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THE LINEAR ALGEBRA OF THE HAN–MONSKY REPRESENTATION RING*

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Abstract. From the perspective of classical linear algebra, we analyze a family of finite-dimensional algebras generated by certain bisymmetric $0,1$-matrices related to to the "Representation Ring" introduced by Han and Monsky [Some surprising Hilbert-Kunz functions. *Math. Z.*, 214:119–135, 1992]. This ring plays a central role in the theory of Hilbert–Kunz multiplicity, a modern numerical invariant of certain positive characteristic commutative rings. The matrices representing these structures have very rich properties with applications to commutative algebra and tantalizing connections to number theory. We construct an eigenbasis that effectively diagonalizes the Han–Monsky product, opening the door for the application of trigonometric sums to Hilbert–Kunz theory.

Key words. Hilbert–Kunz Multiplicity, Jacobi Symbol, Finite Real Algebra, (0,1)-Matrices.

AMS subject classifications. 1502, 13H15, 13M05.

1. Introduction.

1.1. Motivation. Suppose $p \geq 2$ is a positive integer. In this paper, we consider the algebra, R_p, which is a p-dimensional $\mathbb{R}$-vector space spanned by elements $\nu_0, \nu_1, ..., \nu_{p-1}$, and with multiplication determined by the following rule:

for $0 \leq i \leq j \leq p-1$,

$$\nu_i \nu_j = \sum_{k=j-i}^{\min(i+j,2p-2-i-j)} \nu_k.$$

Extending this product linearly endows R_p with the structure of a commutative ring, with ν_0 acting as the multiplicative identity — in other words, R_p is a finite-dimensional, unital, commutative $\mathbb{R}$-algebra (the authors recommend the texts [16] and [6] for an introduction to any abstract algebra discussed in this paper).

Our interest in this structure began with work on a very different looking problem. There is a well known, but somewhat mysterious, invariant in positive characteristic algebra called the Hilbert–Kunz multiplicity (see [13] for a nice discussion). This multiplicity is a real number that gives a useful measure of the 'singularness' of the points on an algebraic variety. Unfortunately, outside of a very limited number of simple examples, Hilbert-Kunz multiplicity is exceedingly difficult (if not impossible) to compute. In the early 1990s, Monsky, the pioneer of Hilbert-Kunz theory, and one of his students, Han, constructed an algebraic object they called the "Representation Ring," which they used to help them to compute certain families of Hilbert–Kunz multiplicities [11].

When p is prime, the ring R_p, which is described above and is our primary interest here, is related to an important sub-algebra of this Han–Monsky representation ring. It turns out that the structure of the entire

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representation ring, which is infinite-dimensional and prohibitively complicated, can be generated from this sub-algebra in an iterative way [22]. This suggested to us that a detailed understanding of R_p would help elucidate the entire Han–Monsky construction.

1.2. Description of the problem. Let $p \geq 2$ be an integer. We continue to denote the basis elements of R_p, as $\nu_0, \nu_1, \ldots, \nu_{p-1}$, and identify R_p with $\mathbb{R}^p$. Under this identification, multiplication by ν_m is identified with a linear transformation $\mathbb{R}^p \to \mathbb{R}^p$. Hence, for $0 \leq m \leq p-1$, each element ν_m is represented by the $0,1$-matrix A_m of size $p \times p$, where $A_m(i,j) = 1$ if and only if[1]

$$|i-j| \leq m \leq \min\{i+j-2, 2p-i-j\},$$

and 0 otherwise.

We note that A_0 is the identity matrix, A_{p-1} is a matrix with ones along the cross diagonal, and when p is odd, $A_{(p-1)/2}$ is the classical diamond matrix in which the non-zero entries form a diamond pattern. So for example, taking $p=5$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since the Han-Monsky ring is commutative, it follows immediately that for $p \geq 2$ that the matrices $A_0, A_1, \ldots, A_{p-1}$ commute.

This representation of the Han-Monsky structure in terms of bisymmetric $0,1$-matrices also appears independently in the notes of Yoshida [26]. Algebras generated by $0,1$-matrices are of interest to specialists in several fields and have been a focus of recent study [3, 9, 14, 18, 15]. The Han–Monsky representation ring is an example of a Grothendieck ring; similar algebraic structures, e.g., fusion rings and partition algebras, are a modern topic of study in mathematical physics, see, for example, [1, 2, 7, 8, 10, 19, 20, 21, 23].

1.3. Outline of results. In algebraic terms, the main contribution of this paper is the "diagonalization" of the $\mathbb{R}$-algebra that we call R_p. Here, we treat the problem of understanding R_p exclusively from the perspective of classical linear algebra. In an upcoming project, we show that, in combination with an iterative procedure based on the θ endomorphism introduced by Monsky and Teixeira (see [22]), the understanding of R_p developed in this paper leads to many interesting applications of trigonometric sums to Hilbert-Kunz Thoery.

Our first objective will be to fully understand the matrices $\{A_0, \ldots, A_{p-1}\}$ for $p \geq 2$. For each value of p, we determine the common set of eigenvectors for the entire algebra, R_p. This leads to formulas for the eigenvalues, which lead immediately to an understanding of invertibility and rank. For those matrices that are nonsingular, it will be shown that the determinants are always either plus or minus one leading to the conclusion that the entries of the inverse matrices are integer valued. We derive a formula for these inverse matrices and show that they are, in fact, composed only of the integers -1, 0, and 1.[2] We then turn our

[1] Note that we number the rows and columns of the A_m in the standard way, from 1 to p, while the indexing of the ν_k on the previous page goes from 0 to $p-1$ — this results in slightly different indexing in the multiplication formulas.

[2] For a discussion of the general problem of computing the inverses of bisymmetric matrices, see, e.g., [25, 27].

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attention to an alternative matrix representation of R_p that is relevant to Hilbert–Kunz computations. We offer a similar account of the eigenstructure of these matrices.

2. Analysis of R_p. We begin our analysis with a simple, but for our purposes, useful lemma followed by the foundational theorem that provides the eigenvectors and eigenvalues of A_1 for arbitrary $p \geq 2$.

2.1. Eigenvalues and eigenvectors for A_m. We begin our analysis of the linear transformations $A_0, A_1, \ldots, A_{p-1}$, with a simple trigonometric lemma, which is related to a well-known result in Fourier analysis.

LEMMA 2.1. *For $m \geq 1$ and $\sin[x] \neq 0$,*

$$\frac{\sin\left[(2m+1)x\right]}{\sin\left[x\right]} = 1 + 2\sum_{k=1}^{m} \cos\left[2kx\right].$$

Proof. For $m \geq 1$ and $a = e^{ix}$,

$$\begin{aligned}\frac{\sin\left[(2m+1)x\right]}{\sin\left[x\right]} &= \frac{e^{(2m+1)ix} - e^{-(2m+1)ix}}{e^{ix} - e^{-ix}} \\ &= \frac{a^{2m+1} - a^{-(2m+1)}}{a - a^{-1}} \\ &= \sum_{k=0}^{2m} a^{2m-2k} \\ &= \sum_{k=0}^{2m} e^{(2m-2k)ix} \\ &= 1 + \sum_{k=1}^{m} \left(e^{2kix} + e^{-2kix}\right) \\ &= 1 + 2\sum_{k=1}^{m} \cos\left[2kx\right].\end{aligned}$$ □

The following theorem is the key to the analysis of R_p. It was discovered by studying one-dimensional plots of the eigenvectors for both small and large values of p.

THEOREM 2.1. *Let $p \geq 2$. For $n = 1, 2, \ldots, p$, the eigenvectors for A_1 are*

$$e_n = \left(\sin\left[\frac{n\pi}{2p}\right], \sin\left[\frac{3n\pi}{2p}\right], \ldots, \sin\left[\frac{(2p-1)n\pi}{2p}\right]\right)^T,$$

with eigenvalues

$$\lambda_{1,n} = \frac{\sin\left[\frac{3n\pi}{2p}\right]}{\sin\left[\frac{n\pi}{2p}\right]} = 1 + 2\cos\left[\frac{n\pi}{p}\right].$$

Proof. We begin by noting that for $n = 1, 2, \ldots, p$,

$$0 < \frac{n\pi}{2p} \leq \frac{\pi}{2},$$

and so $\sin\left[\frac{n\pi}{2p}\right] \neq 0$.

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We then observe that each A_m has a single nonzero entry in the first row. In particular, $[A_1]_{1,2} = 1$, and

$$\lambda_{1,n} e_{n,1} = \lambda_{1,n} \sin\left[\frac{n\pi}{2p}\right] = \frac{\sin\left[\frac{3n\pi}{2p}\right]}{\sin\left[\frac{n\pi}{2p}\right]} \sin\left[\frac{n\pi}{2p}\right] = \sin\left[\frac{3n\pi}{2p}\right] = (A_1 e_n)_1.$$

Next we note that for $2 \le j \le p-1$ and $x = \dfrac{n\pi}{2p}$ that

$$\begin{aligned}
\lambda_{1,n} e_{n,j} &= \left(e^{2ix} + 1 + e^{-2ix}\right) \sin\left[(2j-1)x\right] \\
&= \left(e^{2ix} + 1 + e^{-2ix}\right) \left(e^{(2j-1)ix} - e^{-(2j-1)ix}\right)/(2i) \\
&= \left(e^{(2j+1)ix} + e^{(2j-1)ix} + e^{(2j-3)ix} - e^{-(2j+1)ix} - e^{-(2j-1)ix} - e^{-(2j-3)ix}\right)/(2i) \\
&= \sin[(2j-3)x] + \sin[(2j-1)x] + \sin[(2j+1)x] \\
&= (A_1 e_n)_j.
\end{aligned}$$

And finally for $j = p$, we observe that

$$\begin{aligned}
\lambda_{1,n} e_{n,p} &= \sin[(2p-3)x] + \sin[(2p-1)x] + \sin[(2p+1)x] \\
&= \sin[(2p-3)x] = (A_1 e_n)_p,
\end{aligned}$$

by observing that

$$\sin[(2p \pm 1)x] = \sin\left[\frac{n\pi}{2p}(2p \pm 1)\right] = \sin\left[n\pi \pm \frac{n\pi}{2p}\right],$$

so that by symmetry

$$\sin\left[n\pi + \frac{n\pi}{2p}\right] + \sin\left[n\pi - \frac{n\pi}{2p}\right] = 0. \qquad \square$$

This completes the proof and leads us directly to the following result.

THEOREM 2.2. *For $p \ge 2$, the eigenvalues for A_1 are distinct and, therefore, the eigenspaces are one-dimensional.*

Proof. For $p \ge 2$ and $n = 1, 2, \ldots, p$, the eigenvalues $\lambda_{1,n} = 1 + 2\cos\left[\dfrac{n\pi}{2p}\right]$ are a strictly decreasing set of values since $\cos[x]$ is strictly decreasing on $[0, \pi/2]$. $\square$

THEOREM 2.3. *For $m = 0, 1, \ldots, p-1$ and $n = 1, 2, \ldots, p$, the n^{th} eigenvalue for A_m is*

$$\lambda_{m,n} = \frac{\sin\left[\frac{n\pi}{2p}(2m+1)\right]}{\sin\left[\frac{n\pi}{2p}\right]} = 1 + 2\sum_{k=1}^{m} \cos\left[\frac{2kn\pi}{2p}\right] = 1 + 2\sum_{k=1}^{m} \cos\left[\frac{kn\pi}{p}\right],$$

with eigenvector

$$e_n = \left(\sin\left[\frac{n\pi}{2p}\right], \sin\left[\frac{3n\pi}{2p}\right], \ldots, \sin\left[\frac{(2p-1)n\pi}{2p}\right]\right)^T.$$

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Proof. That A_m have a common set of eigenvectors follows from the fact that for a given p, the family of matrices, R_p, commute and the eigenspaces for A_1 are one-dimensional. Hence, the eigenvectors for A_1 are all unique up to a nonzero scalar multiple.

Given that the eigenvectors for A_m are known, it is straightforward to determine the eigenvalues. We use the fact that for each $m = 0, 1, \ldots, p-1$, A_m has a single nonzero term in the first row, This value is always 1, and it occurs in the $(m+1)$-st column. So,

$$(A_m e_n)_1 = \sin\left[\frac{n\pi}{2p}(2m+1)\right] = \lambda_{m,n} \sin\left[\frac{n\pi}{2p}\right],$$

which establishes the result. □

Since the matrices commute, they are simultaneously diagonalizable, and it will be useful to normalize the eigenvectors to have Euclidean length 1. With this in mind, we compute the norms.

THEOREM 2.4. *For $p \geq 1$ and $||\cdot||$ denoting the Euclidean norm,*

$$||e_n||^2 = \begin{cases} p/2 & \text{if } 1 \leq n < p \\ p & \text{if } n = p. \end{cases}$$

Proof. The case that $n = p$ is straightforward since $\sin[k\pi/2] = \pm 1$ when k is odd. So assume that $1 \leq n < p$ and let $\xi = e^{in\pi/2p}$ and $\xi_k = \xi^k$. Then observe that

$$\begin{aligned}\sum_{j=1}^{p} \xi^{4j-2} &= \sum_{j=1}^{p} \xi_2^{2j-1} \\ &= \sum_{j=0}^{p-1} \xi_2^{2j+1} \\ &= \xi_2 \sum_{j=0}^{p-1} \xi_2^{2j} \\ &= \xi_2 \sum_{j=0}^{p-1} \xi_4^{j} \\ &= \xi_2 \frac{1-\xi_4^p}{1-\xi_4} \\ &= 0,\end{aligned}$$

since $\xi_4^p = \xi^{4p} = 1$. Similarly

$$\sum_{j=1}^{p} \xi^{-4j+2} = 0.$$

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So, we then have

$$\begin{aligned} ||e_n||^2 &= \sum_{j=1}^{p} \sin^2\left[\frac{n(2j-1)\pi}{2p}\right] \\ &= \sum_{j=1}^{p}\left(\frac{e^{in(2j-1)\pi/2p} - e^{-in(2j-1)\pi/2p}}{2i}\right)^2 \\ &= -\frac{1}{4}\sum_{j=1}^{p}\left(\xi^{2j-1} - \xi^{-(2j-1)}\right)^2 \\ &= -\frac{1}{4}\sum_{j=1}^{p}\left(\xi^{4j-2} - 2 + \xi^{-(4j-2)}\right) \\ &= \frac{p}{2} - \frac{1}{4}\sum_{j=1}^{p}\left(\xi^{4j-2} + \xi^{-(4j-2)}\right) \\ &= \frac{p}{2}. \end{aligned}$$ ∎

2.2. Spectral radius. In the motivating Hilbert–Kunz multiplicity problem, the asymptotic value of the spectral radius as $p \to \infty$ is meaningful. Next we prove that, for any $p \geq 2$ and $0 \leq m \leq p-1$, the spectral radius of A_m is $\lambda_{m,1}$.[3]

We note that the spectral radius of $A_0 = \mathrm{I}$ and A_{p-1} are both one.

THEOREM 2.5. *For $p \geq 3$ and $1 \leq m \leq p-2$, the spectral radius of A_m is $\lambda_{m,1}$.*

Proof. We first observe that for each $1 \leq m \leq p-2$ that A_m is indeed nonnegative, aperiodic, primitive, and irreducible. Second, we observe at for any $p \geq 3$ that the elements of

$$e_1 = \left(\sin\left[\frac{\pi}{2p}\right], \sin\left[\frac{3\pi}{2p}\right], \ldots, \sin\left[\frac{(2p-1)\pi}{2p}\right]\right)^T,$$

are nonnegative since $0 < \frac{\pi}{2p} < \frac{(2p-1)\pi}{2p} < \frac{\pi}{2}$. Hence, $\lambda_{m,1} = \frac{\sin\left[\frac{\pi}{2p}(2m+1)\right]}{\sin\left[\frac{\pi}{2p}\right]}$ is the leading eigenvalue and spectral radius of A_m. ∎

As an illustration of Theorem 2.5 consider the following graph of the eigenvalues for $p = 20$ and $m = 5$. Note that the function

$$f(x) = \frac{\sin\left[\frac{x\pi}{2p}(2m+1)\right]}{\sin\left[\frac{x\pi}{2p}\right]},$$

has been superimposed on the plot of the individual points.

[3]This provides a nice illustration of the Perron–Frobenius theorem (see [24] for details).

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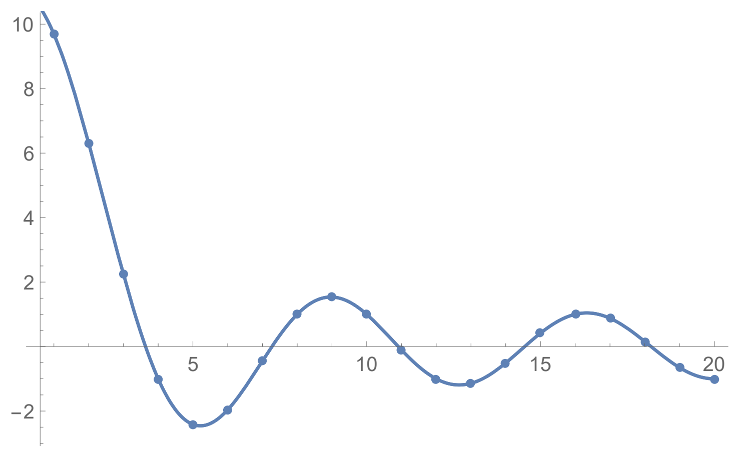

FIGURE 1. *Eigenvalues for A_5 when $p = 20$.*

Since $\lim_{x\to 0} \sin[ax]/x = a$, it is easy to see that for a given value of m (m fixed and not changing with p) that

$$\lim_{p\to\infty} \lambda_{m,1} = \lim_{p\to\infty} \frac{\sin\left[\frac{\pi}{2p}(2m+1)\right]}{\sin\left[\frac{\pi}{2p}\right]} = 2m+1.$$

Somewhat more interesting are the following two results where we allow m to vary with p. We denote this by using m_p to indicate that m varies with p. First of all we consider a simple case.

COROLLARY 2.1. *The spectral radius of the diamond matrix is asymptotically* $\dfrac{2p}{\pi}$.

Proof. Let p be an odd number. The diamond matrix corresponds to $2m_p + 1 = p$, in which case

$$\lambda_{m_p,1} = \frac{\sin\left[\frac{\pi}{2p}p\right]}{\sin\left[\frac{\pi}{2p}\right]} = \frac{1}{\sin\left[\frac{\pi}{2p}\right]}.$$

Taking the limit $p \to \infty$ gives the result. ∎

This result holds more generally.

COROLLARY 2.2. *Fix an integer α, and for each sufficiently large p, let m_p, with $1 \leq m_p \leq p-2$, be such that $2m_p + 1 = p + \alpha$. Then, the assymptotic spectral radius of $A_{m_p} \in R_p$, as $p \to \infty$, is again* $\dfrac{2p}{\pi}$.

Proof. Since

$$\begin{aligned} \lambda_{m_p,1} &= \frac{\sin\left[\frac{(p+\alpha)\pi}{2p}\right]}{\sin\left[\frac{\pi}{2p}\right]}, \\ &= \frac{\sin\left[\frac{\pi}{2} + \frac{\alpha\pi}{2p}\right]}{\sin\left[\frac{\pi}{2p}\right]} \\ &= \frac{\cos\left[\frac{\alpha\pi}{2p}\right]}{\sin\left[\frac{\pi}{2p}\right]}, \end{aligned}$$

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and $\lim_{p\to\infty} \cos\left[\frac{\alpha\pi}{2p}\right] = 1$, the assymptotic value of the spectral radius of A_{m_p} is $\frac{2p}{\pi}$. ∎

As a final comment, we observe that when $p \geq 3$ is odd, the eigenvectors for the A_m described in Theorem 2.3 have an alternative description. In this case, the eigenvectors can be divided up according to whether they exhibit even or odd symmetry about the $r = (p-1)/2$ (central) row/column. This gives a convenient description of the null space of the diamond matrix.

THEOREM 2.6. *Suppose that $p \geq 3$ is odd and let $r = \frac{p-1}{2}$. Then, the eigenvectors of Theorem 2.3 can be written:*

$$e_c(n) = \left\{\cos\left[\frac{nj\pi}{p}\right]\right\}_{j=-r}^{r},$$

for $n = 1, 3, \ldots, p$ along with the vectors

$$e_s(n) = \left\{(-1)^{j+1}\sin\left[\frac{nj\pi}{p}\right]\right\}_{j=-r}^{r},$$

for $n = 1, 3, \ldots, p-2$.

The vectors $e_s(1), e_s(3), \ldots, e_s(p-2)$ are a basis for the null space of the diamond matrix.

2.3. Dimension of the null space. The dimension of the null space of A_m in general for a given value of p is determined by the zeros of $\sin\left[\frac{n\pi}{2p}(2m+1)\right]$, where $m = 0, 1, \ldots, p-1$ and $n = 1, 2, \ldots, p$.

For the following result, we will write $p = 2^l s$, where s is an odd number.

THEOREM 2.7. *Let $p \geq 2$. Then, A_m is invertible if and only if $2m+1$ is relatively prime to p or equivalently $\gcd(2m+1, p) = 1$. Furthermore, the dimension of the null space is $\left\lfloor \frac{p}{n_1} \right\rfloor$, where $n_1 = 2^{l+1}s/c$ and $c = \gcd(2m+1, p)$.*

Proof. Let $p = 2^l s$, where s is odd. Then,

$$\sin\left[\frac{n\pi}{2p}(2m+1)\right] = \sin\left[\frac{n\pi}{2^{l+1}}\frac{2m+1}{s}\right] = 0,$$

if and only if $\frac{n}{2^{l+1}}\frac{2m+1}{s}$ is an integer. If $\gcd(2m+1, s) = 1$, then $\frac{n}{2^{l+1}}\frac{2m+1}{s}$ is an integer only if $|n| \geq 2^{l+1}s = 2p > p$.

On the other hand if $(2m+1, p) = c \geq 3$, then

$$\frac{n}{2^{l+1}}\frac{2m+1}{s} = \frac{n}{2^{l+1}}\frac{(2m+1)/c}{s/c},$$

where $(2m+1)/c$ and s/c are integers. It follows that $n_1 = 2^{l+1}(s/c) < 2^l s = p$, and that $\frac{n_1}{2^{l+1}}\frac{2m+1}{s}$ is an integer. Moreover, n_1 is the smallest natural number with this property. It follows immediately that $\frac{n}{2^{l+1}}\frac{2m+1}{s}$ is an integer for $n = n_1, 2n_1, \ldots, \left\lfloor \frac{p}{n_1} \right\rfloor n_1$ and since $\left\lfloor \frac{p}{n_1} \right\rfloor n_1 \leq p$, the dimension of the null space is $\left\lfloor \frac{p}{n_1} \right\rfloor$. ∎

This result immediately implies the following.

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COROLLARY 2.3. *If $p = 2^l$ is a power of 2, then A_m is invertible for any $m = 0, 1, \dots, p-1$.*

COROLLARY 2.4. *If p is an odd prime, then A_m is invertible except when $2m + 1 = p$ or equivalently when*

$$m = \frac{p-1}{2},$$

in which case A_m is the diamond matrix.

COROLLARY 2.5. *The rank of the diamond matrix for odd values of p is $(p + 1)/2$.*

Proof. The diamond matrix corresponds to the case that $2m + 1 = p$ and so in that case

$$\frac{n}{2p}(2m+1) = \frac{n}{2},$$

which is an integer whenever n is even. When p is odd, the number of even integers between $n = 1$ and $n = p$ is $(p + 1)/2$. ∎

Note: The $(p+1)/2$ cosine eigenvectors from Theorem 2.6 correspond to the nonzero eigenvalues of the diamond matrix, while the $(p-1)/2$ sine eigenvectors from Theorem 2.6 are a basis for it's null space.

2.4. Evaluation of the inverse matrices, A_ℓ^{-1}. We apply the formulas developed above to explicitly compute A_ℓ^{-1}, when it exists. Before proving our formula, we introduce some notation that will help to streamline the calculations.

We use δ to denote the Kronecker symbol, $\delta_{m,j} = \begin{cases} 1 & \text{if } m = j \\ 0 & \text{if } m \neq j \end{cases}$.

Given a fixed value of p, the following function appears in our expression for A_ℓ^{-1} :
for $k \in \mathbb{Z}$, we write

$$g_k = \begin{cases} 1 & \text{if } k \ \equiv \ a(2\ell+1) \mod 2p, \quad \text{for some } a \text{ with } 1 \le a \le b \\ 0 & \text{otherwise} \end{cases} .$$

Here, the value of b is determined according to the requirement that $1 \le b \le 2p - 1$ and

$$(2\ell+1)(2b+1) \equiv 1 \mod 4p.$$

g_k plays an important role in the derivation below.

THEOREM 2.8. *Assume that A_ℓ is invertible in the p-Han–Monsky Algebra.*

Define b, with $1 \le b \le 2p - 1$, by the relation

$$(2b+1)(2\ell+1) \cong 1 \mod 4p.$$

Then, the elements of A_ℓ^{-1} are given by:

$$(2.1) \qquad \left(A_\ell^{-1}\right)_{n+1,j+1} = \frac{(-1)^{\ell+n+j}}{p} + \frac{2}{p}\sum_{k=1}^{p-1} \sin\left[\frac{k\pi}{2p}\right] \frac{\sin\left[\frac{(2j+1)k\pi}{2p}\right]\sin\left[\frac{(2n+1)k\pi}{2p}\right]}{\sin\left[\frac{(2\ell+1)k\pi}{2p}\right]},$$

$$(2.2) \qquad = \delta_{n,j} + g_{n-j} + g_{-(n-j)} - g_{n+j+1} - g_{-(n+j+1)},$$

where $0 \le n, j \le p - 1$.

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For the first equality (2.1), observe that, by Theorems 2.3 and 2.4, the matrices $\frac{2}{p}\ e_k e_k^T$ for $k = 1, \ldots, p-1$ and $\frac{1}{p}\ e_p e_p^T$ are mutually orthogonal idempotent matrices, and

$$\mathrm{I} = \frac{2}{p}\left(e_1 e_1^T + e_2 e_2^T + \cdots + e_{p-1} e_{p-1}^T\right) + \frac{1}{p}\ e_p e_p^T.$$

Thus, when A_ℓ^{-1} exists, we have

$$\begin{aligned} A_\ell^{-1} &= \frac{2}{p}\left(A_\ell^{-1} e_1 e_1^T + A_\ell^{-1} e_2 e_2^T + \cdots + A_\ell^{-1} e_{p-1} e_{p-1}^T\right) + \frac{1}{p}\ A_\ell^{-1} e_p e_p^T \\ &= \frac{2}{p}\left(\frac{1}{\lambda_{\ell,1}} e_1 e_1^T + \frac{1}{\lambda_{\ell,2}} e_2 e_2^T + \cdots + \frac{1}{\lambda_{\ell,p-1}} e_{p-1} e_{p-1}^T\right) + \frac{1}{p}\ \frac{1}{\lambda_{\ell,p}} e_p e_p^T, \end{aligned}$$

where

$$\frac{1}{\lambda_{\ell,k}} = \frac{\sin\left[\frac{k\pi}{2p}\right]}{\sin\left[\frac{k\pi}{2p}(2\ell+1)\right]},$$

for $k = 1, \ldots, p$ and

$$e_k = \left(\sin\left[\frac{k\pi}{2p}\right], \sin\left[\frac{3k\pi}{2p}\right], \ldots, \sin\left[\frac{(2p-1)k\pi}{2p}\right]\right)^T,$$

are the eigenvectors introduced above.

This yields, for $k = 1, \ldots, p-1$ and $0 \le n, j \le p-1$,

$$\frac{1}{\lambda_{\ell,k}}\left(e_k e_k^T\right)_{n+1,j+1} = \frac{\sin\left[\frac{k\pi}{2p}\right]}{\sin\left[(2\ell+1)\frac{k\pi}{2p}\right]} \sin\left[(2n+1)\frac{k\pi}{2p}\right] \sin\left[(2j+1)\frac{k\pi}{2p}\right],$$

and

$$\frac{1}{\lambda_{\ell,p}}\left(e_p e_p^T\right)_{n+1,j+1} = (-1)^{\ell+n+j}.$$

Plugging this information into the idempotent decomposition of A_ℓ^{-1} above gives the first equality, (2.1), in the statement of the Theorem.

Our proof of (2.2) proceeds by directly evaluating (1). This involves the following well-known identity, originally due to Lagrange which is easily verified by writing the trigonometric sum in terms of the primitive $2p$th root of unity, $\xi = e^{i\pi/p}$:

Lemma 2.2. *Suppose that x, y and $p > 1$ are integers. Then,*

$$\sum_{k=1}^{p-1} \cos\left[\frac{(x+y)k\pi}{p}\right] = p\bar{\delta}_{2p}\left[x+y\right] - \frac{(-1)^{x+y}+1}{2}, \tag{2.3}$$

where

$$\bar{\delta}_{2p}\left[x+y\right] = \begin{cases} 1 & \textit{if } 2p|(x+y) \\ 0 & \textit{otherwise.} \end{cases}$$

Now we prove Theorem 2.8.

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Proof. We are evaluating the sum

$$\frac{(-1)^{\ell+n+j}}{p}+\frac{2}{p}\sum_{k=1}^{p-1}\frac{\sin\left[\frac{k\pi}{2p}\right]}{\sin\left[(2\ell+1)\frac{k\pi}{2p}\right]}\sin\left[(2j+1)\frac{k\pi}{2p}\right]\sin\left[(2n+1)\frac{k\pi}{2p}\right]. \tag{2.4}$$

First, let b be an integer such that[4]

$$(2\ell+1)(2b+1)\equiv 1 \mod 4p,$$

and note that it follows that

$$\ell\equiv b \mod 2. \tag{2.5}$$

By construction, we have $(2b+1)(2\ell+1)=1+4pj$, for some $j\in\mathbb{Z}$, and, therefore

$$\sin\left[\frac{k\pi}{2p}\right]=\sin\left[(2b+1)(2\ell+1)\frac{k\pi}{2p}\right].$$

By Lemma 2.1, we may rewrite the first term in the trigonometric sum above as

$$\begin{aligned}\frac{\sin\left[\frac{k\pi}{2p}\right]}{\sin\left[(2\ell+1)\frac{k\pi}{2p}\right]}&=\frac{\sin\left[(b+\frac{1}{2})(2\ell+1)\frac{k\pi}{p}\right]}{\sin\left[\frac{1}{2}(2\ell+1)\frac{k\pi}{p}\right]}\\&=1+2\sum_{a=1}^{a=b}\cos\left[a(2\ell+1)\frac{k\pi}{p}\right].\end{aligned}$$

Moreover, the product of sines can be written as

$$\begin{aligned}&\sin\left[(2j+1)\frac{k\pi}{2p}\right]\sin\left[(2n+1)\frac{k\pi}{2p}\right]\\&\qquad=\frac{1}{2}\left(\cos\left[(n-j)\frac{k\pi}{p}\right]-\cos\left[(n+j+1)\frac{k\pi}{p}\right]\right).\end{aligned}$$

Hence, the expression (2.4) is equal to

$$\frac{(-1)^{\ell+n+j}}{p}+\frac{1}{p}\sum_{k=1}^{p-1}\left(1+2\sum_{a=1}^{b}\cos\left[a(2\ell+1)\frac{k\pi}{p}\right]\right)\left(\cos\left[(n-j)\frac{k\pi}{p}\right]-\cos\left[(n+j+1)\frac{k\pi}{p}\right]\right).$$

Distributing this, we have shown

$$\left(A_\ell^{-1}\right)_{n+1,j+1}=\frac{1}{p}\left((-1)^{\ell+n+j}+\mathrm{I}+\mathrm{II}\right),$$

where

[4] Such a b exists, since $s\ell+1$ is odd, and by Theorem 2.7, A_ℓ is invertible exactly when $2\ell+1$ is invertible modulo p. Moreover, we may always chose b such that $1\leq b\leq 2p-1$.

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$$\mathrm{I} = \sum_{k=1}^{p-1} \cos\left[(n-j)\frac{k\pi}{p}\right] - \sum_{k=1}^{p-1} \cos\left[(n+j+1)\frac{k\pi}{p}\right],$$

and

$$\mathrm{II} = \sum_{a=1}^{b} \left(\sum_{k=1}^{p-1} 2\cos\left[a(2\ell+1)\frac{k\pi}{p}\right] \cos\left[(n-j)\frac{k\pi}{p}\right] - \sum_{k=1}^{p-1} 2\cos\left[a(2\ell+1)\frac{k\pi}{p}\right] \cos\left[(n+j+1)\frac{k\pi}{p}\right] \right).$$

This first expression, I, can be directly evaluated using formula (2.3) of Lemma 2.2, to give

$$\begin{aligned} \mathrm{I} &= p\overline{\delta}_{2p}\left[n-j\right] - \frac{(-1)^{n-j}+1}{2} - p\overline{\delta}_{2p}\left[n+j+1\right] + \frac{(-1)^{n+j+1}+1}{2} \\ &= p\delta_{n,j} - (-1)^{n+j}, \end{aligned}$$

where we have used that fact that $0 \le n, j \le p-1$, so that $\overline{\delta}_{2p}\left[n+j+1\right] = 0$ for all n and j, and $\overline{\delta}_{2p}\left[n-j\right] = \delta_{n,j}$ is only nonzero if $n-j=0$.

The evaluation of II also uses Lemma 2.2, but first we rewrite the products of cosines as sums to get

$$\mathrm{II} = \sum_{a=1}^{b} \left(\sum_{k=1}^{p-1} \left(\cos\left[(a(2\ell+1)+(n-j))\frac{k\pi}{p}\right] + \cos\left[(a(2\ell+1)-(n-j))\frac{k\pi}{p}\right]\right) - \sum_{k=1}^{p-1} \left(\cos\left[(a(2\ell+1)+(n+j+1))\frac{k\pi}{p}\right] + \cos\left[(a(2\ell+1)-(n+j+1))\frac{k\pi}{p}\right]\right) \right).$$

Applying formula (2.3) of Lemma 2.2 to all four sums over k and combining terms gives

$$\begin{aligned} \mathrm{II} = \sum_{a=1}^{b} \Bigg(& p\Big(\overline{\delta}_{2p}\left[a(2\ell+1)+(n-j)\right] + \overline{\delta}_{2p}\left[a(2\ell+1)-(n-j)\right] \\ & - \overline{\delta}_{2p}\left[a(2\ell+1)+(n+j+1)\right] - \overline{\delta}_{2p}\left[a(2\ell+1)-(n+j+1)\right]\Big) \\ & + \frac{(-1)^{a(2\ell+1)+(n+j+1)}+1}{2} + \frac{(-1)^{a(2\ell+1)-(n+j+1)}+1}{2} \\ & - \frac{(-1)^{a(2\ell+1)+(n-j)}+1}{2} - \frac{(-1)^{a(2\ell+1)-(n-j)}+1}{2} \Bigg). \end{aligned}$$

The sum of the four terms with a 2 in the denominator can be written, after a bit of careful manipulation, as

$$(-1)^{a(2\ell+1)+(n+j+1)} - (-1)^{a(2\ell+1)+(n+j)} = -2(-1)^{n+j}(-1)^{a}.$$

Summing this over a, we get

$$\begin{aligned} -2(-1)^{n+j}\sum_{a=1}^{b}(-1)^{a} &= -2(-1)^{n+j}\frac{(-1)^{b}-1}{2} \\ &= -(-1)^{n+j+\ell} + (-1)^{n+j}, \end{aligned}$$

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where, for the final expression, we have used the fact established in (2.5) that ℓ and b have the same parity.

Next, we sum the remaining terms in our expression for II over a—for this, observe that, for any integer q,

$$\sum_{a=1}^{b} \overline{\delta}_{2p}\left[a(2\ell+1)+q\right] = \begin{cases} 1 & \text{if } q \equiv a(2\ell+1) \mod 2p, \quad \text{for some } a \text{ with } 1 \le a \le b \\ 0 & \text{otherwise.} \end{cases}$$

In terms of Theorem 2.8, we write this as

$$\sum_{a=1}^{b} \overline{\delta}_{2p}\left[a(2\ell+1)+q\right] = g_q.$$

Putting all of this together, we conclude

$$\mathrm{II} = -(-1)^{n+j+\ell} + (-1)^{n+j} + p\Big(g_{n-j} + g_{-(n-j)} - g_{n+j+1} - g_{-(n+j+1)}\Big).$$

Now, we reassemble the pieces and simplify to get the desired formula:

$$\begin{aligned} \left(A_\ell^{-1}\right)_{n+1,j+1} &= \frac{1}{p}\left((-1)^{\ell+n+j} + \mathrm{I} + \mathrm{II}\right) \\ &= \frac{1}{p}\left((-1)^{\ell+n+j} + p\delta_{n,j} - (-1)^{n+j} - (-1)^{n+j+\ell} + (-1)^{n+j} + p\Big(g_{n-j} + g_{-(n-j)} - g_{n+j+1} - g_{-(n+j+1)}\Big)\right) \\ &= \delta_{n,j} + g_{n-j} + g_{-(n-j)} - g_{n+j+1} - g_{-(n+j+1)}. \end{aligned}$$

∎

Corollary 2.6. *The elements of A_ℓ^{-1} take values in the set $\{-1, 0, 1\}$.*

Proof. Note that if $n = j$, then $\delta_{n,j} = 1$ while both g_{n-j} and $g_{-(n-j)}$ are equal to 0 since $\gcd(p, 2\ell+1) = 1$.

On the other hand, if $n \neq j$, then $\delta_{n,j} = 0$. Recall that $1 \le b \le 2p-1$ and notice that g_x can be equivalently defined by,

$$g_x = \begin{cases} 1 & \text{if } (2b+1)x \equiv a \mod 2p, \quad \text{for some } a \text{ with } 1 \le a \le b \\ 0 & \text{otherwise.} \end{cases}$$

From this it is clear that if g_{n-j} and $g_{-(n-j)}$ are both 1, then the residues $\{1, 2, \ldots, b\}$ modulo $2p$ must contain at least one of the residues of $\pm(2b+1)(n+j+1)$—and similarly, if both g_{n+j+1} and $g_{-(n+j+1)}$ are equal to 1, there is at least one solution to

$$\pm(2b+1)(n-j) \equiv a \mod 2p,$$

with $1 \le a \le b$.

In any case, the total,

$$\left(A_\ell^{-1}\right)_{n+1,j+1} = \delta_{n,j} + g_{n-j} + g_{-(n-j)} - g_{n+j+1} - g_{-(n+j+1)},$$

always lies between -1 and 1. ∎

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The identity in Theorem 2.8 can also be written

$$A_\ell^{-1} = \mathrm{Id} + \sum_{j=0}^{p-1} g_j A_{j+1} + \sum_{j=0}^{p-1} g_{-j} A_{j+1} - \sum_{j=0}^{p-1} g_{j+1} A_{j+1} - \sum_{j=0}^{p-1} g_{-(j+1)} A_{j+1}.$$

It is also possible to establish the formula for A_ℓ^{-1} in this form algebraically, from the multiplication rules for R_p.

2.5. Determinants. The determinant of a matrix can be written as the product of the eigenvalues and so

$$|A_m| = \prod_{n=1}^{p} \frac{\sin\left[\frac{n\pi}{2p}(2m+1)\right]}{\sin\left[\frac{n\pi}{2p}\right]}.$$

When $\gcd(2m+1,p) \neq 1$, then at least one of these terms is zero and so in that case $|A_m| = 0$.

In [17], it is noted that when $\gcd(2m+1,p) = 1$ the product of the terms in the numerator equal that of the denominator except for the sign of the product. In addition, it is proven that

$$\prod_{n=1}^{p-1} \frac{\sin\left[\frac{n\pi}{2p}(2m+1)\right]}{\sin\left[\frac{n\pi}{2p}\right]} = (-1)^{m(p-1)} \left(\frac{p}{2m+1}\right),$$

where $\left(\frac{p}{2m+1}\right)$ is the Jacobi symbol [12] evaluated at p modulo $2m+1$.

Since $2m+1$ is odd, we have

$$\sin\left[\frac{p\,\pi}{2p}(2m+1)\right] = \sin\left[\frac{\pi}{2}(2m+1)\right] = (-1)^m,$$

giving us the following result.

Theorem 2.9. *Let $p \geq 1$ and $0 \leq m \leq p-1$. Then,*

$$|A_m| = \prod_{n=1}^{p} \frac{\sin\left[\frac{n\pi}{2p}(2m+1)\right]}{\sin\left[\frac{n\pi}{2p}\right]} = (-1)^{mp} \left(\frac{p}{2m+1}\right).$$

As an extra bonus we note that we have the following formula for the products in the previous expression.

Corollary 2.7. *For $p \geq 1$,*

$$\prod_{n=1}^{p} \sin\left[\frac{n\pi}{2p}\right] = \frac{\sqrt{p}}{2^{p-1}},$$

and hence for $0 \leq m \leq p-1$,

$$\prod_{n=1}^{p} \sin\left[\frac{n(2m+1)\pi}{2p}\right] = (-1)^{mp} \left(\frac{p}{2m+1}\right) \frac{\sqrt{p}}{2^{p-1}}.$$

We now turn to another family of matrices that corresponds to another useful basis for R_p.

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3. The δ-basis. The ν-basis for R_p that led to the formulation of our first linear algebra problem, in terms of the A_i matrices, is not the primary basis used by Han and Monsky in [11]. Indeed, they introduce the ν_i's because they simplify certain induction arguments.[5] The structure encoded in the representation ring is more naturally understood in terms of the δ_l of [11], related to ν_i according to the relations

$$\begin{aligned}\delta_0 &= 0,\\ \delta_l &= \sum_{j=0}^{l-1}(-1)^j \nu_j,\end{aligned}$$

for $l = 1, 2, \dots, p$. The eigenvector analysis of section 2 continues to work in this situation, and as a second linear algebra problem, we consider the matrices corresponding to this delta basis.

Definition. For each $p \geq 2$ and $l = 1, 2, \dots, p$, let

$$B_l = \sum_{j=0}^{l-1}(-1)^j A_j.$$

We note that since for each p, the matrices A_j, for $j = 0, 1, \dots, p-1$ have a common set of eigenvectors, it follows immediately that these are also a common set eigenvectors for $B_1, B_2, \dots, B_p$. Hence, it also follows immediately that the eigenvalues for B_l for each l are given by

$$\gamma_{l,k} = \sum_{j=0}^{l-1}(-1)^j \frac{\sin\left[\frac{k\pi}{2p}(2j+1)\right]}{\sin\left[\frac{k\pi}{2p}\right]},$$

for $k = 1, 2, \dots, p$.

We will now show that this formula simplifies.

THEOREM 3.1. *Let $p \geq 2$. Then for $k = 1, 2, \dots, p-1$ and $l = 1, 2, \dots, p$, the k^{th} eigenvalue for B_l is*

$$\gamma_{l,k} = (-1)^{l+1}\frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]}.$$

for $k = 1, 2, \dots, p-1$ and $\gamma_p = l$.

Proof. We first observe that

$$\begin{aligned}\gamma_{l,k} &= \sum_{j=0}^{l-1}(-1)^j \frac{\sin\left[\frac{(2j+1)k\pi}{2p}\right]}{\sin\left[\frac{k\pi}{2p}\right]}\\ &= 1 + \sum_{j=1}^{l-1}(-1)^j\left(1 + 2\sum_{n=1}^{j}\cos\left[\frac{nk\pi}{p}\right]\right)\\ &= \left(\sum_{j=0}^{l-1}(-1)^j\right) + 2\sum_{j=1}^{l-1}(-1)^j\sum_{n=1}^{j}\cos\left[\frac{nk\pi}{p}\right].\end{aligned}$$

[5] Our ν_i is denoted by λ_i in [11].

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The double sum can be indexed as follows:

$$\begin{aligned}
&\sum_{j=1}^{l-1}(-1)^j\sum_{n=1}^{j}\cos\left[\frac{nk\pi}{p}\right]\\
&\quad=(-1)^1\cos\left[\frac{k\pi}{p}\right]+(-1)^2\cos\left[\frac{k\pi}{p}\right]+(-1)^2\cos\left[\frac{2k\pi}{p}\right]+...\\
&\quad=\left[(-1)^1+(-1)^2+(-1)^3+...\right]\cos\left[\frac{k\pi}{p}\right]+\left[(-1)^2+(-1)^3+...\right]\cos\left[\frac{2k\pi}{p}\right]+...\\
&\quad=\sum_{n=1}^{l-1}\left(\sum_{j=n}^{l-1}(-1)^j\right)\cos\left[\frac{nk\pi}{p}\right].
\end{aligned}$$

Now consider two cases: either l is even or l is odd.

Case 1. $l=2h+1$
In this case the expression above becomes

$$\begin{aligned}
\gamma_{l,k}&=\left(\sum_{j=0}^{2h}(-1)^j\right)+2\sum_{j=1}^{2h}(-1)^j\sum_{n=1}^{j}\cos\left[\frac{nk\pi}{p}\right]\\
&=1+2\sum_{n=1}^{2h}\left(\sum_{j=n}^{2h}(-1)^j\right)\cos\left[\frac{nk\pi}{p}\right]\\
&=1+2\sum_{m=1}^{h}\cos\left[\frac{2mk\pi}{p}\right]\\
&=\frac{\sin\left[\frac{(2h+1)k\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]}=\frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]}.
\end{aligned}$$

Case 2. $l=2h$
Here, the expression becomes

$$\begin{aligned}
\gamma_{l,k}&=\left(\sum_{j=0}^{2h-1}(-1)^j\right)+2\sum_{j=1}^{2h-1}(-1)^j\sum_{n=1}^{j}\cos\left[\frac{nk\pi}{p}\right]\\
&=0+2\sum_{n=1}^{2h-1}\left(\sum_{j=n}^{2h-1}(-1)^j\right)\cos\left[\frac{nk\pi}{p}\right]\\
&=-2\sum_{m=0}^{h-1}\cos\left[\frac{(2m+1)k\pi}{p}\right]\\
&=(-1)^{l+1}2\sum_{m=0}^{h-1}\cos\left[\frac{(2m+1)k\pi}{p}\right].
\end{aligned}$$

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To evaluate this sum of cosines, let $\xi = e^{\pi i/p}$ and note that $0 < 2k < 2p$. We have

$$\begin{aligned}
2\sum_{m=0}^{h-1} \cos\left[\frac{(2m+1)k\pi}{p}\right] &= \sum_{m=0}^{h-1}\left(\xi^{(2m+1)k} + \xi^{-(2m+1)k}\right) \\
&= \xi^k \sum_{m=0}^{h-1}\left(\xi^{2k}\right)^m + \xi^{-k}\sum_{m=0}^{h-1}\left(\xi^{-2k}\right)^m \\
&= \xi^k\left(\frac{1-\xi^{2kh}}{1-\xi^{2k}}\right) + \xi^{-k}\left(\frac{1-\xi^{-2kh}}{1-\xi^{-2k}}\right) \\
&= \frac{1-\xi^{2kh}}{\xi^{-k}-\xi^{k}} + \frac{1-\xi^{-2kh}}{\xi^{k}-\xi^{-k}} \\
&= \frac{\xi^{2kh}-\xi^{-2kh}}{\xi^{k}-\xi^{-k}} = \frac{\sin\left[\frac{2hk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]},
\end{aligned}$$

noting that $2h = l$, we get the result.

And finally we note that when $k = p$ that

$$\gamma_{l,p} = \sum_{j=0}^{l-1}(-1)^j \frac{\sin\left[(2j+1)\pi/2\right]}{\sin\left[\pi/2\right]} = \sum_{j=0}^{l-1}(-1)^j(-1)^j = l.$$ ∎

Corollary 3.1. *For each p and $l = 1, 2, \ldots, p$, the matrix B_l is invertible if and only if* $\gcd(l,p) \neq 1$.

Proof. This follows immediately by recalling that for $1 \leq k < p$ that

$$\gamma_{l,k} = (-1)^{l+1}\frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]},$$

is zero, whenever lk is zero mod p. Hence, the product of eigenvalues is zero if and only if $\gcd(l,p) \neq 1$. ∎

Corollary 3.2. *For $p \geq 2$ and $1 \leq l \leq p$ the dimension of the null space of B_l is* $\gcd(l,p) - 1$.

Proof. For $1 \leq l \leq p-1$, $\gamma_k = 0$ if and only if $\sin\left[\frac{lk\pi}{p}\right] = 0$, which occurs only when lk is a multiple of p. We also observe that the smallest number k for which kl/p is an integer is $k = p/c$ for $c = \gcd(l,p)$. Since $\gamma_p = l > 0$, the number of zero eigenvalues is therefore $c - 1$. ∎

We can now determine $|B_l|$ in the interesting case that $\gcd(l,p) = 1$.

Corollary 3.3. *Let $p \geq 2$ and $1 \leq l \leq p$. If $\gcd(l,p) \neq 1$, then $|B_l| = 0$. Otherwise,*

$$|B_l| = (-1)^{\frac{(l-1)(p-1)}{2}}\, l.$$

Proof. We are grateful to the anonymous reviewer for simplifying the statement of this corollary and suggesting the following argument. [6]

[6] Our original proof followed an argument of Van Lint [17].

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From Theorem 3.1, we have for $1 \leq k < p$,

$$\gamma_{l,k} = (-1)^{l+1} \frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]},$$

and $\gamma_p = l$. Hence, it is sufficient to determine

$$\prod_{k=1}^{p-1} \frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]}.$$

In the case that $\gcd(l,p) = 1$, this value is plus or minus one, and

$$\begin{aligned}\prod_{k=1}^{p-1} \frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]} &= \prod_{k=1}^{p-1} \frac{e^{ilk\pi/p} - e^{-ilk\pi/p}}{e^{ik\pi/p} - e^{-ik\pi/p}} \\ &= \prod_{k=1}^{p-1} e^{-i(l-1)\frac{k\pi}{p}} \frac{1 - e^{2ilk\pi/p}}{1 - e^{2ik\pi/p}}.\end{aligned}$$

Since l is prime to p, the mapping $k \longmapsto lk$ is a permutation of the set of nonzero residues $\{1, 2, \ldots, p-1\}$ modulo p. It follows that the following sets

$$\{e^{2ilk\pi/p}\}_{k=1}^{p-1} = \{e^{2ik\pi/p}\}_{k=1}^{p-1},$$

are equal, and thus that

$$\begin{aligned}\prod_{k=1}^{p-1} \frac{\sin\left[\frac{lk\pi}{p}\right]}{\sin\left[\frac{k\pi}{p}\right]} &= \prod_{k=1}^{p-1} e^{-i(l-1)\frac{k\pi}{p}} \frac{1 - e^{2ilk\pi/p}}{1 - e^{2ik\pi/p}} \\ &= \prod_{k=1}^{p-1} e^{-i(l-1)\frac{k\pi}{p}} \\ &= e^{-i(l-1)\frac{\pi}{p}\sum_{k=1}^{p-1} k} \\ &= e^{-i(l-1)\frac{\pi}{p}\frac{p(p-1)}{2}} \\ &= e^{-i\pi(l-1)(p-1)/2}.\end{aligned}$$

Since l and p are relatively prime, at least one of them must be odd, and hence this last term is equal to $(-1)^{\frac{(l-1)(p-1)}{2}}$.
Therefore, we have shown that

$$|B_l| = (-1)^{(l+1)(p-1) + \frac{(l-1)(p-1)}{2}} l.$$

Again noting that l are p relatively prime, either $l+1$ or $p-1$ must be even, simplifying the result to

$$|B_l| = (-1)^{\frac{(l-1)(p-1)}{2}} l.$$

□

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4. Closing remarks. We conclude by mentioning a direct application of these results to Hilbert–Kunz theory that will be of interest to experts.

4.1. Applications to Hilbert–Kunz multiplicity. The results of Sections 2 and 3 can be used to construct a complete idempotent decomposition of the algebra R_p. This gives, with a little work, an explicit splitting of the Han–Monsky representation ring, Γ_p, over the real field $\mathbb{Q}\left(\zeta_p + \zeta_p^{-1}\right)$. Diagonalizing the product structure of Γ_p in this way has led directly to new and exciting explicit formulas for Hilbert–Kunz multiplicities of some diagonal hypersurfaces.

For example, suppose that $d \geq 2$, $p > 2$ is an odd prime, and that κ is a field of characteristic p. Let $R_{p,d}$ denote the d-dimensional quadratic hypersurface singularity

$$R_{p,d} = \kappa[[x_0, \ldots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2).$$

Computing according to Theorems 5.2 and 5.3 of [11] using the eigenvalue formulas from Theorems 2.3 and 3.1 leads, after some standard trigonometric simplifications, to the following remarkable formula:

$$\begin{aligned} e_{HK}\left(R_{p,d}\right) &= \frac{\left[p^d + \frac{(-1)^{\left(\frac{p-1}{2}\right)(d+1)}}{p}\Big(\sigma_{d+1}(p) - \sigma_{d-1}(p)\Big)\right] - \left[\frac{(-1)^{\left(\frac{p-1}{2}\right)(d+1)}}{p}\sigma_{d-1}(p)\right]}{p^d - \left[\frac{(-1)^{\left(\frac{p-1}{2}\right)(d+1)}}{p}\sigma_{d-1}(p)\right]} \\ &= \frac{p^{d+1} + (-1)^{\left(\frac{p-1}{2}\right)(d+1)}\Big(\sigma_{d+1}(p) - 2\sigma_{d-1}(p)\Big)}{p^{d+1} - (-1)^{\left(\frac{p-1}{2}\right)(d+1)}\sigma_{d-1}(p)}, \end{aligned}$$

where $\sigma_d(p)$ denotes the trigonometric sum[7]

$$\sigma_d(p) = \sum_{k=0}^{p-1} \sec^d\left[\frac{2k\pi}{p}\right].$$

This result, and further application of trigonometric sums to Hilbert–Kunz Theory are the topic of a forthcoming project.

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[7] Note that, remarkably, this function is an *integer valued polynomial* of p. Explicit forms for $\sigma_d(p)$ can be derived from Theorem 1 of [5] or formulas A1 and A2 of [4]. We also mention that the numerator and denominator in the formula above turn out to be divisible by $p(p^2-1)$ or $p^2(p^2-1)$, depending on whether the dimension, d, is even or odd.

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