we obtain

$$0 = \tilde{u}' + \left(\frac{1}{\gamma(t)}\lambda\gamma(t) + \frac{1}{\gamma(t)}\frac{2\eta^2 t - 2\eta}{2\gamma(t)}\right)\tilde{u} = \tilde{u}' + \left(\lambda + \frac{\eta^2 t - \eta}{\gamma(t)^2}\right)\tilde{u}.$$

With these examples we underline that, as soon as time-varying transformations are involved, the stability behavior might change arbitrarily. This motivates the following definition.

DEFINITION 2.9. *If a DAE is equivalently transformable into SCF, then we call every explicit ODE part (2.8) living in the configuration space $\mathbb{R}^d$ for $d > 0$ and resulting from such a transformation a* pure *ODE.*

Observe that pure ODEs do not involve any derivatives of the right-hand side q and are not unique for a given DAE.

REMARK 2.10. *Above we preferred the name pure ODE part for (2.8) and do not use the notation essential underlying ODE (EUODE) for good reasons. The label EUODE itself was first used in [1] for special DAEs without any reference to an SCF and generalized in [15] for arbitrary regular DAEs via a very special transformation into a structured SCF. EUODEs are, so to speak, compressed versions of the so-called projector-based inherent ODEs in the context of the tractability framework. By definition, EUODEs are also pure ODE parts; however, they are strictly bound to specific transformations L and K, which in turn arise from complete projector-based decouplings. An EUODE shares with any pure ODE part the dimension d and the property that it does not involve any derivatives of the right-hand side q. As it is verified in [15], to each regular DAE, one can form the compression in such a way that the resulting EUODE reflects the Lyapunov spectrum of the DAE. Therefore, among the pure ODEs parts of a DAE, there is a spectrum preserving one. Of course, if one can do with Lyapunov transformations, then the stability behavior will be maintained.*

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2.5. Other approaches to compute an SCF. There are several constructive approaches that provide transformations into canonical forms (1.2).

- Time-invariant ordered matrix pairs $\{E, F\}$ and the associated matrix pencils $\sigma E + F$, $\sigma \in \mathbb{R}$, are closely related to generalized eigenproblems. Let the pencil be regular, E be singular and $\alpha \in \mathbb{R}$ be such that the matrix $\alpha E + F$ is nonsingular. An SCF, which is at the same time an SSCF[5], when using merely time-invariant transformations L, K, separates the finite and infinite eigenspaces of the pencil, and in the associated DAE, the pure ODE and the pure DAE part. The Drazin inverse is a well-known tool in this context, see, e.g., [9]. For $\hat{E} = (\alpha E + F)^{-1}E$, it holds that $\mu = \mathrm{ind}(\{E, F\}) = \mathrm{ind}\, \hat{E}$ and
$$\mathbb{R}^m \supset \mathrm{im}\, \hat{E} \supset \cdots \supset \mathrm{im}\, \hat{E}^{\mu} = \mathrm{im}\, \hat{E}^{\mu+1},$$
$$\{0\} \subset \ker \hat{E} \subset \cdots \subset \ker \hat{E}^{\mu} = \ker \hat{E}^{\mu+1}.$$
Note that $\mathcal{V}_i = \mathrm{im}\, \hat{E}^i$, $\mathcal{W}_i = \ker \hat{E}^i$, $i \geq 0$, are also known as Wong-sequences, see, e.g., [3]. Naturally, in all these cases, the transformation $K := [V\ W]^{-1}$, V, W bases of $\mathrm{im}\, \hat{E}^{\mu}$ and $\ker \hat{E}^{\mu}$, respectively, $d = \mathrm{rank}\, \hat{E}^{\mu}$ plays its role.
With regard to a generalization for higher-index time-varying DAEs, we recall that the regularity of the so-called local pencil $\sigma E(t) + F(t)$ is irrelevant for the characterization of the DAE, see, e.g., [4], [9].
- The procedure sketched in [2], [7], which traces back to [6], requires analytically solvable DAEs with real-analytic coefficients $E, F : \mathcal{I} \to \mathbb{R}^{m \times m}$. To transform a DAE into SCF, it repeatedly reduces the size of the system by isolating algebraic and differential components. In contrast to our approach, they do not require constant canonical characteristics, but only a constant d. This implies that the rank of N and the powers of N do not have to be constant in the interval $\mathcal{I}$.
- The projector-based decoupling from [14] works for regular time-varying DAEs, and an SCF is obtained by means of a so-called complete decoupling. This procedure is also constructive; however, it comes with several derivatives for algorithmically determined projector functions.

The approach presented here starts from a different setting, namely the existence of an SSCF, and is based on the properties of particularly block-structured matrix functions that are given in the regular case. To our knowledge, there is no comparable algorithm in the literature, not even for the constant coefficient case.

Let us now focus on these particularly structured matrix functions with size a.

3. Block structured upper triangular matrix functions. In this section, we present a compilation of technical details concerning block upper triangular (BUT[6]) matrix functions that will play their role later on for the last a rows and columns of particular regular matrix pairs.

Suppose that for $\mu \in \mathbb{N}, \mu \geq 2$ some integers
$$\kappa_0 \geq \theta_0 \geq \cdots \geq \theta_{\mu-2} > 0,$$
are given and consider

[5] Quasi-Weierstraß form in [3], Weierstraß-Kronecker form in [14].
[6] Unlike the appendices of [17] and [18], our focus is not limited solely to strictly block upper triangular (SUT) and hence structurally nilpotent matrix functions, although these are included in our discussion as well.

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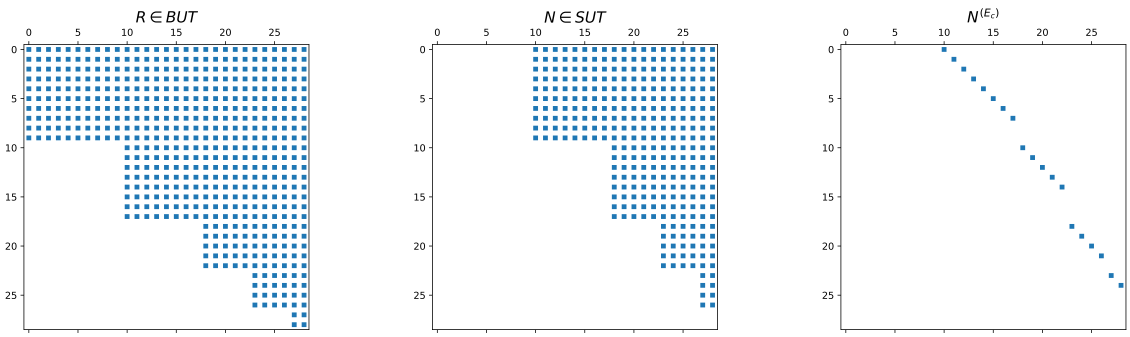

FIGURE 1. *Visualization of the block structure for decreasing block sizes* $\ell_1 = 10, \ell_2 = 8, \ell_3 = 5, \ell_4 = 4, \ell_5 = 2$.

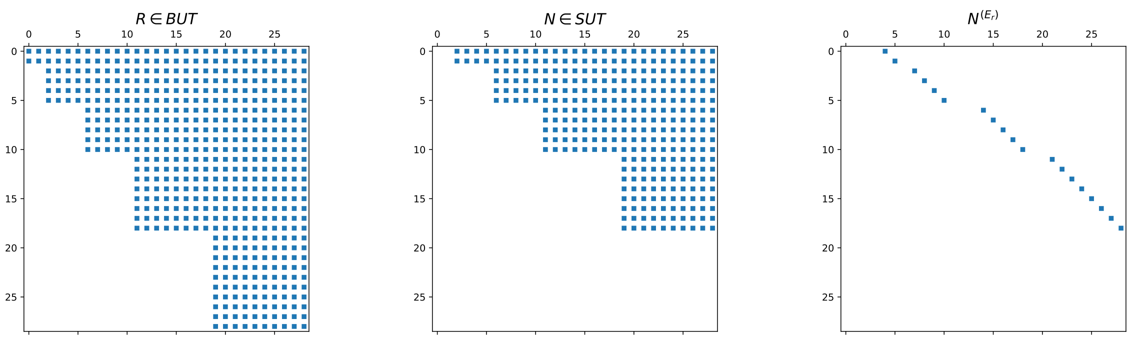

FIGURE 2. *Visualization of the block structure for increasing block sizes* $\ell_1 = 2, \ell_2 = 4, \ell_3 = 5, \ell_4 = 8, \ell_5 = 10$.

- either decreasing $\ell_1 \geq \cdots \geq \ell_\mu > 0$:

$$\ell_1 = \kappa_0, \quad \ell_i = \theta_{i-2}, \quad i = 2, \ldots, \mu,$$

- or increasing $0 < \ell_1 \leq \cdots \leq \ell_\mu$:

$$\ell_i = \theta_{\mu-i-1}, \quad i = 1, \ldots, \mu - 1, \quad \ell_\mu = \kappa_0,$$

and $\ell := \sum_{i=1}^{\mu} \ell_i = \kappa_0 + \sum_{j=0}^{\mu-2} \theta_j$. These two different orders appear in the well-established T-canonical and S-canonical forms discussed in Section 8 that have several common properties that we summarize here.

For $\mathcal{I} \subseteq \mathbb{R}$ and fixed $\ell, \mu, \ell_1, \ldots, \ell_\mu$ we denote by

- $BUT = BUT(\ell, \mu, \ell_1, \ldots, \ell_\mu)$ the set of all block upper triangular matrix functions $B : \mathcal{I} \to \mathbb{R}^{\ell \times \ell}$

$$B = \begin{array}{ll} \begin{bmatrix} B_{11} & B_{12} & & \cdots & B_{1,\mu} \\ 0 & B_{22} & B_{23} & & B_{2,\mu} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & B_{\mu-1,\mu-1} & B_{\mu-1,\mu} \\ 0 & 0 & \cdots & 0 & B_{\mu,\mu} \end{bmatrix} & \begin{array}{l} \} \ \ell_1 \\ \} \ \ell_2 \\ \ \ \vdots \\ \} \ \ell_{\mu-1} \\ \} \ \ell_\mu \end{array} \end{array},$$

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with blocks

$$B_{i,j} : \mathcal{I} \to \mathbb{R}^{\ell_i \times \ell_j}, \quad \text{and} \quad B_{i,j} = 0 \quad \text{for} \quad i > j, \quad \text{see Figs. 1 and 2.}$$

- $BUT_{nonsingular} = BUT_{nonsingular}(\ell, \mu, \ell_1, \dots, \ell_\mu)$ the set of all nonsingular $R \in BUT$ that therefore have pointwise nonsingular diagonal blocks

$$R_{i,i} : \mathcal{I} \to \mathbb{R}^{\ell_i \times \ell_i}.$$

- $SUT = SUT(\ell, \mu, \ell_1, \dots, \ell_\mu)$ the set of all strictly block upper triangular matrix functions $N \in BUT$ with zero diagonal blocks $N_{i,i} = 0$, i.e., in the end

$$N_{i,j} = 0 \quad \text{for} \quad i \geq j, \quad \text{see Figs. 1 and 2.}$$

- In case of $\ell_1 \geq \dots \geq \ell_\mu$, we denote by
  - $SUT_{column} \subset SUT$ the set of all $N \in SUT$ having exclusively blocks $(N)_{i,i+1}$ with full column rank, that is

$$\operatorname{rank} (N(t))_{i,i+1} = \ell_{i+1}, \quad i = 1, \dots, \mu - 1 \quad \text{for all} \quad t \in \mathcal{I}. \tag{3.11}$$

  - $N^{(E_c)} \in SUT_{column}$ (**E**lementary form with full **c**olumn rank) the matrix

$$N^{(E_c)} := \begin{array}{c} \begin{bmatrix} 0 & N_{12}^{(E_c)} & 0 & \cdots & 0 \\ & 0 & N_{23}^{(E_c)} & 0 & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & N_{\mu-1,\mu}^{(E_c)} \\ & & & & 0 \end{bmatrix} \end{array} \begin{array}{lcl} \}\ \ell_1 & = & \kappa_0 \\ \}\ \ell_2 & = & \theta_0 \\ \vdots & & \vdots \\ \}\ \ell_{\mu-1} & = & \theta_{\mu-3} \\ \}\ \ell_\mu & = & \theta_{\mu-2} \end{array},$$

  with blocks

$$N_{i,i+1}^{(E_c)} = \begin{bmatrix} I_{\ell_{i+1}} \\ 0 \end{bmatrix} \in \mathbb{R}^{\ell_i \times \ell_{i+1}}.$$

- In case of $\ell_1 \leq \dots \leq \ell_\mu$, we denote by
  - $SUT_{row} \subset SUT$ the set of all $N \in SUT$ having exclusively blocks $(N)_{i,i+1}$ with full row rank, that is

$$\operatorname{rank} (N(t))_{i,i+1} = \ell_i, \quad i = 1, \dots, \mu - 1 \quad \text{for all} \quad t \in \mathcal{I}. \tag{3.12}$$

  - $N^{(E_r)} \in SUT_{row}$ (**E**lementary form with full **r**ow rank) the matrix

$$N^{(E_r)} = \begin{array}{c} \begin{bmatrix} 0 & N_{12}^{(E_r)} & 0 & \cdots & 0 \\ & 0 & N_{23}^{(E_r)} & 0 & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & N_{\mu-1,\mu}^{(E_r)} \\ & & & & 0 \end{bmatrix} \end{array} \begin{array}{lcl} \}\ \ell_1 & = & \theta_{\mu-2} \\ \}\ \ell_2 & = & \theta_{\mu-3} \\ \vdots & & \vdots \\ \}\ \ell_{\mu-1} & = & \theta_0 \\ \}\ \ell_\mu & = & \kappa_0 \end{array},$$

  with blocks

$$N_{i,i+1}^{(E_r)} = \begin{bmatrix} 0 & I_{\ell_i} \end{bmatrix} \in \mathbb{R}^{\ell_i \times \ell_{i+1}}.$$

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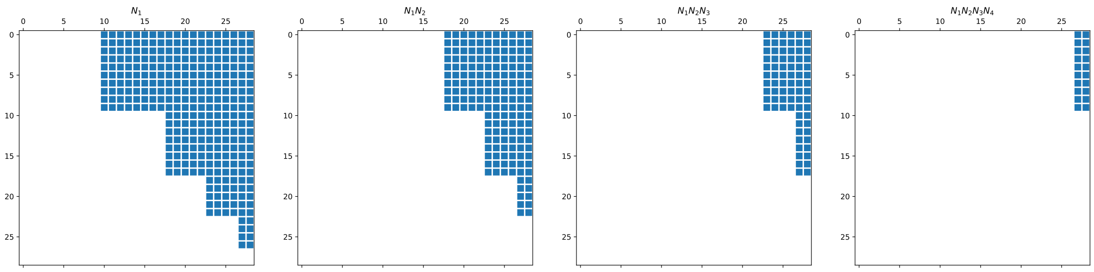

FIGURE 3. *Visualization of products of $N_i \in SUT_{column}$ with $\ell_1 = 10, \ell_2 = 8, \ell_3 = 5, \ell_4 = 4, \ell_5 = 2$ leading to $N_1 \cdots N_5 = 0$, cf. Lemma 3.1 (e).*

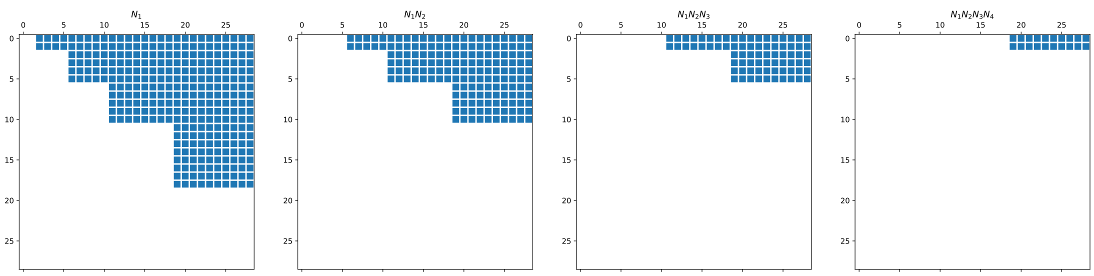

FIGURE 4. *Visualization of products of $N_i \in SUT_{row}$ with $\ell_1 = 2, \ell_2 = 4, \ell_3 = 5, \ell_4 = 8, \ell_5 = 10$ leading to $N_1 \cdots N_5 = 0$, cf. Lemma 3.1 (e).*

We refer to [17], [18] for some useful properties of SUT, SUT_{column}, SUT_{row}, $N^{(E_c)}$, and $N^{(E_r)}$ and use also the notations

$$SUT_{c/r}, \quad N^{c/r}, \quad N^{(E_{c/r})},$$

if $SUT_{column}, N^c, N^{(E_c)}$ is meant for decreasing ℓ_i and $SUT_{row}, N^r, N^{(E_r)}$ for increasing ℓ_i, respectively.

With this notation, all $N^{c/r} \in SUT_{c/r}$ are pointwise nilpotent of order μ and for all $t \in \mathcal{I}$

$$\operatorname{rank} N^{c/r}(t) = \ell - \kappa_0, \quad \theta_i = \operatorname{rank}(N^{c/r}(t))^{i+1} - \operatorname{rank}(N^{c/r}(t))^{i+2}, \quad i = 0, \dots, \mu - 2.$$

LEMMA 3.1. *For $R \in BUT_{nonsingular}$, $N, N_1, \dots, N_k \in SUT_{c/r}$ with the same block sizes, it holds*

(a) $R^{-1} \in BUT_{nonsingular}$, $(R^{-1})_{ii} = (R_{ii})^{-1}$,
(b) $N^{(E_{c/r})}R \in SUT_{c/r}$,
(c) $RN^{(E_{c/r})} \in SUT_{c/r}$,
(d) $R + N \in BUT_{nonsingular}$,
(e) $N^k = 0$ *and* $N_1 \cdots N_k = 0$ *for* $k \geq \mu$. *(SUT-nilpotency)*

Proof. $(a) - (d)$ follow by straight forward computation using the nonsingularity of the diagonal blocks R_{ii}. The SUT-nilpotency (e) follows from the SUT-structure and is illustrated in Figures 3 and 4. ∎

Moreover, according to [18], for a regular pair, we know

$$\{E, F\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\}, \quad \Omega : \mathcal{I} \to \mathbb{R}^{d \times d}, \quad N^{(E_{c/r})} \in \mathbb{R}^{a \times a}.$$

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This will be the starting point here, where we suppose that, by an initial equivalence transformation (a kind of preprocessing)

$$\{E,F\} \overset{L_0,K_0}{\sim} \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, F_0 \right\}, \tag{3.13}$$

is already given, along with certain assumptions on the block structure of F_0. Then, a direct computation of an SSCF becomes possible in four main steps with L_i, K_i, $i = 1, \dots, 4$. Consequently, the transformation $K := K_0K_1K_2K_3K_4$ permits a description of the so-called canonical subspaces and the canonical projector.

4. Pairs in QuasiSCF. As mentioned before, in [17], a couple of different definitions of linear regular DAEs were shown to be equivalent. Here, we focus only on those that involve block structured SCFs and its variants from [18], dealing with the block structures from last section. For simplicity, if either strictly upper block matrices $SUT_{columns}$ or SUT_{row} can be considered, we use the abbreviation $SUT_{c/r}$.

PROPOSITION 4.1. *(Special case of [17, Theorem 8.1]) A pair of matrix functions $\{E,F\}$ is regular, iff*

$$\{E,F\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{c/r} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\},$$

with

$$N^{c/r} \in SUT_{c/r}, \quad \Omega : \mathcal{I} \to \mathbb{R}^{d\times d}.$$

The starting point for the following considerations is based on the existence of a block structured SSCF with decreasing or increasing block sizes of the constant nilpotent matrix. In particular, we can assume that the constant matrix $N^{(C)}$ has an elementary form $N^{(E_{c/r})}$ introduced in Section 3. Both elementary matrices are illustrated in Figs. 1 and 2 and permit the following characterization of regularity, where $N^{(E_{c/r})}$ means either $N^{(E_c)}$ or $N^{(E_r)}$.

PROPOSITION 4.2. *A pair of matrix functions $\{E,F\}$ is regular, iff*

$$\{E,F\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\},$$

with $\Omega : \mathcal{I} \to \mathbb{R}^{d\times d}$.

The equivalence of these two possible definitions of regularity has been shown in [18]. With regard to our purposes here, we show yet another possibility.

PROPOSITION 4.3. *A pair of matrix functions $\{E,F\}$ is regular, iff*

$$\{E,F\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & R \end{bmatrix} \right\},$$

with

- $\Omega : \mathcal{I} \to \mathbb{R}^{d\times d}$,
- $R : \mathcal{I} \to \mathbb{R}^{a\times a}$, *whereas R is block-structured upper triangular with either ascending or descending block sizes corresponding to $N^{(E_{c/r})}$ and nonsingular, i.e., using the notation from Section 3, $R \in BUT_{nonsingular}$.*

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Proof. One direction of the equivalence to Proposition 4.1 follows with

$$L = \begin{bmatrix} I_d & 0 \\ 0 & R^{-1} \end{bmatrix}, \quad K = I_m,$$

and Lemma 3.1. For the other direction, we use specific R^c and R^r, which can be constructed as described in [18], see Appendix B. With these matrix functions, we can choose

$$L = \begin{bmatrix} I_d & 0 \\ 0 & (R^c)^{-1} \end{bmatrix}, \quad K = I_m,$$

or

$$L = \begin{bmatrix} I_d & 0 \\ 0 & (I_a + N^{(E_r)}((R^r)^{-1})')^{-1} \end{bmatrix}, \quad K = \begin{bmatrix} I_d & 0 \\ 0 & (R^r)^{-1} \end{bmatrix}. \qquad \square$$

This motivates the following definition

Definition 4.4. *A pair*

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & R \end{bmatrix} \right\},$$

is in QuasiSCF, iff $R \in BUT_{nonsingular}$, with block sizes corresponding to $N^{(E_{c/r})}$.

From the proof of Proposition 4.3, it follows that for the K used to transform an SCF into a QuasiSCF (or vice versa), it holds

$$\operatorname{im} K \begin{bmatrix} I_d \\ 0 \end{bmatrix} = \operatorname{im} \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \quad \operatorname{im} K \begin{bmatrix} 0 \\ I_a \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 \\ I_a \end{bmatrix},$$

such that S_{can}, N_{can}, $\mathit{\Pi}_{can}$, a pure ODE and a pure DAE, can be characterized using the QuasiSCF instead of an SCF, as we will see later in more detail.

5. DAEs in PreSCF.

We have seen that a regular DAE possesses an equivalent QuasiSCF that is structured according to the canonical values. If the canonical values are known, then our idea is to use specially structured pairs as starting point for a new algorithm[7].

For our computation of an SCF or even an SSCF, we first realize that a transformation into a form

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right\},$$

is always possible for regular DAEs, since indeed a transformation into an SSCF exists. However, since directly finding an SSCF is difficult, we think of transformations that leave the first matrix unchanged. For nonsingular F_{22}, we obtain the following results that are decisive for the next steps.

Lemma 5.1. *For every regular pair in the form*

$$\{E_0, F_0\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right\},$$

[7] This is also new for matrix pencils.

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with nonsingular F_{22} *and the transformation matrix functions*

$$L = \begin{bmatrix} I_d & -F_{12}F_{22}^{-1} \\ 0 & I_a \end{bmatrix}, \quad K = \begin{bmatrix} I_d & F_{12}F_{22}^{-1}N^{(E_{c/r})} \\ 0 & I_a \end{bmatrix},$$

it holds

$$\begin{aligned} LE_0K &= E_0, \\ LF_0K + LE_0K' &= \begin{bmatrix} \bar{F}_{11} & \tilde{F}_{12}N^{(E_{c/r})} \\ F_{21} & \bar{F}_{22} \end{bmatrix}, \end{aligned}$$

for

$$\begin{aligned} \bar{F}_{11} &= F_{11} - F_{12}F_{22}^{-1}F_{21}, \\ \tilde{F}_{12} &= (F_{11} - F_{12}F_{22}^{-1}F_{21})F_{12}F_{22}^{-1} + (F_{12}F_{22}^{-1})', \\ \bar{F}_{22} &= F_{22} + F_{21}F_{12}F_{22}^{-1}N^{(E_{c/r})}. \end{aligned}$$

Proof. Straight forward computation gives the result. ∎

LEMMA 5.2. *For every regular pair in the form*

$$\{E_0, F_0\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right\},$$

with nonsingular F_{22} *and the transformation matrix functions*

$$L = \begin{bmatrix} I_d & 0 \\ N^{(E_{c/r})}F_{22}^{-1}F_{21} & I_a \end{bmatrix}, \quad K = \begin{bmatrix} I_d & 0 \\ -F_{22}^{-1}F_{21} & I_a \end{bmatrix},$$

it holds

$$\begin{aligned} LE_0K &= E_0, \\ LF_0K + LE_0K' &= \begin{bmatrix} \bar{F}_{11} & F_{12} \\ N^{(E_{c/r})}\tilde{F}_{21} & \bar{F}_{22} \end{bmatrix}, \end{aligned}$$

for

$$\begin{aligned} \bar{F}_{11} &= F_{11} - F_{12}F_{22}^{-1}F_{21}, \\ \tilde{F}_{21} &= F_{22}^{-1}F_{21}(F_{11} - F_{12}F_{22}^{-1}F_{21}) - (F_{22}^{-1}F_{21})', \\ \bar{F}_{22} &= F_{22} + N^{(E_{c/r})}F_{22}^{-1}F_{21}F_{12}. \end{aligned}$$

Proof. Straight forward computation gives the result. ∎

It is easy to recognize now that $N^{(E_{c/r})}$ is introduced in expressions of the second matrix function, such that repeating such transformations makes it possible to take advantage of the SUT-nilpotency. Indeed, these two lemmas permit an iterative computation of a QuasiSCF if in all steps $F_{22} \in BUT_{nonsingular}$ is given and the SUT-nilpotency guarantees a finite number of iterations, see Section 6. To ensure the feasibility, we again use the block structures introduced in Section 3, see also Figs. 1 and 2.

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Definition 5.3. *A pair*

$$\left\{\begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right\},$$

is in a preliminary stage of an SCF (PreSCF) iff F_{22} and $F_{21}F_{12}$ are block upper triangular matrices with the same block sizes as $N^{(E_{c/r})}$ and F_{22} is nonsingular.

In short, this means $F_{22} \in BUT_{nonsingular}$, $F_{21}F_{12} \in BUT$ in terms of the notation from Section 3.

For DAEs in a PreSCF, the algorithm below will produce an SCF after a finite number of steps due to Lemma 5.1, Lemma 5.2 and the SUT-nilpotency, since all involved pairs of matrix functions have nonsingular F_{22} thanks to the structural properties.

6. A subsequent procedure to compute an SSCF.

To obtain an SSCF, we consider several transformation steps with $E_0 = E_1 = E_2 = E_4$:

$$\{E, F\} \underset{\text{Step 0}}{\overset{L_0,K_0}{\underset{\longrightarrow}{\sim}}} \underbrace{\{E_0, F_0\}}_{PreSCF} \underset{\text{Step 1}}{\overset{L_1,K_1}{\underset{\longrightarrow}{\sim}}} \underbrace{\{E_1, F_1\}}_{(F_1)_{12}=0} \underset{\text{Step 2}}{\overset{L_2,K_2}{\underset{\longrightarrow}{\sim}}} \underbrace{\{E_2, F_2\}}_{QuasiSCF} \underset{\text{Step 3}}{\overset{L_3,K_3}{\underset{\longrightarrow}{\sim}}} \underbrace{\{E_3, F_3\}}_{SCF} \underset{\text{Step 4}}{\overset{L_4,K_4}{\underset{\longrightarrow}{\sim}}} \underbrace{\{E_4, F_4\}}_{SSCF}.$$

The transformation matrices L_0, K_0 depend on the structure of the given DAE. Here, we discuss them for some particular cases later on.

Assuming that a pair of matrix functions is in a PreSCF, the transformation matrices L_i, K_i, $i = 1, 2, 3, 4$ are the ones used in the i-th step according to the notation below and result systematically from further intermediate steps.

For Steps 1 and 2, we use the two elementary equivalence transformations that preserve the structure of the first constant matrix from Lemmas 5.1 and 5.2 and take advantage of the SUT-nilpotency in iterations.

6.1. Step 1: Obtaining a pure ODE part.

In this step, we construct the equivalence transformation

$$\underbrace{\{E_0, F_0\}}_{PreSCF} \overset{L_1,K_1}{\sim} \{E_1, F_1\}, \quad \text{with} \quad (F_1)_{12} = 0.$$

To this end, we use Lemma 5.1 to construct a sequence $F^{(k)}$, $k \in \mathbb{N}$, with $F_{12}^{(k)} \to 0$, $F^{(0)} = F_0$. In a first iteration, we obtain

$$F_{12}^{(1)} = ((F_{11} - F_{12}F_{22}^{-1}F_{21})F_{12}F_{22}^{-1} + (F_{12}F_{22}^{-1})')N^{(E_{c/r})}, \quad F_{22}^{(1)} = F_{22} + F_{21}F_{12}F_{22}^{-1}N^{(E_{c/r})}.$$

With Lemma 3.1, we observe two aspects:

- Since $F_{21}F_{12}F_{22}^{-1}N^{(E_{c/r})}$ is strictly upper block triangular, the diagonal blocks of F_{22} are preserved in $F_{22}^{(1)}$, such that it is nonsingular.
- $F_{12}^{(1)}$ consists of summands that at the right-hand side present one of the particularly structured nilpotent matrices

$$N_1^{(0)} := F_{22}^{-1}N^{(E_{c/r})}, \quad N_2^{(0)} := (F_{22}^{-1})'N^{(E_{c/r})} = (N_1^{(0)})' \in SUT,$$

see Lemma 3.1. Repeating this procedure for

$$G^{(k)} := (F_{11}^{(k)} - F_{12}^{(k)}(F_{22}^{(k)})^{-1}F_{21}^{(k)}), \quad N_1^{(k)} := (F_{22}^{(k)})^{-1}N^{(E_{c/r})}, \quad N_2^{(k)} := ((F_{22}^{(k)})^{-1})'N^{(E_{c/r})}$$

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with $N_1^{(k)}, N_2^{(k)} \in SUT$, we recognize that

$$F_{12}^{(k+1)} = (G^{(k)} F_{12}^{(k)} + (F_{12}^{(k)})')N_1^{(k)} + F_{12}^{(k)} N_2^{(k)}.$$

From the SUT-nilpotency, it follows $N_{i_0}^{(0)} \cdots N_{i_{\mu-1}}^{(\mu-1)} = 0$ for any $i_k \in \{1,2\}$, such that $F_{12}^{(\mu)} = 0$ and $R := F_{22}^{(\mu)}$, $\Omega := F_{11}^{(\mu)}$.

Therefore,

$$\{E_1, F_1\} = \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{r/c})} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ F_{21} & R \end{bmatrix} \right\}.$$

We see that with this step, we already obtained a pure ODE.

6.2. Step 2: Obtaining a pure DAE part. In this step, we construct the equivalence transformation

$$\{E_1, F_1\} \overset{L_2,K_2}{\sim} \underbrace{\{E_2, F_2\}}_{QuasiSCF}.$$

We use Lemma 5.2 to construct a sequence $F^{(k)}$, $k \in \mathbb{N}$ with $F_{21}^{(k)} \to 0$, $F^{(0)} = F_1$.

- Let us first note that for $F^{(0)} = F_0$ (i.e. $F_{12}^{(0)}$ possibly nonzero), a first iteration would lead to

  $$F_{21}^{(1)} = N^{(E_{c/r})}(F_{22}^{-1} F_{21}(F_{11} - F_{12} F_{22}^{-1} F_{21}) - (F_{22}^{-1} F_{21})'), \quad F_{22}^{(1)} = F_{22} + N^{(E_{c/r})} F_{22}^{-1} F_{21} F_{12}.$$

  Now, with Lemma 3.1, we observe that
  - Since $N^{(E_{c/r})} F_{22}^{-1} F_{21} F_{12}$ is strictly upper block triangular, the diagonal blocks of F_{22} are preserved in $F_{22}^{(1)}$, such that it is nonsingular.
  - Each summand of $F_{21}^{(1)}$ starts with one of the particularly structured nilpotent matrices

    $$N_3^{(0)} := N^{(E_{c/r})} F_{22}^{-1}, \quad N_4^{(0)} := N^{(E_{c/r})} (F_{22}^{-1})' = (N_3^{(0)})' \in SUT.$$

    Repeating this procedure for

    $$G^{(k)} := (F_{11}^{(k)} - F_{12}^{(k)} (F_{22}^{(k)})^{-1} F_{21}^{(k)}), \quad N_3^{(k)} := N^{(E_{c/r})} (F_{22}^{(k)})^{-1}, \quad N_4^{(k)} := N^{(E_{c/r})} ((F_{22}^{(k)})^{-1})',$$

    with $N_3^{(k)}, N_4^{(k)} \in SUT$, we recognize that

    $$F_{21}^{(k+1)} = N_3^{(k)} (F_{21}^{(k)} G^{(k)} - (F_{21}^{(k)})') - N_4^{(k)} F_{21}^{(k)}.$$

    From the SUT-nilpotency it follows $N_{i_{\mu-1}}^{(\mu-1)} \cdots N_{i_0}^{(0)} = 0$ for any $i_k \in \{3,4\}$, such that $F_{21}^{(\mu)} = 0$.
- If Step 2 is done after Step 1, i.e., $F^{(0)} = F_1$, then $F_{12}^{(j)} = 0$, $F_{22}^{(j)} = R$, $F_{11}^{(j)} = \Omega$ remain unchanged for $j = 0, 1, \ldots, \mu$.

Note that the order of Step 1 and Step 2 could also have been inverted. We prefer this order, since then we first obtain a pure ODE with its final Ω.

6.3. Steps 3 and 4: Computation of an SCF and an SSCF. As soon as we have a representation of the form

$$E_2 = \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{c/r})} \end{bmatrix}, \quad F_2 = \begin{bmatrix} \Omega & 0 \\ 0 & R \end{bmatrix},$$

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for a nonsingular block upper triangular R, with

$$L_3 = \begin{bmatrix} I_d & 0 \\ 0 & R^{-1} \end{bmatrix}, \quad K_3 = I_m,$$

we obtain an SCF. In a last step, if desired, an SSCF can be computed with particular L_{SSCF} and K_{SSCF} for the pure DAE as described in [18] and

$$L_4 = \begin{bmatrix} I_d & 0 \\ 0 & L_{SSCF} \end{bmatrix}, \quad K_4 = \begin{bmatrix} I_d & 0 \\ 0 & K_{SSCF} \end{bmatrix}.$$

Since K_3 and K_4 transform algebraic variables only, they are not crucial for the canonical spaces and therefore we will not go into them any further, cf. Section 7.

6.4. A detailed example. We consider the following example to illustrate on the one hand all the described steps and on the other hand the importance of the regularity requirement.

EXAMPLE 6.1. *[2, Example 5.6.]*

$$E = \begin{bmatrix} \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 \\ -\sin(t)\cos(t) & \sin^2(t) & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -\sin(t)+\cos(t) & -\sin(t)-\cos(t) & 0 \\ \cos(t) & -\sin(t) & 0 \\ \sin^2(t) & \sin(t)\cos(t) & -t^2-1 \end{bmatrix}.$$

In [2], this example is used to illustrate their transformation into SCF that does not presuppose regularity, but real analytic E, F. Indeed, the nilpotent matrix function of the SCF found there has rank drops at zeros of $\sin$*, just like the matrix E. Therefore, it is obvious that on $\mathbb{R}$ no equivalent transformation into an SSCF is possible.*

Here, for regularity, we require $\mathcal{I} \subset \mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$, such that $m = 3, \mu = 2, r = 2, \theta_0 = 1, d = 1, a = 2$ on $\mathcal{I}$, $N^{(E_c)} = N^{(E_r)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Step 0: With

$$L_0 = \begin{bmatrix} 1 & -1-\frac{\cos(t)}{\sin(t)} & 0 \\ \cos(t) & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad K_0 = \begin{bmatrix} \frac{1}{\sin(t)} & 0 & -\frac{\cos(t)}{\sin(t)} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

we obtain the required form of a PreSCF:

$$E_0 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad F_0 = \left[\begin{array}{c|cc} \frac{2\sin(t)\cos(t)+2}{-\sin^2(t)+\cos^2(t)-1} & 0 & \frac{\sin(t)+\cos(t)}{\sin^2(t)} \\ \hline -\cos(t)+\frac{1}{\sin(t)} & -t^2-1 & -\frac{\cos(t)}{\sin(t)} \\ \frac{\cos(t)}{\sin(t)} & 0 & -\frac{1}{\sin(t)} \end{array}\right].$$

Step 1:

1. *First (and in this case last) iteration step:*

$$L_1^{(1)} = \left[\begin{array}{c|cc} 1 & 0 & 1+\frac{\cos(t)}{\sin(t)} \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_1^{(1)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

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$E_1 := E_1^{(1)} = E_0$, $L_1 := L_1^{(1)}$, $K_1 := K_1^{(1)}$,

$$F_1 := F_1^{(1)} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline -\cos(t) + \frac{1}{\sin(t)} & -t^2-1 & -\frac{\cos(t)}{\sin(t)} \\ \frac{\cos(t)}{\sin(t)} & 0 & -\frac{1}{\sin(t)} \end{array}\right].$$

Step 2:

1. *First iteration step:*

$$L_2^{(1)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -\cos(t) & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_2^{(1)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -\frac{-\sin(t)+\cos(t)}{t^2+1} & 1 & 0 \\ \cos(t) & 0 & 1 \end{array}\right],$$

$E_2^{(1)} = E_1 = E_0$ *and*

$$F_2^{(1)} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline -\sin(t) + \cos(t) & -t^2-1 & -\frac{\cos(t)}{\sin(t)} \\ 0 & 0 & -\frac{1}{\sin(t)} \end{array}\right].$$

2. *Second (and last) iteration step:*

$$L_2^{(2)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_2^{(2)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \frac{-\sin(t)+\cos(t)}{t^2+1} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$E_2 := E_2^{(2)} = E_2^{(1)} = E_1 = E_0$, $L_2 := L_2^{(2)} L_2^{(1)}$, $K_2 := K_2^{(1)} K_2^{(2)}$, *results in a QuasiSCF with*

$$F_2 := F_2^{(2)} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & -t^2-1 & -\frac{\cos(t)}{\sin(t)} \\ 0 & 0 & -\frac{1}{\sin(t)} \end{array}\right].$$

Step 3:

$$L_3 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -\frac{1}{t^2+1} & \frac{\cos(t)}{t^2+1} \\ 0 & 0 & -\sin(t) \end{array}\right], \quad K_3 = I_3,$$

leads to the SCF

$$E_3 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{t^2+1} \\ 0 & 0 & 0 \end{array}\right], \quad F_3 = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Step 4: Finally, with

$$L_4 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -t^2+1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_4 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -\frac{1}{t^2+1} & 0 \\ 0 & 0 & 1 \end{array}\right],$$

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we obtain the SSCF

$$E_4 = E_0 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad F_4 = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Note that for the homogenous DAE the pure ODE $u' - u = 0$, also given in [2], is found already after Step 1. However, it is not unique and different L_0, K_0 may provide different pure ODEs. We illustrate and discuss this in Example C.1.

In the examples from the Appendix C, for shortness, we will not specify the intermediate iteration steps of Step 1 and Step 2 anymore.

7. Consequences for the canonical projector and pure ODEs. If $K := K_0K_1K_2K_3K_4$ is the transformation of the unknowns that was used above to transform an arbitrary DAE into an SSCF, then the canonical projector

$$\Pi_{can} := K \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} K^{-1},$$

can be computed as

$$\Pi_{can} := K_0K_1K_2 \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} K_2^{-1}K_1^{-1}K_0^{-1},$$

since K_3 and K_4 transform algebraic variables only. Therefore, these two last steps will not be that important for latter considerations, such that we will focus on Steps 0, 1, and 2.

Proposition 7.1. *For every sufficiently smooth pair of matrix functions in a PreSCF,*

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{r/c})} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right\},$$

the canonical spaces are

$$S_{can} = \operatorname{im} \begin{bmatrix} I_d + AB \\ B \end{bmatrix}, \quad N_{can} = \operatorname{im} \begin{bmatrix} A \\ I_a \end{bmatrix},$$

and the canonical projector Π_{can} is

$$\Pi_{can} = \begin{bmatrix} I_d + AB & -A - ABA \\ B & -BA \end{bmatrix},$$

for $A := \sum_{i=1}^{\mu} A_i : \mathcal{I} \to \mathbb{R}^{d\times a}$, $B := \sum_{j=1}^{\mu} B_j : \mathcal{I} \to \mathbb{R}^{a\times d}$, resulting from

$$\begin{aligned}
F^{(0)} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \\
F^{(i)} &= \begin{bmatrix} I_d & -F_{12}^{(i-1)}(F_{22}^{(i-1)})^{-1} \\ 0 & I_a \end{bmatrix} F^{(i-1)} \begin{bmatrix} I_d & A_i \\ 0 & I_a \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & A_i' \\ 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, \mu, \\
F^{(\mu+j)} &= \begin{bmatrix} I_d & 0 \\ -(F_{22}^{(\mu+j-1)})^{-1}F_{21}^{(\mu+j-1)}N^{(E_{c/r})} & I_a \end{bmatrix} F^{(\mu+j-1)} \begin{bmatrix} I_d & 0 \\ B_j & I_d \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 \\ N^{(E_{c/r})}B_j' & 0 \end{bmatrix}, \quad j = 1, \ldots, \mu,
\end{aligned}$$

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for

$$A_i = F_{12}^{(i-1)}(F_{22}^{(i-1)})^{-1}N^{(E_{c/r})}, \quad B_j = (F_{22}^{(\mu+j-1)})^{-1}F_{21}^{(\mu+j-1)}.$$

Proof. The assertion follows from

$$\begin{aligned} K_{1-2} := K_1K_2 &= \begin{bmatrix} I_d & A_1 \\ 0 & I_a \end{bmatrix} \cdots \begin{bmatrix} I_d & A_\mu \\ 0 & I_a \end{bmatrix} \cdot \begin{bmatrix} I_d & 0 \\ B_1 & I_a \end{bmatrix} \cdots \begin{bmatrix} I_d & 0 \\ B_\mu & I_a \end{bmatrix} \\ &= \begin{bmatrix} I_d & A \\ 0 & I_a \end{bmatrix} \cdot \begin{bmatrix} I_d & 0 \\ B & I_a \end{bmatrix} = \begin{bmatrix} I_d + AB & A \\ B & I_a \end{bmatrix}, \end{aligned}$$

and

$$K_{1-2}^{-1} := K_2^{-1}K_1^{-1} = \begin{bmatrix} I_d & 0 \\ -B & I_a \end{bmatrix} \begin{bmatrix} I_d & -A \\ 0 & I_a \end{bmatrix} = \begin{bmatrix} I_d & -A \\ -B & BA + I_a \end{bmatrix}.$$

∎

Note that this Proposition enables us to compute in general

$$\Pi_{can} = K_0 \begin{bmatrix} I_d + AB & -A - ABA \\ B & -BA \end{bmatrix} K_0^{-1},$$

directly from K_0, K_1, K_2, since $A = (K_1)_{12} = (K_1K_2)_{12}$, $B = (K_2)_{21} = (K_1K_2)_{21}$. Moreover, if needed, a closer look provides the precise smoothness assumptions required for parts of F_0 that we will not go into here.

From the above proof, we recognize that, as soon as we finish Step 1, a pure ODE can be represented, since it means that for every pair

$$\{E_1, F_1\} = \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{r/c})} \end{bmatrix}, \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \right\},$$

it holds

$$\{E_1, F_1\} \overset{L_2,K_2}{\sim} \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_{r/c})} \end{bmatrix}, \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \right\} = \{E_2, F_2\},$$

with

$$L_2 = \begin{bmatrix} I_d & 0 \\ N^{(E_{c/r})}B & I_a \end{bmatrix}, \quad K_2 = \begin{bmatrix} I_d & 0 \\ B & I_a \end{bmatrix}.$$

Corollary 7.2. *Let L_0, K_0 be matrix functions that transform a pair $\{E, F\}$ into a PreSCF, and L_1, K_1 matrix functions that transform this PreSCF according to Step 1, leading to the matrix function $\Omega : \mathcal{I} \to \mathbb{R}^{d\times d}$.*

A pure ODE (2.8) of the DAE (1.1) is then obtained for

$$u := \begin{bmatrix} I_d & 0 \end{bmatrix} (K_0K_1)^{-1}x, \quad \bar{q}_1 := \begin{bmatrix} I_d & 0 \end{bmatrix} L_1L_0q.$$

Proof. The assertion follows from the particular form of K_3, K_2, L_3, L_2:

$$\begin{bmatrix} I_d & 0 \end{bmatrix} K_3^{-1}K_2^{-1}K_1^{-1}K_0^{-1} = \begin{bmatrix} I_d & 0 \end{bmatrix} K_1^{-1}K_0^{-1}, \quad \begin{bmatrix} I_d & 0 \end{bmatrix} L_3L_2L_1L_0 = \begin{bmatrix} I_d & 0 \end{bmatrix} L_1L_0.$$

∎

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8. Application to T-canonical and S-canonical forms.

8.1. Block-structured canonical forms of regular DAEs. There are further canonical forms, which play their role within the projector-based [14] and the strangeness-reduction [13] frameworks, both of them designed in a constructive way. We show here that they (almost) fulfill the structural assumptions from Section 5.

Following the notation of [14, Section 2.10.1], a time-varying pair

$$\left\{ \begin{bmatrix} I_d & E_{12} \\ 0 & N \end{bmatrix}, \quad \begin{bmatrix} F_{11} & 0 \\ F_{21} & I_a \end{bmatrix} \right\},$$

with a strictly block upper triangular matrix function

$$N = \begin{array}{c} \begin{bmatrix} 0 & N_{12} & \cdots & N_{1\mu} \\ & \ddots & \ddots & \vdots \\ & & \ddots & N_{\mu-1,\mu} \\ & & & 0 \end{bmatrix} \end{array} \begin{array}{c} \ell_1 \\ \vdots \\ \ell_{\mu-1} \\ \ell_\mu \end{array}, \qquad \ell_i \in \mathbb{N}, \quad \ell = \sum_{i=1}^{\mu} \ell_i = a,$$

is said to be in

1. **T-canonical form** if
$$E_{12} = 0,$$
and each $N_{i,i+1}$ has full *column* rank for $i = 1, \ldots, \mu - 1$, i.e., $N = N^c \in SUT_{column}$ using the notation from Section 3.
Note that this implies decreasing $\ell_1 \geq \cdots \geq \ell_\mu$ and rank $N_{i,i+1} = \ell_{i+1}$.
For DAEs, a T-canonical form primarily shows a pure ODE part and provides the subspace N_{can}:
$$\begin{aligned} x_1' + F_{11}x_1 &= q_1 \\ N^c x_2' + x_2 &= q_2 - F_{21}x_1, \end{aligned}$$
whereas in case of a complete decoupling $F_{21} = 0$ is given, and therefore, the T-canonical form is already in SCF [14].
2. **S-canonical form** if
$$F_{21} = 0, \quad E_{12} = \begin{bmatrix} 0 & (E_{12})_2 & \cdots & (E_{12})_\mu \end{bmatrix},$$
and each $N_{i,i+1}$ has full *row* rank for $i = 1, \ldots, \mu - 1$, i.e., $N = N^r \in SUT_{row}$ using the notation from Section 3.
Note that this implies increasing $\ell_1 \leq \cdots \leq \ell_\mu$ and rank $N_{i,i+1} = \ell_i$.
For DAEs, an S-canonical form primarily shows a pure DAE part and provides the subspace S_{can}:
$$\begin{aligned} x_1' + E_{12}x_2' + F_{11}x_1 &= q_1, \\ N^r x_2' + x_2 &= q_2. \end{aligned}$$

Having the T- or S-canonical form with block sizes $d, \ell_1, \ldots, \ell_\mu$, we know immediately the canonical characteristics $r, \theta_0, \ldots, \theta_\mu$, see Section 3.

We show now how each of these canonical forms can be transformed into a preliminary stage of an SCF according to Section 5. In this context, once again the properties of block structured matrix valued functions with decreasing or increasing block sizes, which are summarized in Section 3, play their role. Note that in both cases, $a = \sum_{i=1}^{\mu} \ell_i = m - d$.

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8.2. T-canonical form. Due to Lemma B.1, we can assume that N has the required secondary diagonal blocks such that according to Lemma B.2

$$R^c := N(N^{(E_c)})^T + (I - N^{(E_c)}(N^{(E_c)})^T) \in BUT_{nonsingular},$$

is nonsingular and $R^{-c} := (R^c)^{-1} \in BUT_{nonsingular}$ fulfills $R^{-c}N = N^{(E_c)}$. Therefore, with

$$L_c = \begin{bmatrix} I_d & \\ & R^{-c} \end{bmatrix}, \quad K_c = \begin{bmatrix} I_d & \\ & I_a \end{bmatrix},$$

it follows

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ F_{21} & I_a \end{bmatrix} \right\} \quad \sim \quad \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_c)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ R^{-c}F_{21} & R^{-c} \end{bmatrix} \right\},$$

that is in a PreSCF according to Definition 5.3 for decreasing block sizes $\ell_1 \geq \cdots \geq \ell_\mu$. Note further that due to $F_{12} = 0$, the transformations in Steps 1 and 2 do not change neither Ω nor R^{-c}, and Steps 3 and 4 lead to

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_c)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & R^{-c} \end{bmatrix} \right\} \quad \sim \quad \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_c)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\}.$$

COROLLARY 8.1. *Let L_T, K_T be matrix functions that transform a regular pair $\{E, F\}$ into a T-canonical form. Then, for the matrix B described in Proposition 7.1, we obtain*

$$\Pi_{can} = K_T \begin{bmatrix} I_d & 0 \\ B & 0 \end{bmatrix} K_T^{-1}.$$

Proof. The assertion follows directly from Proposition 7.1, since due to $F_{12} = 0$ also $A = 0$. ☐

8.3. S-canonical form. Due to Lemma B.1, we can assume that N has the required secondary diagonal blocks such that according to Lemma B.2

$$R^r := (N^{(E_r)})^T N + (I - (N^{(E_r)})^T N^{(E_r)}) \in BUT_{nonsingular},$$

is nonsingular and $R^{-r} := (R^r)^{-1} \in BUT_{nonsingular}$ fulfills $NR^{-r} = N^{(E_r)}$. Therefore, with

$$L_r = \begin{bmatrix} I_d & \\ & I_a \end{bmatrix}, \quad K_r = \begin{bmatrix} I_d & -E_{12}R^{-r} \\ 0 & R^{-r} \end{bmatrix}, \tag{8.14}$$

it follows

$$\left\{ \begin{bmatrix} I_d & E_{12} \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_r)} \end{bmatrix}, \begin{bmatrix} \Omega & -\Omega E_{12}R^{-r} - (E_{12}R^{-r})' + E_{12}(R^{-r})' \\ 0 & R^{-r} + N(R^{-r})' \end{bmatrix} \right\},$$

that is in a PreSCF according to Definition 5.3 for increasing block sizes $\ell_1 \leq \cdots \leq \ell_\mu$. Note further that due to $F_{21} = 0$, the transformations in Steps 1 and 2 do not change either Ω or $R^{-r} + N(R^{-r})' \in BUT_{nonsingular}$, and Step 3 and 4 lead to

$$\left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_r)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & R^{-r} + N(R^{-r})' \end{bmatrix} \right\} \sim \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(E_r)} \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & I_a \end{bmatrix} \right\}.$$

COROLLARY 8.2. *Let L_S, K_S be the matrix functions that transform a regular pair $\{E, F\}$ into an S-canonical form and L_r, K_r be the matrix functions described in (8.14). Then, for the matrix A described in Proposition 7.1, we obtain*

$$\Pi_{can} = K_S K_r \begin{bmatrix} I_d & -A \\ 0 & 0 \end{bmatrix} K_r^{-1} K_S^{-1}.$$

Proof. The assertion follows from Proposition 7.1 due to $F_{21} = 0$ and therefore $B = 0$. ☐

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9. A closer look to DAEs in Hessenberg form.

9.1. Definition and canonical characteristics. Let us consider pairs $\{H_E, H_F\}$ in Hessenberg form for $\mu \geq 2$. This means that there exist $m_j \in \mathbb{N}$, $j = 1, \ldots, \mu$, $m = m_1 + \ldots + m_\mu$, $r = m_1 + \ldots + m_{\mu-1}$, $m_\mu = m - r$, matrix functions

$$H_{ij} : \mathcal{I} \to \mathbb{R}^{m_i \times m_j},$$

and that the pattern

$$H_E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad H_F = \begin{array}{cl} \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1,\mu-1} & H_{1\mu} \\ H_{21} & H_{22} & \cdots & H_{2,\mu-1} & 0 \\ & \ddots & \vdots & \vdots & \vdots \\ & & H_{\mu-1,\mu-2} & H_{\mu-1,\mu-1} & 0 \\ & & & H_{\mu,\mu-1} & 0 \end{bmatrix} & \begin{array}{l} \}\ m_1 \\ \}\ m_2 \\ \ \vdots \\ \}\ m_{\mu-1} \\ \}\ m_\mu \quad = \quad m - r \end{array} \end{array}, \tag{9.15}$$

with pointwise nonsingular

$$H_{\mu,\mu-1} H_{\mu-1,\mu-2} \cdots H_{21} H_{1\mu} : \mathcal{I} \to \mathbb{R}^{m_\mu \times m_\mu},$$

is given, such that $m_1 \geq m_2 \geq \cdots \geq m_\mu > 0$.

Using [17, Theorem 8.1], the rank properties proven in [14, Theorem 3.42] imply

$$\theta_0 = \cdots = \theta_{\mu-2} = m_\mu = m - r =: \theta, \quad d = r - (\mu - 1)\theta = m - \mu\theta.$$

In particular, this means that

$$N^{(E)} := \begin{bmatrix} 0 & I_\theta & & \\ & \ddots & \ddots & \\ & & 0 & I_\theta \\ & & & 0 \end{bmatrix} \quad \text{with} \quad \ell_1 = \cdots = \ell_\mu = \theta = m - r,$$

can be interpreted as $N^{(E_c)}$ or $N^{(E_r)}$. Of course, H_E can easily be transformed into $\begin{bmatrix} I_d & 0 \\ 0 & N^{(E)} \end{bmatrix}$. Below, if particular orthogonal L_0 and K_0 are used, then the matrix function F_0 in (3.13) presents a block structure that corresponds to a PreSCF.

9.2. The index-2 case. Let us first focus on $\mu = 2$ and therefore

$$H_E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad H_F = \begin{array}{cl} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & 0 \end{bmatrix} & \begin{array}{l} \}\ m_1 \\ \}\ m_2 \quad = \quad m - r = \theta \end{array} \end{array}, \tag{9.16}$$

with pointwise nonsingular

$$H_{21} H_{12} : \mathcal{I} \to \mathbb{R}^{m_2 \times m_2}.$$

We begin by focusing on Step 0, an initial orthogonal equivalence transformation into a PreSCF.

To decouple equations presenting this structure, we use a matrix function $B_{d,2}$ whose columns are an orthonormal basis of $\ker H_{21}$ and a matrix function $B_{a,2}$ whose columns are an orthonormal basis of the orthogonal complement, such that

$$B_{d,2} : \mathcal{I} \to \mathbb{R}^{m_1 \times (m_1 - \theta)}, \quad B_{a,2} : \mathcal{I} \to \mathbb{R}^{m_1 \times \theta}, \quad \text{im } B_{d,2} \oplus \text{im } B_{a,2} = \mathbb{R}^{m_1},$$

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and

$$B_{d,2}^T B_{d,2} = I_{m_1-\theta}, \quad B_{a,2}^T B_{a,2} = I_\theta, \quad B_{d,2}^T B_{a,2} = 0.$$

With these rectangular matrices functions, we construct the orthogonal matrix functions

$$L_B = \begin{bmatrix} B_{d,2}^T & 0 \\ B_{a,2}^T & 0 \\ 0 & I_\theta \end{bmatrix}, \quad K_B = \begin{bmatrix} B_{d,2} & 0 & B_{a,2} \\ 0 & I_\theta & 0 \end{bmatrix}, \tag{9.17}$$

such that $\{H_E, H_F\} \sim \{E_0, F_0\}$ with

$$E_0 = L_B H_E K_B = \left[\begin{array}{c|cc} I_{m_1-\theta} & 0 & 0 \\ \hline 0 & 0 & I_\theta \\ 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_d & 0 \\ 0 & N^{(E)} \end{bmatrix},$$

$$L_B H_E K_B' = \left[\begin{array}{c|cc} B_{d,2}^T B_{d,2}' & 0 & B_{d,2}^T B_{a,2} \\ \hline B_{a,2}^T B_{d,2} & 0 & B_{a,2}^T B_{a,2}' \\ 0 & 0 & 0 \end{array}\right],$$

$$F_0 = L_B H_F K_B + L_B H_E K_B' = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{array}{l} \} \quad d \\ \} \quad a \end{array},$$

with $d = m - 2\theta = m_1 - m_2$, $F_{11} : \mathcal{I} \to \mathbb{R}^{d\times d}$ and

$$F_{22} = \begin{bmatrix} (F_{22})_{11} & (F_{22})_{12} \\ 0 & (F_{22})_{22} \end{bmatrix} \begin{array}{lcl} \} \; \ell_1 & = & \theta \\ \} \; \ell_2 & = & \theta = m_2 = m - r \end{array},$$

having quadratic blocks

$$(F_{22})_{11} = B_{a,2}^T H_{12} : \mathcal{I} \to \mathbb{R}^{\theta\times\theta},$$

$$(F_{22})_{22} = H_{21} B_{a,2} : \mathcal{I} \to \mathbb{R}^{\theta\times\theta}.$$

These blocks are nonsingular, since

- H_{12} has full column rank m_2, $B_{a,2}^T$ has full row rank m_2, $m_2 = \theta$.
- H_{21} has full row rank m_2, $B_{a,2}$ has full column rank m_2, $m_2 = \theta$,
- $\theta = m_2 = \text{rank } H_{21}H_{12} = \text{rank } H_{21}B_{a,2}B_{a,2}^T H_{12} \le \min\{\text{rank } H_{21}B_{a,2}, \text{rank } B_{a,2}^T H_{12}\} \le \theta$, whereas we used that $B_{a,2}B_{a,2}^T$ is an orthogonal projector onto $(\ker H_{21})^\perp$.

Moreover, we have the structure

$$F_{12} = \begin{array}{c} \begin{array}{cc} \ell_1 & \ell_2 \end{array} \\ \begin{bmatrix} * & * \end{bmatrix} \end{array} \; m_1 - \theta,$$

as well as

$$F_{21} = \begin{array}{c} m_1 - \theta \\ \begin{bmatrix} * \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \ell_1 \\ \ell_2 \end{array},$$

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whereas $*$ stands for possibly nonzero and time-varying blocks.

Because of this pattern, we have the block structure

$$F_{21}F_{12} = \begin{array}{cc} & \begin{array}{cc} \ell_1 & \ell_2 \end{array} \\ & \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \begin{array}{l} \ell_1 = \theta \\ \ell_2 = \theta \end{array} \end{array},$$

such that a PreSCF is given. Therefore, $F_{21}F_{12}F_{22}^{-1}N^{(E)}$ and $N^{(E)}F_{22}^{-1}F_{21}F_{12}$ have the strictly upper block structure

$$\begin{array}{c} \begin{array}{cc} \ell_1 & \ell_2 \end{array} \\ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \begin{array}{l} \ell_1 = \theta \\ \ell_2 = \theta \end{array} \end{array},$$

as required for Steps 1 and 2.

Once Step 0 is available, we compute an SCF and an SSCF as outlined in the previous sections.

Corollary 9.1. *The orthogonal matrix functions L_B, K_B from (9.17) transform Hessenberg index-2 pairs $\{H_E, H_F\}$ into a PreSCF. Consequently, an SCF and an SSCF can be computed as described in Section 6.*

An example can be found in Appendix C.1.

9.3. The particular structure resulting in multibody dynamics. To motivate the structure, we consider in the next section, let us first have closer look at the DAEs resulting in multibody dynamics.

9.3.1. Structure without angular velocities. Considering linear (or linearized) constrained multibody systems with p and v describing the n_p position and velocity coordinates, respectively, leads to equations of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p' \\ v' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -I & 0 \\ K & D & G^T \\ G & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \\ \lambda \end{bmatrix} = q,$$

where G is the $n_\lambda \times n_p$ constraint matrix, K, D, M are the stiffness-, damping-, and mass matrices, q may contain time-dependent excitations, and the n_λ unknowns λ are the so-called Lagrange multipliers [8]. For constant G, K, D, M, the state space form is well understood. Here, we focus on the case that these matrices may vary with time, although we drop the argument t for the sake of clarity. We start permuting and scaling the equations with

$$L_H = \begin{bmatrix} 0 & M^{-1} & 0 \\ I_{n_p} & 0 & 0 \\ 0 & 0 & I_{n_\lambda} \end{bmatrix}, \quad K_H = \begin{bmatrix} 0 & I_{n_p} & 0 \\ I_{n_p} & 0 & 0 \\ 0 & 0 & I_{n_\lambda} \end{bmatrix},$$

to obtain the pair in Hessenberg form

$$\{H_E, H_F\} = \left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M^{-1}D & M^{-1}K & M^{-1}G^T \\ -I & 0 & 0 \\ 0 & G & 0 \end{bmatrix} \right\},$$

with nonsingular $GM^{-1}G^T$, $m_1 = m_2 = n_p$, $m_3 = n_\lambda = \theta$, $x_1 = v$, $x_2 = p$, $x_3 = \lambda$.

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9.3.2. Structure with angular velocities. We also want to emphasize that if in the 3D-case the angular velocities are considered as well, then in the linear (or linearized) case, we consider

$$x_1 = \begin{bmatrix} v \\ s \end{bmatrix}, \quad x_2 = p, \quad x_3 = \lambda, \quad m_1 = n_p + n_s,\ m_2 = n_p,\ m_3 = n_\lambda = \theta,$$

and $p' = Zv$ for a regular transformation matrix function Z, leading to

$$H_{13} = \begin{bmatrix} M^{-1}Z^T G^T \\ 0 \end{bmatrix}, \quad H_{21} = \begin{bmatrix} -Z & 0 \end{bmatrix}, \quad H_{32} = G,$$

see [8], [14], with nonsingular

$$H_{32}H_{21}H_{13} = -GZM^{-1}Z^TG^T.$$

9.4. Transformation for a particular Hessenberg form of order 3. From the preceding considerations, it results the special Hessenberg form

$$\left\{ \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ [Z_{21}\ 0] & 0 & 0 \\ 0 & G & 0 \end{bmatrix} \right\},$$

for a quadratic, nonsingular Z_{21} . Notice now that the equivalence transformation with

$$L_Z = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & Z_{21}^{-1} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix}, \quad K_Z = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & Z_{21} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix},$$

leads to

$$\left\{ \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} H_{11} & H_{12}Z_{21} & H_{13} \\ [I_{m_2}\ 0] & Z_{21}^{-1}Z_{21}' & 0 \\ 0 & H_{32}Z_{21} & 0 \end{bmatrix} \right\},$$

with nonsingular

$$H_{32}Z_{21}[I_{m_2}\ 0]H_{13} = -GZM^{-1}Z^TG^T.$$

Therefore, for the sake of simplicity in the following, we assume that the particular Hessenberg structure

$$\{H_E, H_F\} = \left\{ \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ [I_{m_2}\ 0] & H_{22} & 0 \\ 0 & H_{32} & 0 \end{bmatrix} \right\}, \tag{9.18}$$

with pointwise nonsingular $H_{32}[I_{m_2}\ 0]H_{13}$, implying rank $H_{32} = m_3 = \theta$, $m = m_1 + m_2 + m_3$, $r = m_1 + m_2$. Of course, for $m_1 = m_2$, we simply assume $[I_{m_2}0] = I_{m_2} = I_{m_1}$.

Finally, we like to note that the transformations from below could also be generalized for Hessenberg forms with $\mu > 3$ assuming

$$H_{i+1,i} = [I_{m_{i+1}}\ 0] \quad \text{for} \quad i = 1, \dots, \mu - 2.$$

Let us focus first on Step 0, i.e. an initial orthogonal equivalence transformation into PreSCF.

To decouple pairs presenting the structure (9.18),

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- we first use a matrix function $B_{d,3}$ whose columns are an orthonormal basis of $\ker H_{32}$ and a matrix function $B_{a,3}$ whose columns are an orthonormal basis of the orthogonal complement, such that

$$B_{d,3} : \mathcal{I} \to \mathbb{R}^{m_2\times(m_2-\theta)}, \quad B_{a,3} : \mathcal{I} \to \mathbb{R}^{m_2\times\theta}, \quad \text{im } B_{d,3} \oplus \text{im } B_{a,3} = \mathbb{R}^{m_2},$$

  and

$$B_{d,3}^T B_{d,3} = I_{m_2-\theta}, \quad B_{a,3}^T B_{a,3} = I_\theta, \quad B_{d,3}^T B_{a,3} = 0,$$

- we second use matrix functions

$$B_{d,2} = \begin{bmatrix} B_{d,3} & 0 \\ 0 & I_{m_1-m_2} \end{bmatrix} : \mathcal{I} \to \mathbb{R}^{m_1\times(m_1-\theta)}, \quad B_{a,2} = \begin{bmatrix} B_{a,3} \\ 0 \end{bmatrix} : \mathcal{I} \to \mathbb{R}^{m_1\times\theta},$$

  fulfilling $\ker H_{32}H_{21} = \ker H_{32}[I_{m_2}\ 0] = \text{im } B_{d,2}$, $\text{im } B_{d,2} \oplus \text{im } B_{a,2} = \mathbb{R}^{m_1}$,

$$B_{d,2}^T B_{d,2} = I_{m_1-\theta}, \quad B_{a,2}^T B_{a,2} = I_\theta, \quad B_{d,2}^T B_{a,2} = 0,$$

- finally, since $\ker H_{32}H_{21}H_{13} = \{0\}$, we set $B_{a,1} = I_{m_3}$.

With this notation, we define orthogonal

$$(9.19)\qquad L_B = \begin{array}{c} \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{bmatrix} B_{d,2}^T & 0 & 0 \\ 0 & B_{d,3}^T & 0 \\ B_{a,2}^T & 0 & 0 \\ 0 & B_{a,3}^T & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix} \end{array} \begin{array}{l} \\ m_1-\theta \\ m_2-\theta \\ \theta \\ \theta \\ m_3=\theta \end{array}, \quad K_B = \begin{array}{c} \begin{array}{ccccc} m_1-\theta & m_2-\theta & m_3 & \theta & \theta \end{array} \\ \begin{bmatrix} B_{d,2} & 0 & 0 & B_{a,2} & 0 \\ 0 & B_{d,3} & 0 & 0 & B_{a,3} \\ 0 & 0 & I_{m_3} & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ m_1 \\ m_2 \\ m_3=\theta \end{array}.$$

For (9.18) with $d = m - 3\theta = m_1 + m_2 - 2m_3$, $L_0 = L_B$ and $K_0 = K_B$ leads to

$$L_0 H_E K_0 = \begin{bmatrix} I_d & 0 \\ 0 & N^{(E)} \end{bmatrix}, \quad N^{(E)} = \begin{bmatrix} 0 & I_\theta & 0 \\ 0 & 0 & I_\theta \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{a\times a}.$$

$$L_0 H_E K_0' = \begin{array}{c} \begin{array}{ccccc} m_1-\theta & m_2-\theta & \theta & \theta & \theta \end{array} \\ \left[\begin{array}{cc|ccc} B_{d,2}^T B_{d,2}' & 0 & 0 & B_{d,2}^T B_{a,2}' & 0 \\ 0 & B_{d,3}^T B_{d,3}' & 0 & 0 & B_{d,3}^T B_{a,3}' \\ B_{a,2}^T B_{d,2}' & 0 & 0 & B_{a,2}^T B_{a,2}' & 0 \\ 0 & B_{a,3}^T B_{d,3}' & 0 & 0 & B_{a,3}^T B_{a,3}' \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ m_1-\theta \\ m_2-\theta \\ \theta \\ \theta \\ \theta=m_3 \end{array},$$

and

$$F_0 := L_0 H_F K_0 + L_0 H_E K_0' =: \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{array}{l} \}\ d \\ \}\ a \end{array},$$

with $d = m - 3\theta = m_1 + m_2 - 2m_3$, $F_{11} : \mathcal{I} \to \mathbb{R}^{d\times d}$ and

$$F_{22} = \begin{bmatrix} (F_{22})_{11} & (F_{22})_{12} & (F_{22})_{13} \\ 0 & (F_{22})_{22} & (F_{22})_{23} \\ 0 & 0 & (F_{22})_{33} \end{bmatrix} \begin{array}{lcl} \}\ \ell_1 & = & \theta \\ \}\ \ell_2 & = & \theta \\ \}\ \ell_3 & = & \theta = m_3 = m - r \end{array},$$

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having quadratic blocks

$$\begin{aligned}(F_{22})_{11} &= B_{a,2}^T H_{13} : \mathcal{I} \to \mathbb{R}^{\theta\times\theta},\\ (F_{22})_{22} &= B_{a,3}^T [I_{m_2}\ 0] B_{a,2} = I_{\ell_2} = I_\theta,\\ (F_{22})_{33} &= H_{32} B_{a,3} : \mathcal{I} \to \mathbb{R}^{\theta\times\theta},\end{aligned}$$

that are nonsingular, since

- H_{13} has full column rank m_3, $B_{a,2}^T$ has full row rank $m_3 = \theta$,
- H_{32} has full row rank m_2, $B_{a,3}$ has full column rank $m_3 = \theta$,
- $H_{32}H_{21}H_{13} = H_{32}B_{a,3}B_{a,3}^T \begin{bmatrix} I_{m_2} & 0 \end{bmatrix} H_{13} = H_{32}B_{a,3}B_{a,2}^T H_{13}$, and therefore

$$\theta = m_3 = \operatorname{rank}\, H_{32}H_{21}H_{13} \le \min\left\{\operatorname{rank}\, H_{32}B_{a,3}, \operatorname{rank}\, B_{a,2}^T H_{13}\right\} \le \theta,$$

whereas we used that $B_{a,3}B_{a,3}^T$ is an orthogonal projector onto $(\ker H_{32})^\perp$.

Moreover, we have the structure

$$F_{12} = \begin{array}{c} \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \end{array} \\ \begin{bmatrix} * & * & * \\ 0 & * & * \end{bmatrix} \end{array} \begin{array}{l} \\ m_1-\theta \\ m_2-\theta \end{array},$$

as well as

$$F_{21} = \begin{array}{c} \begin{array}{cc} m_1-\theta & m_2-\theta \end{array} \\ \begin{bmatrix} * & * \\ 0 & * \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \ell_1=\theta \\ \ell_2=\theta \\ \ell_3=\theta \end{array},$$

whereas $*$ stands for possibly nonzero and time-varying blocks.

Because of this pattern, we have the block structure

$$F_{21}F_{12} = \begin{array}{c} \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \end{array} \\ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \ell_1=\theta \\ \ell_2=\theta \\ \ell_3=\theta \end{array},$$

such that a PreSCF is given. Therefore, $F_{21}F_{12}F_{22}^{-1}N^{(E)}$ and $N^{(E)}F_{22}^{-1}F_{21}F_{12}$ have the strictly upper block structure

$$\begin{array}{c} \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \end{array} \\ \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \ell_1=\theta \\ \ell_2=\theta \\ \ell_3=\theta \end{array},$$

that is crucial for Steps 1 and 2. Recall that for arbitrary $\{E, F\}$, the transformation matrix functions L_0 and K_0 to obtain the block structure $\{E_0, F_0\}$ may be composed step by step. Indeed, for multibody systems, we showed

$$L_0 = L_B L_Z L_H, \quad K_0 = K_H K_Z K_B. \tag{9.20}$$

Now that Step 0 is given, we can compute an SCF and SSCF as described before.

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Corollary 9.2.

- *The orthogonal matrix functions L_B, K_B from (9.19) transform the Hessenberg index-3 pairs $\{H_E, H_F\}$ with the particular structure (9.18) into a PreSCF.*
- *Pairs resulting from linear (or linearized) constrained multibody systems as described in Section 9.3 can be transformed into a PreSCF with the nonsingular matrix functions L_0, K_0 from (9.20).*

Consequently, in both cases, an SCF and an SSCF can be computed as described in Section 6.

An example can be found in Appendix C.2.

10. Conclusion. DAEs in SCF or even SSCF are easy to understand and to handle. However, the equivalence transformation of linear time-varying DAEs into an SCF is usually rather difficult. Fortunately, for some classes of structured DAEs with known canonical characteristics, a transformation into the preliminary stage PreSCF introduced in Definition 5.3 results to be simpler and as soon as we have it, the computation of the SCF becomes possible with the described steps, which take advantage of the nilpotency of particular block structured strictly upper matrices.

Of course, in general, the computation of this preliminary stage is not trivial either, but, at least for some classes of DAEs, relatively easy. We showed how to transform the T-canonical and the S-canonical forms from the tractability and strangeness frameworks into PreSCF. Moreover, we considered Hessenberg forms that have a completely different structure and computed the SCF step-by-step via the PreSCF. We are confident that for further classes of DAEs, appropriate general descriptions of suitable transformations may be found.

Once the SCF and the corresponding transformation matrix function K that describes the transformation of unknowns is found, a description of the canonical subspaces S_{can} and N_{can}, as well as the canonical projector is possible. In particular, we provide this for DAEs arising from linear or linearized multibody systems.

Appendix A. A note on transformations of explicit (pure) ODEs.

Consider an explicit ODE

$$x' + \Omega x = 0,$$

with a time-varying matrix function $\Omega : \mathcal{I} \to \mathbb{R}^{d\times d}$, and an unknown function x. A transformation $x = K\tilde{x}$ by a pointwise nonsingular smooth matrix function K leads to

$$x' + \Omega x = K'\tilde{x} + K\tilde{x}' + \Omega K\tilde{x} = K\tilde{x}' + (\Omega K + K')\tilde{x},$$

such that the premultiplication by $L := K^{-1}$ provides the transformed ODE $\tilde{x}' + \tilde{\Omega}\tilde{x} = 0$ with $\tilde{\Omega} = K^{-1}(\Omega K + K')$.

Let us now suppose that for any given $\alpha \in \mathbb{R}$, we want to compute K such that $\tilde{\Omega} = \alpha I$. From $\alpha I = K^{-1}(\Omega K + K')$, it follows that such a matrix function K has to fulfill the system of ODEs

$$K' = (\alpha I - \Omega)K.$$

If we choose a nonsingular initial value $K(0) = K_0$, the solution K is nonsingular on $\mathcal{I}$ due to the formula of Liouville. Indeed, K is the fundamental matrix of the above ODE. Consequently,

$$\{I, \Omega\} \overset{K^{-1},K}{\sim} \{I, \alpha I\}.$$

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Notice that for $\alpha_1 > 0$ and $\alpha_2 < 0$, we therefore can find nonsingular K_1 and K_2, such that the equivalent ODEs present completely different stability properties. This is a crucial difference between constant transformations of coordinates and time-varying transformations.

This means that for DAEs, as showed also in Examples 2.7 and 2.8, different pure ODEs may present completely different stability properties. However, in both Examples, either $|K_{11}|$ or $\left|K_{11}^{-1}\right|$ grows unboundedly for $t \to \infty$.

Moreover, the above considerations imply that with the scope of the here permitted transformations, we can formulate a further possible definition of regular DAEs.

Proposition A.1. *A pair of matrix functions $\{E, F\}$ is regular, iff*

$$\{E, F\} \;\sim\; \left\{ \begin{bmatrix} I_d & 0 \\ 0 & N^{(C)} \end{bmatrix}, \begin{bmatrix} \alpha I_d & 0 \\ 0 & I_a \end{bmatrix} \right\},$$

for any $\alpha \in \mathbb{R}$ and a constant, nilpotent $N^{(C)} \in \mathbb{R}^{a \times a}$.

This proposition means that a time-varying linear DAE is regular, iff it is equivalent to a time-invariant regular DAE. Moreover, by choosing $\alpha = 0$ under the considered equivalence, the explicit ODE part can always be simplified to the trivial case $u' = \bar{q}_1$.

Appendix B. Some technical details of SUT matrix functions. As a supplement to Section 3, this appendix section summarizes certain properties of particular SUT matrix functions established in [18] that are used in the present article.

Lemma B.1. *[18]*

- *Every pair $\{N^c, I\}$ with a sufficiently smooth matrix function $N^c \in SUT_{column}$ can be equivalently transformed into a pair $\{\hat{N}^c, I\}$ with a matrix function $\hat{N}^c \in SUT_{column}$ showing special secondary diagonal blocks*

$$\hat{N}^c_{i,i+1} = \begin{bmatrix} R_{i+1,i+1} \\ 0 \end{bmatrix},$$

  and pointwise nonsingular $R_{i+1,i+1}$.
- *Every pair $\{N^r, I\}$ with a sufficiently smooth matrix function $N^r \in SUT_{row}$ can be equivalently transformed into a pair $\{\hat{N}^r, I\}$ with a matrix function $\hat{N}^r \in SUT_{row}$ showing special secondary diagonal blocks*

$$\hat{N}^r_{i,i+1} = \begin{bmatrix} R_{i,i} & 0 \end{bmatrix},$$

  and pointwise nonsingular $R_{i,i}$.

Proof. A proof that uses the smooth SVD can be found in [18]. A visualization of the pattern is given in Figure 5. □

Lemma B.2. *[18]*

- *In case of decreasing $\ell_1 \geq \cdots \geq \ell_\mu$, for any $N^c \in SUT_{column}$ it holds that*

$$R^c := N^c (N^{(E_c)})^T + (I - N^{(E_c)} (N^{(E_c)})^T) \in BUT_{nonsingular}$$

  and

$$R^c (N^{(E_c)}) = N^c.$$

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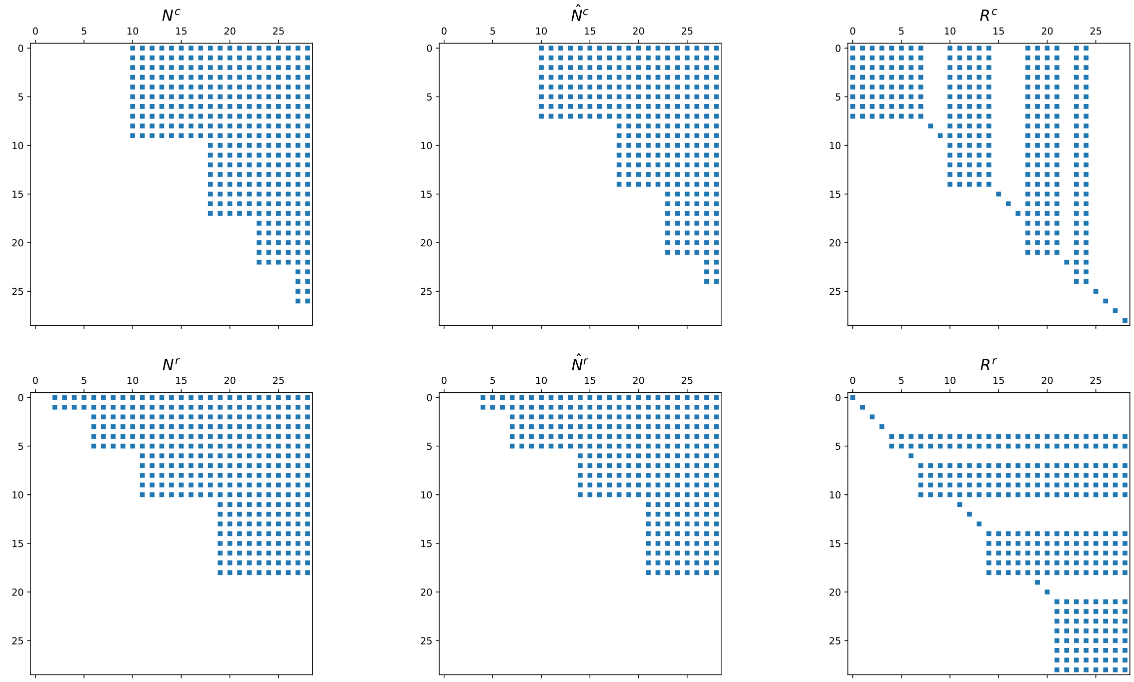

FIGURE 5. *Visualization of the pattern of N^c, $\hat{N}^c$ and a nonsingular R^c constructed with $\hat{N}^c$ for decreasing block sizes (top) as well as N^r, $\hat{N}^r$ and a corresponding nonsingular R^r for increasing block sizes (bottom), cf. Lemma B.1 and Lemma B.2.*

Moreover, if the secondary diagonal blocks have the pattern

$$N^c_{i,i+1} = \begin{bmatrix} R_{i+1,i+1} \\ 0 \end{bmatrix},$$

with pointwise nonsingular $R_{i+1,i+1}$, then R^c is pointwise nonsingular and $(R^c)^{-1}N^c = N^{(E_c)}$.

- *In case of increasing $\ell_1 \leq \cdots \leq \ell_\mu$, for any $N^r \in SUT_{row}$, it holds that*

$$R^r := (N^{(E_r)})^T N^r + (I - (N^{(E_r)})^T N^{(E_r)}) \in BUT_{nonsingular},$$

and

$$(N^{(E_r)})R^r = N^r.$$

Moreover, if the secondary diagonal blocks have the pattern

$$N^r_{i,i+1} = \begin{bmatrix} R_{i+1,i+1} & 0 \end{bmatrix},$$

with pointwise nonsingular $R_{i+1,i+1}$, then R^r is nonsingular and $N^r(R^r)^{-1} = N^{(E_r)}$.

Proof. A proof can be found in [18] (Lemmas 4.2 and 5.2 therein), and the reasoning thereafter. A visualization of the pattern of the nonsingular matrix functions is given in Fig. 5. □

LEMMA B.3. *For every $N \in SUT_{c/r}$, there exists a $R^{c/r} \in BUT_{nonsingular}$ such that*

$$\{N, I\} \sim \{N^{(E_{c/r})}, (R^{c/r})^{-1}\}.$$

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Proof. The assertions follows from Lemmas B.1 and B.2. ∎

Appendix C. Illustrative examples in Hessenberg form. In this appendix section, we present an illustrative step-by-step computation of the SSCF and the canonical projector for two examples in Hessenberg form fulfilling the assumptions described in Section 9.

C.1. An index-2 example.

Example C.1. *[10, Section 5], [14, Example 8.11] Hessenberg form, index $\mu = 2$, $m = 3$, $r = 2$, $\theta_0 = 1$, $d = 1$.*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \lambda & -1 & -1 \\ \eta\left(-\eta t^2 + t - 1\right) & \lambda & -\eta t \\ -\eta t + 1 & 1 & 0 \end{bmatrix}.$$

For shortness, let us define $\gamma(t) := \sqrt{(-\eta t + 1)^2 + 1} \neq 0$.

Step 0: $\{E, F\} \sim \{E_0, F_0\}$

$$L_0 = \begin{bmatrix} \frac{1}{\gamma(t)} & \frac{\eta t - 1}{\gamma(t)} & 0 \\ \frac{-\eta t + 1}{\gamma(t)} & \frac{1}{\gamma(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K_0 = \begin{bmatrix} \frac{1}{\gamma(t)} & 0 & \frac{-\eta t + 1}{\gamma(t)} \\ \frac{\eta t - 1}{\gamma(t)} & 0 & \frac{1}{\gamma(t)} \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_0 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_d & 0 \\ 0 & N^{(E)} \end{bmatrix},$$

$$F_0 = \left[\begin{array}{c|cc} \frac{-\eta(\eta t-1)(t(\eta t-1)+1)+\lambda+(\eta t-1)(\lambda(\eta t-1)-1)}{\gamma(t)^2} & \frac{-\eta t(\eta t-1)-1}{\gamma(t)} & \frac{\eta^4 t^4-3\eta^3 t^3+\eta^3 t^2+3\eta^2 t^2-2\eta^2 t-\eta t-1}{\gamma(t)^2} \\ \hline \frac{-\eta t+1}{\gamma(t)^2} & -\frac{1}{\gamma(t)} & \frac{\eta t+\lambda+(\eta t-1)(\eta t(\eta t-1)+\eta+\lambda(\eta t-1))-1}{\gamma(t)^2} \\ 0 & 0 & \gamma(t) \end{array}\right].$$

As expected, by construction F_{22} is nonsingular and (block) triangular. Now we show how starting from $\{E_0, F_0\}$ an SCF and an SSCF can be constructed.

Step 1: $\{E_0, F_0\} \sim \{E_1, F_1\}$ with $E_1 = E_0$ and

$$L_1 = \left[\begin{array}{c|cc} 1 & -\eta^2 t^2 + \eta t - 1 & \eta t \gamma(t) \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_1 = \left[\begin{array}{c|cc} 1 & 0 & \eta^2 t^2 - \eta t + 1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$$F_1 = \left[\begin{array}{c|cc} \omega(t) & 0 & 0 \\ \hline \frac{-\eta t+1}{\gamma(t)^2} & -\frac{1}{\gamma(t)} & \frac{\eta^2\lambda t^2+\eta^2 t-2\eta\lambda t-\eta+2\lambda}{\gamma(t)^2} \\ 0 & 0 & \gamma(t) \end{array}\right],$$

for

$$\omega(t) := \frac{\eta^2 \lambda t^2 - \eta^2 t - 2\eta\lambda t + \eta + 2\lambda}{\gamma(t)^2} = \lambda + \frac{-\eta^2 t + \eta}{\gamma(t)^2}.$$

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Step 2: $\{E_1, F_1\} \sim \{E_2, F_2\}$ with $E_2 = E_1 = E_0$ and

$$L_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \frac{-\eta t+1}{\gamma(t)} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$$F_2 = \left[\begin{array}{c|cc} \omega(t) & 0 & 0 \\ \hline 0 & -\frac{1}{\gamma(t)} & \frac{\eta^2\lambda t^2+\eta^2 t-2\eta\lambda t-\eta+2\lambda}{\gamma(t)^2} \\ 0 & 0 & \gamma(t) \end{array}\right].$$

Step 3: $\{E_2, F_2\} \sim \{E_{SCF}, F_{SCF}\}$

$$L_3 = \begin{bmatrix} 1 & 0 \\ 0 & (F_2)_{22}^{-1} \end{bmatrix}, \quad K_3 = I_3,$$

$$E_{SCF} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -\gamma(t) \\ 0 & 0 & 0 \end{array}\right], \quad F_{SCF} = \left[\begin{array}{c|cc} \omega(t) & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Step 4: Following the procedure from [18], with

$$L_4 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{1}{-\gamma(t)} & 0 \\ 0 & 0 & 1 \end{array}\right], \quad K_4 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -\gamma(t) & 0 \\ 0 & 0 & 1 \end{array}\right],$$

we finally obtain

$$\{E_{SCF}, F_{SCF}\} \sim \{E_{SSCF}, F_{SSCF}\},$$

with

$$E_{SSCF} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad F_{SSCF} = F_{SCF} = \left[\begin{array}{c|cc} \omega(t) & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Note that the existence of an SCF and an SSCF for the same DAE corresponds to the uniqueness properties of the SCF, see Theorem 2.5 in the Appendix 2.4.

Since

$$L := L_2 L_1 L_0 = \begin{bmatrix} \eta t\gamma(t) & \frac{-\eta^2 t^2+2\eta t-2}{\gamma(t)} & \eta t\gamma(t) \\ \frac{-\eta t+1}{\gamma(t)} & \frac{1}{\gamma(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$K := K_0 K_1 K_2 = \begin{bmatrix} \frac{1}{\gamma(t)} & 0 & \gamma(t) \\ \frac{\eta t-1}{\gamma(t)} & 0 & \eta t\gamma(t) \\ \frac{-\eta t+1}{\gamma(t)} & 1 & 0 \end{bmatrix},$$

provide $\{E_2, F_2\}$, with

$$K^{-1} = \begin{bmatrix} \eta t\gamma(t) & -\gamma(t) & 0 \\ \eta t\,(\eta t-1) & -\eta t+1 & 1 \\ \frac{-\eta t+1}{\gamma(t)} & \frac{1}{\gamma(t)} & 0 \end{bmatrix},$$

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we can represent the canonical projector

$$\Pi_{can} = K \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} K^{-1} = \begin{bmatrix} \eta t & -1 & 0 \\ \eta t\,(\eta t - 1) & -\eta t + 1 & 0 \\ \eta t\,(-\eta t + 1) & \eta t - 1 & 0 \end{bmatrix}. \tag{C.21}$$

Alternatively, we can compute Π_{can} with Proposition 7.1. From the representation of K_1 and K_2, we know

$$A = \begin{bmatrix} 0 & \eta^2 t^2 - \eta t + 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-\eta t + 1}{\gamma(t)} \\ 0 \end{bmatrix},$$

such that $AB = 0$, therefore also $ABA = 0$, and

$$\Pi_{can} = K_0 \begin{bmatrix} I_d & -A \\ B & -BA \end{bmatrix} K_0^{-1} = K_0 \begin{bmatrix} 1 & 0 & -\eta^2 t^2 + \eta t - 1 \\ \frac{-\eta t+1}{\gamma(t)} & 0 & \frac{(-\eta t+1)(\eta^2 t^2 - \eta t + 1)}{\gamma(t)} \\ 0 & 0 & 0 \end{bmatrix} K_0^{-1},$$

that also leads to the expression (C.21).

Let us finally compute the solution of the resulting homogeneous DAE $E_{SSCF} z' + F_{SSCF} z = 0$ for $z = K^{-1}x$. The solution of the pure ODE

$$z_1' = -\omega(t) z_1,$$

reads

$$z_1(t) = C \cdot e^{-\lambda t} \cdot \gamma(t).$$

and the pure DAE leads to

$$z_3 = 0, \quad z_2 = 0,$$

such that

$$x = Kz = \begin{bmatrix} \frac{1}{\gamma(t)} & 0 & \gamma(t) \\ \frac{\eta t - 1}{\gamma(t)} & 0 & \eta t \gamma(t) \\ \frac{-\eta t + 1}{\gamma(t)} & 1 & 0 \end{bmatrix} \begin{bmatrix} C \cdot e^{-\lambda t} \cdot \gamma(t) \\ 0 \\ 0 \end{bmatrix} = C \cdot e^{-\lambda t} \begin{bmatrix} 1 \\ \eta t - 1 \\ -\eta t + 1 \end{bmatrix}.$$

This coincides with the known solution from literature. However, note that the pure ODE obtained in [10] from an IERODE is $\tilde{z}_1' = -\lambda\, \tilde{z}_1$. This apparent difference results from the choice of K_0 only and corresponds to $z_1 = \gamma\, \tilde{z}_1$, since also the pure ODE is not unique. Indeed, in terms of Corollary 2.6, for

$$\bar{L} = \begin{bmatrix} -\gamma(t) & 0 \\ 0 & I_2 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} \frac{1}{-\gamma(t)} & 0 \\ 0 & I_2 \end{bmatrix},$$

it holds

$$\{E_{SSCF}, F_{SSCF}\} \overset{\bar{L},\bar{K}}{\sim} \left\{ \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \right\}.$$

We want to emphasize also that choosing at the beginning

$$\tilde{L}_0 = \begin{bmatrix} \frac{1}{\gamma(t)^2} & \frac{\eta t - 1}{\gamma(t)^2} & 0 \\ \frac{-\eta t + 1}{\gamma(t)^2} & \frac{1}{\gamma(t)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{K}_0 = \begin{bmatrix} 1 & 0 & -\eta t + 1 \\ \eta t - 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

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Steps 0, 1, 2 lead to the pure ODE $\tilde{z}_1' = -\lambda \tilde{z}_1$ straight forward. This makes clear that, although in general we proposed to choose orthogonal L_0 and K_0, a different scaling may be reasonable to define K_0 as simple as possible.

C.2. An index-3 example: The linearized Campbell–Moore DAE. The following example is a linearized version of a test problem proposed in [5] and has been discussed extensively in [11].

EXAMPLE C.2. *Semi-explicit, index $\mu = 3$, $m = 7$, $r = 6$, $\theta_0 = \theta_1 = 1$, $d = 4$, assuming $\alpha \neq 0$.*

$$E = \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \sin(t) & 0 & 1 & -\cos(t) & -\alpha\cos^2(t) \\ 0 & 0 & -\cos(t) & -1 & 0 & -\sin(t) & -\alpha\sin(t)\cos(t) \\ 0 & 0 & 1 & 0 & 0 & 0 & \alpha\sin(t) \\ \alpha\cos^2(t) & \alpha\sin(t)\cos(t) & -\alpha\sin(t) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This DAE is not directly in Hessenberg form. To decouple it step by step, we use several equivalence transformations

$$\{E, F\} \quad \sim \quad \{LEK, LFK + LEK'\}.$$

C.2.1. Permutation into Hessenberg form. With the block permutation matrix

$$P = \begin{bmatrix} 0 & I_3 & 0 \\ I_3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain a pair in Hessenberg form $\{H_E, H_F\}$ for $L_H = K_H = P = P^T$, such that

$$\{E, F\} \sim \{H_E, H_F\},$$

and

$$H_E = L_H R K_H = E = \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_F = L_H F K_H,$$

$$H_F = \left[\begin{array}{ccc|ccc|c} 0 & 1 & -\cos(t) & 0 & 0 & \sin(t) & -\alpha\cos^2(t) \\ -1 & 0 & -\sin(t) & 0 & 0 & -\cos(t) & -\alpha\sin(t)\cos(t) \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha\sin(t) \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha\cos^2(t) & \alpha\sin(t)\cos(t) & -\alpha\sin(t) & 0 \end{array}\right].$$

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C.2.2. Transformation into a PreSCF. The goal of this step is to construct $\{E_0, F_0\}$ that are in a PreSCF and fulfill

$$\{H_E, H_F\} \sim \{E_0, F_0\},$$

according to Section 9.4. To this end, we consider a particular basis.

- Since $H_{32} = \begin{bmatrix} \alpha\cos^2(t) & \alpha\sin(t)\cos(t) & -\alpha\sin(t) \end{bmatrix}$, we may choose

$$B_{d3} = \begin{bmatrix} \sin(t)\cos(t) & \sin(t) \\ \sin^2(t) & -\cos(t) \\ \cos(t) & 0 \end{bmatrix}, \quad B_{a3} = \begin{bmatrix} \cos^2(t) \\ \sin(t)\cos(t) \\ -\sin(t) \end{bmatrix}.$$

  Note that B_{d3} is not uniquely determined and could have been chosen in a different way.
- Since $H_{32}H_{21} = -H_{32}$ we can choose $B_{d2} = B_{d3}$ and $B_{a2} = B_{a3}$.
- Since $H_{32}H_{21}H_{13} = -\alpha^2 \neq 0$, $B_{a1} = 1$.

Consider

$$L_B = \left[\begin{array}{ccc|ccc|c} \sin(t)\cos(t) & \sin^2(t) & \cos(t) & 0 & 0 & 0 & 0 \\ \sin(t) & -\cos(t) & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \sin(t)\cos(t) & \sin^2(t) & \cos(t) & 0 \\ 0 & 0 & 0 & \sin(t) & -\cos(t) & 0 & 0 \\ \hline \cos^2(t) & \sin(t)\cos(t) & -\sin(t) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos^2(t) & \sin(t)\cos(t) & -\sin(t) & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$K_B = \left[\begin{array}{cc|cc|c|c|c} \sin(t)\cos(t) & \sin(t) & 0 & 0 & 0 & \cos^2(t) & 0 \\ \sin^2(t) & -\cos(t) & 0 & 0 & 0 & \sin(t)\cos(t) & 0 \\ \cos(t) & 0 & 0 & 0 & 0 & -\sin(t) & 0 \\ \hline 0 & 0 & \sin(t)\cos(t) & \sin(t) & 0 & 0 & \cos^2(t) \\ 0 & 0 & \sin^2(t) & -\cos(t) & 0 & 0 & \sin(t)\cos(t) \\ 0 & 0 & \cos(t) & 0 & 0 & 0 & -\sin(t) \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Step 0: With $L_0 = L_B L_P$, $K_0 = K_P K_B$, we obtain

$$E_0 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

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$$F_0 = \left[\begin{array}{cccc|ccc} -\sin(t)\cos(t) & 0 & \cos^2(t) & 0 & 0 & -\cos^2(t) & -\sin(t)\cos(t) \\ 0 & 0 & \cos(t) & 0 & 0 & 0 & -\sin(t) \\ -1 & 0 & 0 & \sin(t) & 0 & 0 & -1 \\ 0 & -1 & -\sin(t) & 0 & 0 & 0 & -\cos(t) \\ \hline \sin^2(t) & 0 & -\sin(t)\cos(t) & 0 & -\alpha & \sin(t)\cos(t) & \sin^2(t) \\ 0 & 0 & 1 & \cos(t) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{array}\right].$$

C.2.3. Steps 1 and 2: Iterative procedures. Step 1: For

$$L_1 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & -\cos^2(t) & \frac{\left(\cos^2(t)+3\right)\sin(t)\cos(t)}{\alpha} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{\sin(t)}{\alpha} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{\cos^2(t)+1}{\alpha} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{\cos(t)}{\alpha} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad K_1 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & \cos^2(t) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

by Lemma 5.1, we obtain $E_1 = E_0$ and

$$F_1 = \left[\begin{array}{cccc|ccc} -\sin(t)\cos(t) & 0 & 0 & -\cos^3(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \sin(t) & 0 & 0 & 0 \\ 0 & -1 & -\sin(t) & 0 & 0 & 0 & 0 \\ \hline \sin^2(t) & 0 & -\sin(t)\cos(t) & 0 & -\alpha & \sin(t)\cos(t) & \left(\cos^2(t)+1\right)\sin^2(t) \\ 0 & 0 & 1 & \cos(t) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{array}\right].$$

Step 2: For

$$L_2 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -\cos(t) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$K_2 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \frac{\sin^2(t)+1}{\alpha} & \frac{\cos(t)}{\alpha} & \frac{\sin(t)\cos(t)}{\alpha} & -\frac{\sin^3(t)+\sin(t)}{\alpha} & 1 & 0 & 0 \\ 0 & 0 & 1 & \cos(t) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

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$E_2 = E_1 = E_0$ and by Lemma 5.2, we obtain $E_2 = E_1 = E_0$ and

$$F_2 = \left[\begin{array}{cccc|ccc} -\sin(t)\cos(t) & 0 & 0 & -\cos^3(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \sin(t) & 0 & 0 & 0 \\ 0 & -1 & -\sin(t) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha & \sin(t)\cos(t) & \left(\cos^2(t)+1\right)\sin^2(t) \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{array}\right].$$

C.2.4. Representation of the canonical projector. From the representation of K_1 and K_2, we know

$$A = \begin{bmatrix} 0 & 0 & \cos^2(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\sin^2(t)+1}{\alpha} & \frac{\cos(t)}{\alpha} & \frac{\sin(t)\cos(t)}{\alpha} & -\frac{\sin^3(t)+\sin(t)}{\alpha} \\ 0 & 0 & 1 & \cos(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also in this case, $AB = 0$, such that

$$\mathit{\Pi}_{can} = K_0 \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -\cos^2(t) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \frac{\sin^2(t)+1}{\alpha} & \frac{\cos(t)}{\alpha} & \frac{\sin(t)\cos(t)}{\alpha} & -\frac{\sin^3(t)+\sin(t)}{\alpha} & 0 & 0 & -\frac{\left(\sin^2(t)+1\right)\cos^2(t)}{\alpha} \\ 0 & 0 & 1 & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] K_0^{-1}.$$

Note that for the considered index-2 and index-3 DAEs in Hessenberg form and orthogonal K_0, $AB = 0$ follows by construction, such that in these cases, we always have

$$\mathit{\Pi}_{can} = K_0 \begin{bmatrix} I_d & -A \\ B & -BA \end{bmatrix} K_0^{-1}.$$

REFERENCES

[1] U.M. Ascher and L.R. Petzold. Projected implicit Runge-Kutta methods for differential-algebraic equations. *SIAM J. Numer. Anal.*, 28(4):1097–1120, 1991.

[2] T. Berger and A. Ilchmann. On the standard canonical form of time-varying linear DAEs. *Q. Appl. Math.*, LXXI(1):69–87, 2013.

[3] T. Berger, A. Ilchmann, and S. Trenn. The quasi–Weierstraß form for regular matrix pencils. *Linear Algebra Appl.*, 436:4052–4069, 2012.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 613-653, July 2026.

[4] K.E. Brenan, S.L. Campbell, and L.R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* North-Holland, New York, Amsterdam, London, 1989.

[5] S.L. Campbell and E. Moore. Constraint preserving integrators for general nonlinear higher index DAEs. *Numer. Math.*, 69(4):383–399, 1995.

[6] S.L. Campbell and L.R. Petzold. Canonical forms and solvable singular systems of differential equations. *SIAM J. Algebr. Discrete Methods*, 4(4):517–521, 1983.

[7] V.F. Chistyakov and A.A. Shcheglova. *Izbrannye glavy teorii algebro-differentsial'nych sistem.* Nauka, Novosibirsk, 2003. In Russian.

[8] E. Eich-Soellner and C. Führer. *Numerical Methods in Multibody Dynamics.* European Consortium for Mathematics in Industry (ECMI Series). Teubner, Stuttgart, 1998.

[9] E. Griepentrog and R. März. *Differential-Algebraic Equations and Their Numerical Treatment.* Teubner Texte zur Mathematik 88. BSB Teubner, Leipzig, 1986.

[10] M. Hanke, E. Izquierdo Macana, and R. März. On asymptotics in case of linear index-2 differential-algebraic equations. *SIAM J. Numer. Anal.*, 35(4):1326–1346, 1998.

[11] M. Hanke and R. März. Canonical subspaces of linear time-varying differential-algebraic equations and their usefulness for formulating accurate initial conditions. *DAE Panel*, 1:1–32, 2023.

[12] M. Hanke and R. März. On the computation of accurate initial conditions for linear higher-index differential-algebraic equations and its application in initial value solvers. *Numer. Algor.*, 101:2849–2909, 2026.

[13] P. Kunkel and V. Mehrmann. *Differential-algebraic equations. Analysis and numerical solution.* EMS Textb. Math. Berlin: European Mathematical Society (EMS), 2nd edition edition, 2024.

[14] R. Lamour, R. März, and C. Tischendorf. *Differential-Algebraic Equations: A Projector Based Analysis*. Differential-Algebraic Equations Forum. Springer-Verlag Berlin Heidelberg New York Dordrecht London, 2013. eds. Achim Ilchmann and Timo Reis.

[15] V. H. Linh and R. März. Adjoint pairs of differential-algebraic equations and their Lyapunov exponents. *J. Dyn. Differ. Equations*, 29(2):655–684, 2017.

[16] R. Riaza. DAEs in circuit modelling: A survey. In: A. Ilchmann and T. Reis (editors), *Surveys in Differential-Algebraic Equations I*, DAE Forum. Springer Verlag, Berlin, Heidelberg, 97–136, 2013.

[17] D. Estévez Schwarz, R. Lamour, and R. März. The common ground of DAE approaches: An overview of diverse DAE frameworks emphasizing their commonalities. *DAE Panel*, 3, 2025. https://doi.org/10.52825/dae-p.v3i.2547.

[18] D. Estévez Schwarz, R. Lamour, and R. März. Regular linear time varying DAEs are equivalent to DAEs in strong standard canonical form, 2025. To appear in DAE Panel, 4, 2026.