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OPERATOR MEAN AND MEAN ITERATIONS OF POSITIVE CONSTANT UPPER TRIANGULAR MATRICES*

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Abstract. Adapting the Kubo–Ando's operator mean, we define the operator mean on the Lie group CP_n of $n \times n$ positive constant upper triangular matrices. We also study the weighted spectral geometric mean on CP_n and provide its binomial expansion. Moreover, we establish the Gauss mean and logarithmic mean on CP_n by proving the convergence of mean iterations. Finally, we investigate two multivariable means, the resolvent mean and the $A\#H$ mean, on CP_n.

Key words. Positive constant upper triangular matrix, Operator mean, Spectral geometric mean, Gauss mean, Logarithmic mean, Resolvent mean, $A\#H$ mean.

AMS subject classifications. 15B30, 22E25.

1. Introduction. Let $B(\mathcal{H})$ be the Banach space of all bounded linear operators on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, and let $S(\mathcal{H}) \subset B(\mathcal{H})$ be the closed subspace of all self-adjoint linear operators. We say that $A \in S(\mathcal{H})$ is positive semi-definite if $\langle x, Ax \rangle \geq 0$ for all vector $x \in \mathcal{H}$ and is positive definite if $\langle x, Ax \rangle > 0$ for all nonzero vector $x \in \mathcal{H}$. We denote as $\mathbb{P} \subset S(\mathcal{H})$ the open convex cone of all positive definite operators. The group GL of all invertible operators transitively acts on $\mathbb{P}$ via congruence transformation.

Note that for $A, B \in S(\mathcal{H})$, $A \leq B$ if and only if $B - A$ is positive semi-definite, and $A < B$ if and only if $B - A$ is positive definite. This gives us a partial order on $S(\mathcal{H})$, known as the Loewner order.

Kubo and Ando [10] have introduced a notion of operator mean, which has a great influence on mean theory. A binary operation $\sigma : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is called an *operator mean* if it satisfies the following conditions: for any $A, B, C, D \in \mathbb{P}$

1. $A\sigma A = A$.
2. $A\sigma B \leq C\sigma D$ whenever $A \leq C$ and $B \leq D$.
3. $T(A\sigma B)T^* \leq TAT^*\sigma TBT^*$ for any $T \in B(\mathcal{H})$.
4. σ is continuous.

The important result of Kubo–Ando's operator mean is that there exists a one-to-one correspondence between operator means on $\mathbb{P}$ and operator monotone functions defined on $(0, \infty)$. The correspondence is given by

$$A\sigma B := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \tag{1.1}$$

where the function f as in above satisfies the following conditions:

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(a) $f(1) = 1$.
(b) $tf(t^{-1}) = f(t)$.
(c) f is operator monotone.
(d) f is continuous.

One of the important Kubo–Ando's operator means is the weighted geometric mean: for $A, B \in \mathbb{P}$

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \quad t \in [0,1].$$

It naturally gives rise to the multi-variable geometric mean, including the Karcher mean [14] as a unique positive definite solution X to the equation

$$\sum_{j=1}^{n} w_j \log(X^{1/2}A_j^{-1}X^{1/2}) = 0, \tag{1.2}$$

for $A_1, \ldots, A_n \in \mathbb{P}$, where $(w_1, \ldots, w_n)$ is a positive probability vector.

Recently, the construction of mean for matrices, which are not positive definite, has been studied [4, 5, 9, 11]. It is natural to study the means on the following two Lie groups:

- the Lie group UP_n of $n \times n$ unit upper triangular matrices, and
- the Lie group CP_n of $n \times n$ positive constant upper triangular matrices.

Note that a positive constant upper triangular matrix is an upper triangular matrix whose main diagonal entries are all same positive constants. Especially, the convergence of mean iterations consisting of the arithmetic and harmonic means to the geometric mean has been shown on UP_n in [5]. Moreover, the Karcher mean of unit upper triangular matrices has been successfully defined in [11] as a unique solution $X \in \mathrm{UP}_n$ to (1.2). As one of the typical non-Kubo–Ando's operator means, the Wasserstein mean of unit upper triangular matrices $A_1, \ldots, A_n$ has been defined in [9] as the only unit upper triangular matrix X that satisfies the equation

$$I = \sum_{j=1}^{n} w_j (X^{-1}\#A_j). \tag{1.3}$$

We review the definition of an operator mean and revisit the weighted geometric mean on UP_n and extend these notations to CP_n in Section 2. As another example of non-Kubo–Ando's operator means, we study the weighted spectral geometric mean of positive constant upper triangular matrices and find its binomial expansion with explicit formulas in Section 3. In Section 4, we show that the (skewed) mean iteration of arithmetic and geometric means converges to the Gauss mean (logarithmic mean, respectively) for positive constant upper triangular matrices. Especially when $t = 1/2$, we define the logarithmic mean using an integral representation, see that it relates to Gregory coefficients, and show that the skewed mean iteration and the integral representation are equivalent. Finally, in Section 5, we study the resolvent mean and $A\#H$ mean as multi-variable matrix means of positive constant upper triangular matrices and verify that mean iteration consisting of the resolvent mean converges to the $A\#H$ mean.

2. Operator mean of positive constant upper triangular matrices. We define in this section the operator mean on CP_n. Denote $\mathfrak{up}_n$ the Lie algebra of UP_n, which consists of $n \times n$ strictly upper triangular matrices (called nilpotent matrices). Denote $\mathfrak{cp}_n$ the Lie algebra of CP_n, which consists of $n \times n$ upper triangular matrices with constant diagonals.

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To transfer between CP_n and UP_n and between their Lie algebras, we define the following maps.

- $\Phi_D : \mathfrak{cp}_n \to \mathbb{R}I_n$ such that $\Phi_D(A) \in \mathbb{R}I_n$ has the same diagonal as A;
- $\Phi_N : \mathfrak{cp}_n \to \mathfrak{up}_n$ such that $\Phi_N(A) = A - \Phi_D(A) \in \mathfrak{up}_n$;
- $\Phi_U : \mathrm{CP}_n \to \mathrm{UP}_n$ such that $\Phi_U(A) = \Phi_D(A)^{-1}A \in \mathrm{UP}_n$.

Lemma 2.1. *The maps Φ_D and Φ_U satisfy the following properties:*

1. *For $A, B \in \mathfrak{cp}_n$ and $u, v \in \mathbb{R}$,*

$$\Phi_D(uA + vB) = u\Phi_D(A) + v\Phi_D(B), \qquad \Phi_D(AB) = \Phi_D(A)\Phi_D(B). \tag{2.4}$$

2. *For $A, B \in \mathrm{CP}_n$, $u \in \mathbb{R}^+$, and $s \in \mathbb{R}$,*

$$\Phi_U(uAB) = \Phi_U(A)\Phi_U(B), \qquad \Phi_U(A^s) = \Phi_U(A)^s. \tag{2.5}$$

3. *For every $c \in \mathbb{R} \setminus \{0\}$, there is a nonsingular real $n \times n$ matrix P such that*

$$A = \Phi_D(A) + \Phi_N(A) = P(\Phi_D(A) + c\Phi_N(A))P^{-1}. \tag{2.6}$$

Proof. The proofs of (1) and (2) are obviously. For (3), note that the nilpotent matrix $\Phi_N(A)$ is similar to $c\Phi_N(A)$ for every $c \neq 0$. There is a nonsingular real $n \times n$ matrix P such that $\Phi_N(A) = P(c\Phi_N(A))P^{-1}$. Write $\Phi_D(A) = aI$ Then, $A = aI + \Phi_N(A) = P(\Phi_D(A) + c\Phi_N(A))P^{-1}$ as desired. ∎

Lemma 2.2. *The map*

$$\Phi : \mathrm{CP}_n \to \mathbb{R}^+I_n \times \mathrm{UP}_n, \qquad \Phi(A) \mapsto (\Phi_D(A), \Phi_U(A)), \tag{2.7}$$

is a Lie group isomorphism between CP_n and $\mathbb{R}^+I_n \times \mathrm{UP}_n$.

The proof is straightforward, and it is skipped here.

Let $A = aI + N \in \mathrm{CP}_n$ where $a \in \mathbb{R}^+$ and N is a strictly upper triangular matrix. The binomial expansion of A^t for $t \in \mathbb{R}$ is

$$A^t = (aI + N)^t = \sum_{i=0}^{n-1} a^{t-i}\binom{t}{i}N^i, \tag{2.8}$$

where

$$\binom{t}{i} = \frac{t(t-1)(t-2)\cdots(t-i+1)}{i!}, \qquad \binom{t}{0} := 1.$$

Definition 2.3. *A binary operation $\sigma : \mathrm{CP}_n \times \mathrm{CP}_n \to \mathrm{CP}_n$ is called an* operator mean *on* CP_n *if it has the same analytic expression of an operator mean on positive operators.*

We can similarly define an *operator mean* on UP_n. Given an operator mean σ on CP_n, the restriction $\sigma|_{\mathrm{UP}_n \times \mathrm{UP}_n}$ is an operator mean on UP_n. Conversely, every operator mean σ on UP_n can be extended to an operator mean on CP_n, since σ is also an operator mean on $\mathbb{R}^+I_n$ and matrices in $\mathbb{R}^+I_n$ are commuting with those in UP_n.

We adapt the original definition of Kubo–Ando's operator mean on $\mathbb{P}$ to the matrix Lie group CP_n so that the following theorem is obviously obtained:

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THEOREM 2.4. *For every operator mean σ on CP_n, there exists an operator monotone function f such that $A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ for every $A, B \in \mathrm{CP}_n$.*

There are numerous examples of operator means on CP_n and UP_n such as the arithmetic, geometric, and harmonic means of which corresponding operator monotone functions $f(t)$ from Theorem 2.4 are given by

$$\frac{1+t}{2}, \qquad t^{1/2}, \text{ and } \quad \left(\frac{1+t^{-1}}{2}\right)^{-1}.$$

See [5] for more information about operator means on UP_n, including the geometric mean.

Recently, the weighted geometric mean of unit upper triangular matrices $A, B \in \mathrm{UP}_n$

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \qquad t \in \mathbb{R}, \tag{2.9}$$

has been established in [4] with its interesting properties. Both the mean and the properties can be migrated to the weighted geometric mean on CP_n. The Lie group CP_n is a Loos symmetric space equipped with symmetries $S_A : \mathrm{CP}_n \to \mathrm{CP}_n$, $S_A(B) := AB^{-1}A = A \bullet B$. The Riccati equation $XA^{-1}X = X \bullet A = B$ for $A, B \in \mathrm{CP}_n$ has a unique solution $A\#B = A\#_{1/2}B$ (as shown in [12] and more details on UP_n in [4]).

A *geodesic of symmetry* on CP_n is a continuous map $\gamma : \mathbb{R} \to \mathrm{CP}_n$ satisfying

$$\gamma(2s-t) = \gamma(s) \bullet \gamma(t), \quad \forall s, t \in \mathbb{R}.$$

Loos symmetric spaces and geodesics of symmetry have been studied in [12], and by Theorem 4.5 therein, $A\#_t B$ is the unique geodesic of symmetry passing through A at $t = 0$ and B at $t = 1$. It also satisfies the affine property of parameters:

$$A\#_{\frac{s+t}{2}} B = (A\#_s B)\#(A\#_t B), \tag{2.10}$$

for all $s, t \in \mathbb{R}$.

Moreover, the binomial expansion of the weighted geometric mean has been shown by using the Taylor expansion for the analytic power map.

THEOREM 2.5. *[4] For $A, B \in \mathrm{UP}_n$ and $t \in \mathbb{R}$,*

$$A\#_t B = \sum_{k=0}^{n-1} \binom{t}{k} A(A^{-1}B - I)^k. \tag{2.11}$$

One can rewrite (2.11) for $A, B \in \mathrm{UP}_n$ as

$$A\#_t B = (1-t)A + tB + \sum_{k=2}^{n-1} \binom{t}{k} A(A^{-1}B - I)^k. \tag{2.12}$$

Note that (2.12) and (2.11) are not true for $A, B \in \mathrm{CP}_n$ with $\Phi_D(A) \neq \Phi_D(B)$.

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EXAMPLE 2.6. *The weighted geometric means for $A, B \in \mathrm{UP}_n$ and $2 \leq n \leq 4$ are*

$$\begin{aligned}
&for\ n=2, \quad A\#_t B = (1-t)A + tB,\\
&for\ n=3, \quad A\#_t B = (1-t)A + tB + \frac{t(t-1)}{2}(A-B)^2,\\
&for\ n=4, \quad A\#_t B = (1-t)A + tB - t(1-t)(A-B)^2\\
&\qquad\qquad + \frac{t(1-t)}{2}(A-B)\left[\frac{1+t}{3}A + \frac{2-t}{3}B\right](A-B).
\end{aligned}$$

The weighted geometric mean on CP_n has the following expression.

THEOREM 2.7. *Let Φ_D and Φ_U be as defined before Lemma 2.1. For $A, B \in \mathrm{CP}_n$ and $t \in \mathbb{R}$,*

$$\Phi_D(A\#_t B) = \Phi_D(A)^{1-t}\Phi_D(B)^t, \qquad \Phi_U(A\#_t B) = \Phi_U(A)\#_t\Phi_U(B), \tag{2.13}$$

where the expansion of $\Phi_U(A)\#_t\Phi_U(B)$ is given by (2.11).

The following are fundamental properties of weighted geometric mean on CP_n.

THEOREM 2.8. *Let $A, B, U, V \in \mathrm{CP}_n$ and $s, t, u \in \mathbb{R}$.*

(1) $A\#_t B = A(A^{-1}B)^t = (AB^{-1})^{1-t}B$.
(2) $A\#_t B = A^{1-t}B^t$ if A and B commute.
(3) $(\alpha A)\#_t(\beta B) = \alpha^{1-t}\beta^t(A\#_t B)$ for any $\alpha, \beta > 0$.
(4) $(A\#_t B)^{-1} = A^{-1}\#_t B^{-1}$.
(5) $A\#_t B = B\#_{1-t}A$.
(6) $(UAV)\#_t(UBV) = U(A\#_t B)V$.
(7) $(A\#_s B)\#_u(A\#_t B) = A\#_{(1-u)s+ut}B$.

Proof. For (1),

$$\begin{aligned}
A\#_t B &= A[A^{-1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}]\\
&= A(A^{-1/2}A^{-1/2}BA^{-1/2}A^{1/2})^t = A(A^{-1}B)^t.
\end{aligned}$$ ∎

Similarly, we can prove that $A\#_t B = (AB^{-1})^{1-t}B$. The proofs of (2), (3), and (4) are straightforward by (2.9) and (1). For (5), (6), (7), they follow from a unique geodesic of symmetry joining A and B.

As a corollary, we extend Theorem 2.5 to a series expansion of $A\#_t B$ for $A, B \in \mathrm{CP}_n$.

COROLLARY 2.9. *For $A, B \in \mathrm{CP}_n$ and $t \in \mathbb{R}$, if each diagonal entry of A is greater than $1/2$ of each diagonal entry of B, then*

$$A\#_t B = \sum_{k=0}^{\infty}\binom{t}{k}A(A^{-1}B - I)^k, \tag{2.14}$$

$$= (1-t)A + tB + \sum_{k=2}^{\infty}\binom{t}{k}A(A^{-1}B - I)^k. \tag{2.15}$$

Proof. Use the series expansions in Theorem 2.8 (1) to prove (2.14) and (2.15). Note that the series converge if and only if the absolute values of the diagonal entries of $A^{-1}B - I$ are less than 1, if and only if $\Phi_D(A) > \frac{1}{2}\Phi_D(B)$. ∎

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3. Spectral geometric mean of positive constant upper triangular matrices. The weighted spectral geometric mean of $A, B \in \mathbb{P}$ and $t \in [0,1]$ is given in [15] by

$$A \natural_t B = (A^{-1} \# B)^t A (A^{-1} \# B)^t. \tag{3.16}$$

For $t = 1/2$, Fiedler and Pták [6] have first introduced the spectral geometric mean of $n \times n$ positive definite Hermitian matrices. This is a typical example of non-Kubo–Ando's operator mean on $\mathbb{P}$, since it does not satisfy the monotonicity for the Loewner order. In this section, we study the weighted spectral geometric mean on CP_n by (3.16) for any $t \in \mathbb{R}$, as same as that of positive definite operators, with its properties and explicit forms.

Since the exponential map $\exp : \mathfrak{cp}_n \to \mathrm{CP}_n$ is a bijection whose inverse is the logarithmic map $\log : \mathrm{CP}_n \to \mathfrak{cp}_n$. We define

$$A^p := \exp(p \log A), \quad p \in \mathbb{R},$$

for any $A \in \mathrm{CP}_n$. Then it is well-defined, and so is (3.16) for matrices $A, B \in \mathrm{CP}_n$ because CP_n is closed under matrix multiplication. Now, we show in Proposition 3.1 and Theorem 3.2 that the weighted spectral geometric mean is the unique solution of certain equations.

Proposition 3.1. *For $A, B \in \mathrm{CP}_n$ and $t \in \mathbb{R}$, $A \natural_t B$ is the unique solution $X \in \mathrm{CP}_n$ to the equation*

$$A^{-1} \# X = (A^{-1} \# B)^t.$$

Proof. We observe that

$$\begin{aligned} A^{-1} \# (A \natural_t B) &= A^{-1/2} (A^{1/2} (A^{-1} \# B)^t A (A^{-1} \# B)^t A^{1/2})^{1/2} A^{-1/2} \\ &= A^{-1/2} (A^{1/2} (A^{-1} \# B)^t A^{1/2}) A^{-1/2} = (A^{-1} \# B)^t. \end{aligned}$$

If we have $A^{-1} \# X = A^{-1} \# Y$ for $X, Y \in \mathrm{CP}_n$, then $X = Y$. ∎

Theorem 3.2. *For nonzero $t \in \mathbb{R}$ let $F_t : \mathrm{CP}_n \times \mathrm{CP}_n \to \mathrm{CP}_n$ be a map satisfying $F_t(X, X) = X$ for all $X \in \mathrm{CP}_n$ and*

$$F_t(X, Y) = I \implies Y = X^{1 - \frac{1}{t}} \text{ for any } X, Y \in \mathrm{CP}_n. \tag{3.17}$$

Then, $A \natural_t B$ for $A, B \in \mathrm{CP}_n$ is the unique solution $X \in \mathrm{CP}_n$ to the equation

$$F_t(A \# X^{-1}, B \# X^{-1}) = I. \tag{3.18}$$

Proof. Fix $A, B \in \mathrm{CP}_n$.

First, let $X = A \natural_t B \in \mathrm{CP}_n$. By Proposition 3.1, $A^{-1} \# X = (A^{-1} \# B)^t$ so that $A \# X^{-1} = (A \# B^{-1})^t$. By the Riccati equation,

$$\begin{aligned} A &= (A \# X^{-1}) X (A \# X^{-1}) = (A \# B^{-1})^t X (A \# B^{-1})^t, \\ B &= (B \# A^{-1}) A (B \# A^{-1}) = (B \# A^{-1}) [(A \# B^{-1})^t X (A \# B^{-1})^t] (B \# A^{-1}) \\ &= (A \# B^{-1})^{t-1} X (A \# B^{-1})^{t-1}, \\ B \# X^{-1} &= (A \# B^{-1})^{t-1}. \end{aligned}$$

Therefore, by (3.17), $X = A \natural_t B$ satisfies that

$$F_t(A \# X^{-1}, B \# X^{-1}) = F_t((A \# B^{-1})^t, (A \# B^{-1})^{t-1}) = I.$$

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Second, let $X \in \mathrm{CP}_n$ satisfy the equation (3.18). Set $U = A\#X^{-1}$ and $V = B\#X^{-1}$. Then, $U, V \in \mathrm{CP}_n$ and $F_t(U, V) = I$, so by (3.17)

$$V = U^{1-\frac{1}{t}}.$$

Since $X = U^{-1}AU^{-1} = V^{-1}BV^{-1}$ by the Riccati equation, we have

$$A = UV^{-1}BV^{-1}U = U^{\frac{1}{t}}BU^{\frac{1}{t}}.$$

Again by the Riccati equation $U^{\frac{1}{t}} = A\#B^{-1}$, and hence, by Theorem 2.8 (4)

$$X = U^{-1}AU^{-1} = (A\#B^{-1})^{-t}A(A\#B^{-1})^{-t} = (A^{-1}\#B)^t A(A^{-1}\#B)^t = A\natural_t B. \qquad \square$$

THEOREM 3.3. *For any $A, B \in \mathrm{CP}_n$ and $t \in \mathbb{R}$*

$$\begin{cases} A = X^{-t}YX^{-t} & (1) \\ B = X^{1-t}YX^{1-t} & (2), \end{cases}$$

has a unique solution $(X, Y) = (A^{-1}\#B, A\natural_t B)$.

Proof. By (1) $Y = X^t AX^t$, and then (2) reduces to $B = XAX$. By the Riccati equation $X = A^{-1}\#B$, so

$$Y = (A^{-1}\#B)^t A(A^{-1}\#B)^t = A\natural_t B. \qquad \square$$

We list some fundamental properties of the weighted spectral geometric mean in CP_n.

PROPOSITION 3.4. *For $A, B, U \in \mathrm{CP}_n$ and $s, t, u \in \mathbb{R}$*

(1) $A\natural_t B = A^{1-t}B^t$ *if A and B commute.*
(2) $(\alpha A)\natural_t(\beta B) = \alpha^{1-t}\beta^t(A\natural_t B)$ *for any $\alpha, \beta > 0$.*
(3) $(A\natural_t B)^{-1} = A^{-1}\natural_t B^{-1}$.
(4) $A\natural_t B = B\natural_{1-t} A$.
(5) $(UAU^{-1})\natural_t(UBU^{-1}) = U(A\natural_t B)U^{-1}$.
(6) $(A\natural_s B)\natural_t(A\natural_u B) = A\natural_{(1-t)s+tu} B$.

Proof. One can easily see from Theorem 2.8 that properties (1), (2), and (3) are true.

(4) By the Riccati equation, Theorem 2.8 (4) and (5)

$$\begin{aligned} B\natural_{1-t} A &= (B^{-1}\#A)^{1-t}B(B^{-1}\#A)^{1-t} \\ &= (B^{-1}\#A)^{-t}[(B^{-1}\#A)B(B^{-1}\#A)](B^{-1}\#A)^{-t} \\ &= (B\#A^{-1})^t A(B\#A^{-1})^t = A\natural_t B. \end{aligned}$$

(5) Since $\log A = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}(A - I)^k$ for any $A \in \mathrm{CP}_n$, we have $U(\log A)U^{-1} = \log(UAU^{-1})$ for any $U \in \mathrm{CP}_n$. So from Theorem 2.8 (6)

$$((UAU^{-1})^{-1}\#(UBU^{-1}))^t = U(A^{-1}\#B)^t U^{-1},$$

and thus, we obtain the identity (5).

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 42, pp. 294-313, March 2026.

(5) Let $X = A^{-1}\#B$. Then, $A\natural_s B = X^s A X^s$, and $(X^sAX^s)^{-1}\#(X^uAX^u) = X^{u-s}$ by the Riccati equation, since

$$X^uAX^u = X^{u-s}(X^sAX^s)X^{u-s}.$$

Thus,

$$\begin{aligned}(A\natural_s B)\natural_t(A\natural_u B) &= [(X^sAX^s)^{-1}\#(X^uAX^u)]^t X^sAX^s[(X^sAX^s)^{-1}\#(X^uAX^u)]^t\\ &= X^{(u-s)t}X^sAX^sX^{(u-s)t}\\ &= X^{(1-t)s+tu}AX^{(1-t)s+tu} = A\natural_{(1-t)s+tu}B.\end{aligned}$$ ◻

The following provides the explicit formula of weighted spectral geometric mean.

THEOREM 3.5. *For any $A, B \in \mathrm{UP}_n$ and $t \in \mathbb{R}$, setting $X = A^{-1}\#B$, we get*

$$A\natural_t B = \sum_{i=0}^{n-1}\sum_{k=0}^{i}\binom{t}{k}\binom{t}{i-k}(X-I)^kA(X-I)^{i-k}. \tag{3.19}$$

For $A, B \in \mathrm{CP}_n$ and $t \in \mathbb{R}$, using Φ_D and Φ_U as defined before Lemma 2.1, we have

$$\Phi_D(A\natural_t B) = \Phi_D(A)^{1-t}\Phi_D(B)^t, \qquad \Phi_U(A\natural_t B) = \Phi_U(A)\natural_t\Phi_U(B). \tag{3.20}$$

Proof. First, suppose that $A, B \in \mathrm{UP}_n$. Let $X = A^{-1}\#B$. Then, $X \in \mathrm{UP}_n$. Since

$$X^t = \sum_{k=0}^{n-1}\binom{t}{k}(X-I)^k, \qquad \forall t \in \mathbb{R},$$

by Equation (11) in [4], we have from (3.16)

$$\begin{aligned}A\natural_t B &= X^tAX^t\\ &= \left[\sum_{k=0}^{n-1}\binom{t}{k}(X-I)^k\right]A\left[\sum_{j=0}^{n-1}\binom{t}{j}(X-I)^j\right]\\ &= \sum_{k+j\le n-1}\binom{t}{k}\binom{t}{j}(X-I)^kA(X-I)^j\\ &= \sum_{i=0}^{n-1}\sum_{k=0}^{i}\binom{t}{k}\binom{t}{i-k}(X-I)^kA(X-I)^{i-k}.\end{aligned}$$

The third equality follows from that $(X-I)^kA(X-I)^j = 0$ for all $k+j \ge n$, due to $X - I \in \mathfrak{up}_n$, and the last follows from substituting $i = k+j$.

Second, let $A, B \in \mathrm{CP}_n$. Then by Lemma 2.1 and Theorem 2.7,

$$\begin{aligned}\Phi_D(A\natural_t B) &= \Phi_D\left((A^{-1}\#B)^tA(A^{-1}\#B)^t\right) = \Phi_D(A)\natural_t\Phi_D(B) = \Phi_D(A)^{1-t}\Phi_D(B)^t,\\ \Phi_U(A\natural_t B) &= \Phi_U\left((A^{-1}\#B)^tA(A^{-1}\#B)^t\right) = \Phi_U(A)\natural_t\Phi_U(B).\end{aligned}$$ ◻

EXAMPLE 3.6. *Here, we give examples of the weighted spectral geometric mean on UP_n for $n = 2, 3$ by using (3.19). Let*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 294-313, March 2026.

$$P = \frac{I + AB}{2} \quad and \quad Q = \frac{I + BA}{2}.$$

1. *For $n = 2$,*

$$X = A^{-1} \# B = \frac{A^{-1} + B}{2},$$

and

$$\begin{aligned} A \natural_t B &= A + tA(X - I) + t(X - I)A \\ &= (1 - 2t)A + t(P + Q). \end{aligned}$$

2. *For $n = 3$,*

$$X = A^{-1} \# B = \frac{A^{-1} + B}{2} - \frac{1}{2} \left[\frac{A^{-1} - B}{2} \right]^2,$$

and

$$\begin{aligned} A \natural_t B &= A + t[A(X - I) + (X - I)A] \\ &\quad + \left[\frac{t(t-1)}{2} A(X - I)^2 + t^2 (X - I)A(X - I) + \frac{t(t-1)}{2} (X - I)^2 A \right] \\ &= (1 - t)(1 - 2t)A + 2t(1 - t)(AX + XA) \\ &\quad + \frac{t(t-1)}{2} (AX^2 + X^2 A) + t^2 XAX. \end{aligned}$$

Note that

$$\begin{aligned} AX &= \frac{I + AB}{2} - \frac{1}{2} A \left(\frac{A^{-1} - B}{2} \right)^2 = P - \frac{1}{2}(P - I)^2, \\ XA &= \frac{I + BA}{2} - \frac{1}{2} \left(\frac{A^{-1} - B}{2} \right)^2 A = Q - \frac{1}{2}(Q - I)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} AX^2 &= \left[P - \frac{1}{2}(P - I)^2 \right] \left[A^{-1}P - \frac{1}{2} A^{-1}(P - I)A^{-1}(P - I) \right] \\ &= PA^{-1}P - \frac{1}{2} PA^{-1}(P - I)A^{-1}(P - I) - \frac{1}{2}(P - I)^2 A^{-1}P \\ &= PA^{-1}P - (P - I)^2, \end{aligned}$$

since

$$PA^{-1}(P - I)A^{-1}(P - I) = \begin{bmatrix} 0 & 0 & \frac{(a_1+b_1)(a_3+b_3)}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (P - I)^2,$$

where $P = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}$, and similarly $(P - I)^2 A^{-1} P = (P - I)^2$. By a similar calculation, we have

$$X^2 A = QA^{-1}Q - (Q - I)^2.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 294-313, March 2026.

Combining all computations together with $(P-I)^2 = (Q-I)^2$ *and* $XAX = B$ *by the Riccati equation, we obtain*

$$\begin{aligned} A\natural_t B &= (1-t)(1-2t)A + t^2 B + 2t(1-t)[P + Q - (P-I)^2] \\ &\quad + \frac{t(t-1)}{2}[PA^{-1}P + QA^{-1}Q - 2(P-I)^2] \\ &= (1-t)(1-2t)A + t^2 B + 2t(1-t)(P+Q) \\ &\quad - \frac{t(1-t)}{2}(PA^{-1}P + QA^{-1}Q) - t(1-t)(P-I)^2. \end{aligned}$$

4. Mean iterations. It has been shown in [5] that the sequences A_k and B_k given by $A_0 = A$ and $B_0 = B$ of unit upper triangular matrices $A, B \in \mathrm{UP}_n$,

$$A_{k+1} = \frac{A_k + B_k}{2} \quad \text{and} \quad B_{k+1} = \left(\frac{A_k^{-1} + B_k^{-1}}{2}\right)^{-1},$$

converge to the geometric mean $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ in a finite step. Motivated by the above result, we study in this section two types of mean iteration, whose common limits are, respectively, the Gauss and logarithmic means.

4.1. Gauss mean. For $n \times n$ positive definite matrices A and B, the *Gauss mean* (or *arithmetic-geometric mean*) [7, 13] is defined by the common limit of iterative compositions of the arithmetic and geometric means: for any $t \in (0, 1)$

$$G_t(A, B) = \lim_{k\to\infty} A_k = \lim_{k\to\infty} B_k, \tag{4.21}$$

where $A_1 = A, B_1 = B$, and

$$A_{k+1} = (1-t)A_k + tB_k, \quad B_{k+1} = A_k \#_t B_k, \tag{4.22}$$

for $k \geq 1$. The main idea of its proof is that every monotone and bounded sequence converges. On the other hand, it does not hold for matrices on UP_n and CP_n since the partial order and monotonicity cannot be applied.

Let $\mathcal{N}^{(p)}$ denote the subspace of $\mathfrak{cp}_n$ consisting of constant diagonal upper triangular matrices vanishing up to p-th subdiagonal entries for each $p \in \{0, 1, 2, \ldots, n\}$. Then,

$$\{0\} = \mathcal{N}^{(n)} \subset \mathcal{N}^{(n-1)} \subset \cdots \subset \mathcal{N}^{(1)} = \mathfrak{up}_n \subset \mathcal{N}^{(0)} = \mathfrak{cp}_n. \tag{4.23}$$

When $p > n$, we denote $\mathcal{N}^{(p)} = \{0\}$ for consistency. Then,

$$p, q \in \mathbb{N},\ A \in \mathcal{N}^{(p)},\ B \in \mathcal{N}^{(q)} \implies A + B \in \mathcal{N}^{(\min\{p,q\})},\ AB \in \mathcal{N}^{(p+q)}. \tag{4.24}$$

We first show that for $A, B \in \mathrm{UP}_n$, the iteration procedure (4.22) terminates in the finite number of steps, $k \geq 1 + \lceil \log_2 n \rceil$. Thus, the Gauss mean is well-defined on unit upper triangular matrices. Note that for every $A \in \mathrm{UP}_n$ there exists an (unique) element $N_A \in \mathcal{N}^{(1)}$ such that

$$A = I + N_A. \tag{4.25}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 294-313, March 2026.

THEOREM 4.1. *Let $A, B \in \mathrm{UP}_n$, and let A_k, B_k be iterative sequences as in (4.22). Then, $A_k = B_k$ for $k \geq 1 + \lceil \log_2 n \rceil$.*

Proof. We have $B_1 - A_1 \in \mathcal{N}^{(1)}$. Assume that $B_k - A_k \in \mathcal{N}^{(2^{k-1})}$ for $k \geq 1$. By (2.12) and substituting $j = i - 2$,

$$\begin{aligned}A_{k+1} - B_{k+1} &= (1-t)A_k + tB_k - A_k \#_t B_k \\ &= (1-t)A_k + tB_k - (1-t)A_k - tB_k - \sum_{i=2}^{n-1} \binom{t}{i} A_k(A_k^{-1}B_k - I)^i \\ &= -(B_k - A_k)A_k^{-1}(B_k - A_k) \sum_{i=2}^{n-1} \binom{t}{i} A_k(A_k^{-1}B_k - I)^{i-2}.\end{aligned}$$

By (4.24), $(B_k - A_k)A_k^{-1}(B_k - A_k) \in \mathcal{N}^{(2^k)}$, so that $A_{k+1} - B_{k+1} \in \mathcal{N}^{(2^k)}$. By induction, we conclude that $A_k = B_k$ for $k \geq 1 + \lceil \log_2 n \rceil$. ∎

We now consider the iteration procedure (4.22) for $A, B \in \mathrm{CP}_n$. In general, the iterative sequence converges in infinite steps. We need the following auxiliary result.

THEOREM 4.2. *Let $A, B \in \mathrm{CP}_n$, and let A_k, B_k be iterative sequences as in (4.22). Then, $\lim_{k\to\infty} A_k = \lim_{k\to\infty} B_k$.*

Proof. Let $Z_k = A_k B_k^{-1} \in \mathrm{CP}_n$. It suffices to prove that $\lim_{k\to\infty} Z_k = I$. By (4.22) and Theorem 2.8 (1),

$$A_{k+1}B_k^{-1} = (1-t)Z_k + tI, \quad B_{k+1}B_k^{-1} = Z_k^{1-t}.$$

Hence,

$$Z_{k+1} = A_{k+1}B_{k+1}^{-1} = [(1-t)Z_k + tI]Z_k^{t-1} = (1-t)Z_k^t + tZ_k^{t-1}. \tag{4.26}$$

We define a function

$$f_t(x) = (1-t)x^t + tx^{t-1}.$$

Then, $Z_{k+m} = f_t^{(m)}(Z_k)$ where $f_t^{(m)}$ is the m-th composed functions of f_t.

Write $Z_k = z_k I + N_k$ for $N_k \in \mathcal{N}^{(1)}$. Then,

$$z_{k+1} = (1-t){z_k}^t + t{z_k}^{t-1} = f_t(z_k).$$

If $z_1 \geq 1$, then $z_1 \geq z_2 \geq z_3 \geq \cdots$ and $\lim_{k\to\infty} z_k = 1$. Similarly, when $z_1 < 1$, we also have $\lim_{k\to\infty} z_k = 1$.

Using ${N_k}^n = 0$, (4.26) can be rewritten as

$$\begin{aligned}N_{k+1} & \\ &= (1-t)(z_k I + N_k)^t + t(z_k I + N_k)^{t-1} - z_{k+1}I \\ &= \sum_{i=1}^{n-1} (1-t)\binom{t}{i} z_k^{t-i}{N_k}^i + \sum_{i=1}^{n-1} t\binom{t-1}{i} z_k^{t-1-i}{N_k}^i \\ &= t(1-t)\left[(z_k^t - z_k^{t-1})N_k + \sum_{i=2}^{n-1}\left(\frac{(t-1)\cdots(t-i+1)}{i!}z_k^t - \frac{(t-2)\cdots(t-i)}{i!}z_k^{t-1}\right)z_k^{-i}N_k^i\right],\end{aligned} \tag{4.27}$$

$$= t(1-t)\left[(z_k^t - z_k^{t-1})N_k + \sum_{i=2}^{n-1}\frac{1}{i}\binom{t-2}{i-2}\frac{iz_k^{t-1} - (1-t)z_k^t - tz_k^{t-1}}{i-1}z_k^{-i}N_k^i\right]. \tag{4.28}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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For $t \in (0,1)$ and $i \in \{2,\dots,n-1\}$, we have $|\frac{1}{i}\binom{t-2}{i-2}| < 1$; and when $z_k \to 1$, we have $z_k^t - z_k^{t-1} \to 0$ and $\frac{iz_k^{t-1}-(1-t)z_k^t-tz_k^{t-1}}{i-1} z_k^{-i} \to 1$. There is m such that for every $k > m$, we have $|z_k^t - z_k^{t-1}| < 1/2$ and $|\frac{iz_k^{t-1}-(1-t)z_k^t-tz_k^{t-1}}{i-1} z_k^{-i}| < 3/2$. Fix a $k > m$. Choose a scalar $c \neq 0$ such that $\sum_{i=2}^{n-1} \|cN_k\|_2^{i-1} < 1$. By Lemma 2.1 (3), there is a nonsingular matrix P such that $P^{-1}N_kP = cN_k$. Hence, by (4.28),

$$P^{-1}N_{k+1}P = t(1-t)\left[(z_k^t - z_k^{t-1})(cN_k) + \sum_{i=2}^{n-1} \frac{1}{i}\binom{t-2}{i-2} \frac{iz_k^{t-1}-(1-t)z_k^t-tz_k^{t-1}}{i-1} z_k^{-i}(cN_k)^i\right].$$

We have the spectral norm inequalities

$$\|P^{-1}N_{k+1}P\|_2 \leq t(1-t)\left[\frac{1}{2}\|cN_k\|_2 + \frac{3}{2}\|cN_k\|_2 \sum_{i=2}^{n-1} \|cN_k\|_2^{i-1}\right] \leq \frac{1}{2}\|cN_k\|_2.$$

Replace k by its sequential numbers and apply (4.28) repeatedly. We get $\|P^{-1}N_{k+i+1}P\|_2 \leq \frac{1}{2}\|P^{-1}N_{k+i}P\|_2$ for every positive integer i. Therefore, $\lim_{k\to\infty} P^{-1}N_kP = 0$, so that $\lim_{k\to\infty} N_k = 0$ and $\lim_{k\to\infty} Z_k = I$. This shows that $\lim_{k\to\infty} A_k = \lim_{k\to\infty} B_k$. ◻

Definition 4.3. *The* Gauss mean $G_t(A,B)$ *of* $A, B \in \mathrm{CP}_n$ *and* $t \in (0,1)$ *is defined as the common limit of mean iterations given in Theorem 4.2.*

Example 4.4. *Here, we give explicit forms of the Gauss mean on* UP_n *for* $2 \leq n \leq 7$ *by using* (2.11).

1. *For* $n = 2$, *one can easily see that*

$$G_t(A,B) = A_2 = B_2 = (1-t)A + tB.$$

2. *For* $n = 3, 4$, *the mean iteration in* (4.22) *terminates at the third step and*

$$G_t(A,B) = A_3 = B_3 = (1-t)A + tB - \frac{t^2(1-t)}{2}(A-B)^2.$$

3. *For* $5 \leq n \leq 7$, *the mean iteration in* (4.22) *terminates at the fourth step and*

$$G_t(A,B) = A_4 = B_4 = (1-t)A + tB - \frac{t^2(1-t)}{2}(A-B)^2 - \frac{t^4(1-t)^3}{2^3}(A-B)^4.$$

The following are properties of Gauss mean on CP_n, which can be derived from Theorem 2.8.

Corollary 4.5. *Let* $A, B, U, V \in \mathrm{CP}_n$ *and* $t \in (0,1)$.

(1) $G_t(A,B) = G_{1-t}(B,A)$.
(2) $G_t(UAV, UBV) = UG_t(A,B)V$.

Remark 4.6. *Note from [13] that the Gauss mean* G_t *is an operator mean of positive constant upper triangular matrices. On the other hand, its corresponding operator monotone function in the sense of Kubo and Ando is unknown.*

4.2. Logarithmic mean. The logarithmic mean of $n \times n$ positive definite Hermitian matrices has two characterizations, which are the common limit of skewed mean iterations and the integral representation with operator monotone function. See [3] for more details. We consider in this section two such constructions of logarithmic mean on CP_n and verify they are the same.

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For $n \times n$ positive definite Hermitian matrices A and B, the logarithmic mean $L(A, B)$ [3, 13] is defined by the common limit of skewed iteration of the arithmetic and geometric means: for given $t \in (0, 1)$

$$L_t(A, B) = \lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \mathcal{B}_k, \tag{4.29}$$

where $\mathcal{A}_1 = (1 - t)A + tB,\ \mathcal{B}_1 = A\#_t B$ and

$$\mathcal{A}_{k+1} = (1 - t)\mathcal{A}_k + t\mathcal{B}_k, \quad \mathcal{B}_{k+1} = \mathcal{A}_{k+1}\#_t\mathcal{B}_k, \tag{4.30}$$

for $k \geq 1$. We simply denote as $L = L_{1/2}$. We show that the skewed mean iterations (4.30) of unit upper triangular matrices A and B converge to the common limit.

For index sets $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, we denote as $A[\alpha, \beta]$ the submatrix of entries that lie in the rows of A indexed by α and the columns indexed by β. The submatrix $A[\alpha] = A[\alpha, \alpha]$ for $\alpha = \beta$ is called a *principal submatrix* of A. Note that an $n \times n$ matrix has $\binom{n}{k}$ distinct principal submatrices of size k, where $1 \leq k \leq n$. Especially, $A[1, \ldots, k]$ for $\alpha = \{1, \ldots, k\}$ and $A[k, \ldots, n]$ for $\alpha = \{k, \ldots, n\}$ are respectively called a *leading principal submatrix* and a *trailing principal submatrix* of A.

Theorem 4.7. *Let $A, B \in \mathrm{CP}_n$, and let $\mathcal{A}_k, \mathcal{B}_k$ be skewed mean iterative sequences as in (4.30). Then, $\lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \mathcal{B}_k$ for all $t \in (0, 1)$.*

Proof. By (4.30) and Theorem 2.8 (1), we have

$$\mathcal{A}_{k+1}\mathcal{B}_k^{-1} = (1 - t)\mathcal{A}_k\mathcal{B}_k^{-1} + tI, \tag{4.31}$$

$$\mathcal{B}_{k+1}\mathcal{B}_k^{-1} = (\mathcal{A}_{k+1}\mathcal{B}_k^{-1})^{1-t} = [(1 - t)\mathcal{A}_k\mathcal{B}_k^{-1} + tI]^{1-t}. \tag{4.32}$$

Let $\mathcal{Z}_k = \mathcal{A}_k\mathcal{B}_k^{-1}$. By (4.31) and (4.32),

$$\mathcal{Z}_{k+1} = \mathcal{A}_{k+1}\mathcal{B}_{k+1}^{-1} = [(1 - t)\mathcal{A}_k\mathcal{B}_k^{-1} + tI]^t = [(1 - t)\mathcal{Z}_k + tI]^t. \tag{4.33}$$

Define a function

$$g_t(x) = [(1 - t)x + t]^t.$$

We get $\mathcal{Z}_{k+1} = g_t(\mathcal{Z}_k)$. It suffices to prove that $\lim_{k\to\infty} \mathcal{Z}_k = I$, or equivalently, $\lim_{k\to\infty} P\mathcal{Z}_kP^{-1} = I$ for certain nonsingular matrix P.

Let $\mathcal{Z}_k = y_kI + L_k$ for $L_k \in \mathcal{N}^{(1)}$. The diagonal entries $y_k \in \mathbb{R}^+$ satisfy that $y_{k+1} = g_t(y_k)$. It is straightforward to see that for any $t \in (0, 1)$, the sequence $\{y_k \mid k = 1, 2, 3, \ldots\}$ approaches 1 monotonically.

By (4.33),

$$\log(\mathcal{Z}_{k+1}) = t \log((1 - t)\mathcal{Z}_k + tI). \tag{4.34}$$

We also have

$$\begin{aligned}
L_{k+1} &= \mathcal{Z}_{k+1} - y_{k+1}I = [(1 - t)(y_kI + L_k) + tI]^t - y_{k+1}I \\
&= \{[(1 - t)y_k + t]I + (1 - t)L_k\}^t - y_{k+1}I \\
&= [(1 - t)y_k + t]^t \left\{I + \frac{1 - t}{(1 - t)y_k + t}L_k\right\}^t - y_{k+1}I \\
&= [(1 - t)y_k + t]^t \left\{\frac{t(1 - t)}{(1 - t)y_k + t}I + \sum_{i=2}^{n-1} \binom{t}{i} \left[\frac{1 - t}{(1 - t)y_k + t}\right]^i L_k^{i-1}\right\} L_k.
\end{aligned}$$

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Choose $\delta > 0$ such that $(1+\delta)(1+2\delta)t(1-t) < 1/2$. There exists a natural number m such that for every $k > m$, we have $|[(1-t)y_k + t]^t - 1| < \delta$ and $|\frac{1}{(1-t)y_k+t} - 1| < \delta$. Fix such a $k > m$. Choose a scalar $c \neq 0$ such that $\sum_{i=2}^{n-1} \left|\binom{t}{i}\right| \left[\frac{1-t}{(1-t)y_k+t}\right]^i \|cL_k\|_2^{i-1} < \delta t(1-t)$. By Lemma 2.1 (3), there is a nonsingular matrix P such that $P^{-1}L_kP = cL_k$. Then,

$$P^{-1}L_{k+1}P = [(1-t)y_k + t]^t \left\{ \frac{t(1-t)}{(1-t)y_k+t} I + \sum_{i=2}^{n-1} \binom{t}{i} \left[\frac{1-t}{(1-t)y_k+t}\right]^i (cL_k)^{i-1} \right\} (P^{-1}L_kP).$$

Hence,

$$\|P^{-1}L_{k+1}P\|_2 \leq (1+\delta)\{(1+\delta)t(1-t) + \delta t(1-t)\}\|P^{-1}L_kP)\|_2 < \frac{1}{2}\|P^{-1}L_kP\|_2.$$

Repeating the process, we get $\|P^{-1}L_{k+i+1}P\|_2 < \frac{1}{2}\|P^{-1}L_{k+i}P\|$ for all $i = 0, 1, 2, \ldots$. Therefore, $\lim_{k\to\infty} PL_kP^{-1} = 0$, $\lim_{k\to\infty} P\mathcal{Z}_kP^{-1} = I$, and $\lim_{k\to\infty} \mathcal{Z}_k = I$. ∎

The integral representation of the logarithmic mean of $n \times n$ positive definite Hermitian matrices A and B is given by

$$\hat{L}(A, B) = \int_0^1 A\#_t B \, dt. \tag{4.35}$$

One can naturally define the logarithmic mean of $A, B \in \mathrm{CP}_n$ as above in (4.35), since $t \in [0, 1] \mapsto A\#_t B$ is a continuous (indeed, polynomial) map by (2.11).

In the following, *Gregory coefficients* g_k are the rational numbers that occur in the Maclaurin series expansion of the reciprocal logarithm

$$\frac{z}{\ln(1+z)} = 1 + \sum_{k=1}^{\infty} g_k z^k,$$

for all $z \in \mathbb{C}$ with $|z| < 1$. Note from [1] that Gregory coefficients have several different formulas

$$\begin{aligned} g_k &= \frac{1}{k!}\left[\frac{d^k}{dz^k}\frac{z}{\ln(1+z)}\right]_{z=0} \\ &= \int_0^1 \binom{x}{k} dx \\ &= (-1)^{k-1}\int_0^{\infty} \frac{dx}{(1+x)^k(\ln^2 x + \pi^2)}, \end{aligned}$$

and their numerical values are

$$g_1 = \frac{1}{2},\ g_2 = -\frac{1}{12},\ g_3 = \frac{1}{24},\ g_4 = -\frac{19}{720}, \cdots$$

These are also known as reciprocal logarithm numbers, Bernoulli numbers of the second kind, or Cauchy number of the first kind.

THEOREM 4.8. *For $A, B \in \mathrm{UP}_n$*

$$\hat{L}(A, B) = A + \sum_{k=1}^{n-1} g_k A(A^{-1}B - I)^k.$$

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Proof. Taking integral on both sides of (2.11) yields

$$\begin{aligned}\int_0^1 A\#_t B\,dt &= \int_0^1 \sum_{k=0}^{n-1} \binom{t}{k} A(A^{-1}B-I)^k\,dt \\ &= \sum_{k=0}^{n-1}\left[\int_0^1 \binom{t}{k}\,dt\right] A(A^{-1}B-I)^k = A + \sum_{k=1}^{n-1} g_k A(A^{-1}B-I)^k. \qquad \square\end{aligned}$$

EXAMPLE 4.9. *Using Theorem 4.8, we give a list of logarithmic means $\hat{L}(A,B)$ of $A,B \in \mathrm{UP}_n$ for $n = 2,3,4$.*

1. *For $n=2$,*
$$\hat{L}(A,B) = A + \frac{1}{2}A(A^{-1}B-I) = \frac{A+B}{2}.$$
2. *For $n=3$,*
$$\hat{L}(A,B) = A + \frac{1}{2}A(A^{-1}B-I) - \frac{1}{12}A(A^{-1}B-I)^2 = \frac{A+B}{2} - \frac{1}{3}\left[\frac{A-B}{2}\right]^2,$$
since $A(A^{-1}B-I)^2 = (A-B)^2$.
3. *For $n=4$,*
$$\begin{aligned}\hat{L}(A,B) &= A + \frac{1}{2}A(A^{-1}B-I) - \frac{1}{12}A(A^{-1}B-I)^2 + \frac{1}{24}A(A^{-1}B-I)^3 \\ &= \frac{A+B}{2} - \frac{1}{3}\left[\frac{A-B}{2}\right]^2 + \left[\frac{A-B}{2}\right]\left[\frac{A-I}{3}\right]\left[\frac{A-B}{2}\right] - \frac{1}{3}\left[\frac{A-B}{2}\right]^3 \\ &= \frac{A+B}{2} - \frac{2}{3}\left[\frac{A-B}{2}\right]^2 + \frac{1}{3}\left[\frac{A-B}{2}\right]\left[\frac{A+B}{2}\right]\left[\frac{A-B}{2}\right],\end{aligned}$$
since $A(A^{-1}B-I)^2 = (A-B)^2 - (A-B)(A-I)(A-B)$ and $A(A^{-1}B-I)^3 = -(A-B)^3$.

Note from [3, Theorem 4.3 (L1)] that the integral representation (4.35) and the common limit of skewed mean iteration given in (4.30) with $t = 1/2$ are the same on $\mathbb{P}$. We have the same result on UP_n.

THEOREM 4.10. *Let $A,B \in \mathrm{UP}_n$. Then, $\hat{L}(A,B) = L_{1/2}(A,B)$.*

Proof. One can see from Definition 2.3 that $\mathcal{A}_k$ and $\mathcal{B}_k$ are operator means on UP_n. By Theorem 2.4, there exist operator monotone functions f_k and g_k such that

$$\mathcal{A}_k = A^{1/2} f_k(A^{-1/2}BA^{-1/2})A^{1/2} \quad \text{and} \quad \mathcal{B}_k = A^{1/2} g_k(A^{-1/2}BA^{-1/2})A^{1/2},$$

for each $k \in \mathbb{N}$. It has been shown in [2] that both $f_k(x)$ and $g_k(x)$ converge to $f(x) = g(x) = \dfrac{x-1}{\log x} = \displaystyle\int_0^1 x^t\,dt$. Therefore,

$$\begin{aligned}L_{1/2}(A,B) &= A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2} \\ &= A^{1/2}\left[\int_0^1 (A^{-1/2}BA^{-1/2})^t\,dt\right]A^{1/2} = \int_0^1 A\#_t B\,dt = \hat{L}(A,B). \qquad \square\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 42, pp. 294-313, March 2026.

Remark 4.11. *One can see from several examples in Example 2.6, Example 4.4, and Example 4.9 that the geometric mean, Gauss mean, and Logarithmic mean for $t = 1/2$ can be written as the polynomials of $M := \dfrac{A+B}{2}$ and $N := \dfrac{A-B}{2}$. It arises two natural questions:*
(1) whether such observation holds in general, and
(2) whether the spectral geometric mean can be written as the polynomial of M and N (see Example 3.6).

5. Resolvent mean and $A\#H$ mean. Let $\mathbb{A} = (A_1, \dots, A_m) \in \mathrm{CP}_n^m$ and $\omega = (w_1, \dots, w_m)$ be a positive probability vector in $\mathbb{R}^m$. Given $\mu > 0$, the *resolvent mean* $\mathcal{R}_\mu$ and *$A\#H$ mean* $\mathcal{L}_\mu$ are defined, respectively, by

$$\mathcal{R}_\mu(\omega; \mathbb{A}) = \left[\sum_{j=1}^m w_j (A_j + \mu I)^{-1}\right]^{-1} - \mu I,$$
$$\mathcal{L}_\mu(\omega; \mathbb{A}) = \left[\sum_{j=1}^m w_j (A_j + \mu I)\right] \# \left[\sum_{j=1}^m w_j (A_j + \mu I)^{-1}\right]^{-1} - \mu I.$$

Some properties of the resolvent mean and $A\#H$ mean on $\mathbb{P}$ including the mean iteration and minimization problem of Kullback–Leibler divergence have been shown in [8].

We list fundamental properties of the resolvent mean and $A\#H$ mean on CP_n. In the following, we denote

$$\mathbb{A}_\sigma := (A_{\sigma(1)}, \dots, A_{\sigma(m)}),$$
$$U\mathbb{A}V := (UA_1V, \dots, UA_mV)$$

for any permutation σ on $\{1, \dots, m\}$ and $U, V \in \mathrm{CP}_n$.

Theorem 5.1. *Let $\mathbb{A} = (A_1, \dots, A_m) \in \mathrm{CP}_n^m$ and let $\omega = (w_1, \dots, w_m)$ be a positive probability vector. Then, the resolvent mean satisfies the following:*

(1) $\mathcal{R}_\mu(\omega; \alpha\mathbb{A}) = \alpha\mathcal{R}_{\frac{\mu}{\alpha}}(\omega; \mathbb{A})$ for any $\alpha > 0$;
(2) $\mathcal{R}_\mu(\omega_\sigma; \mathbb{A}_\sigma) = \mathcal{R}_\mu(\omega; \mathbb{A})$ for any permutation σ on $\{1, \dots, m\}$;
(3) $\mathcal{R}_\mu(\omega; U\mathbb{A}U^{-1}) = U\mathcal{R}_\mu(\omega; \mathbb{A})U^{-1}$ for any $U \in \mathrm{CP}_n$;
(4) $\mathcal{R}_\mu(\omega; \mathbb{A}^{-1})^{-1} = \mathcal{R}_{\mu^{-1}}(\omega; \mathbb{A})$.

The $A\#H$ mean $\mathcal{L}_\mu$ also satisfies (1)–(3), and moreover, $\mathcal{L}_\mu(\omega; \mathbb{A}) + \mu I$ is the unique solution $X \in \mathrm{CP}_n$ to the equation

$$\sum_{j=1}^m w_j X (A_j + \mu I)^{-1} X = \sum_{j=1}^m w_j A_j + \mu I. \tag{5.36}$$

Proof. Properties (1)–(3) are straightforward. For (4) note from [16, Section 2.2] that

$$(A + B)^{-1} = A^{-1} - A^{-1}(A^{-1} + B^{-1})^{-1}A^{-1},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 42, pp. 294-313, March 2026.

when the involved inverses exist. Then

$$\begin{aligned}\mathcal{R}_\mu(\omega;\mathbb{A}^{-1})^{-1} &= -\frac{1}{\mu}I - \frac{1}{\mu^2}\left[\sum_{j=1}^m w_j(A_j^{-1}+\mu I)^{-1} - \frac{1}{\mu}I\right]^{-1}\\ &= -\frac{1}{\mu}I - \frac{1}{\mu^2}\left[\sum_{j=1}^m w_j\left\{\frac{1}{\mu}I - \frac{1}{\mu^2}\left(A_j+\frac{1}{\mu}I\right)^{-1}\right\} - \frac{1}{\mu}I\right]^{-1}\\ &= -\frac{1}{\mu}I + \left[\sum_{j=1}^m w_j\left(A_j+\frac{1}{\mu}I\right)^{-1}\right]^{-1} = \mathcal{R}_{\mu^{-1}}(\omega;\mathbb{A}).\end{aligned}$$

Furthermore, the equation (5.36) is equivalent that

$$X\left[\sum_{j=1}^m w_j(A_j+\mu I)^{-1}\right]X = \sum_{j=1}^m w_j(A_j+\mu I).$$

Thus,

$$X = \left[\sum_{j=1}^m w_j(A_j+\mu I)\right]\#\left[\sum_{j=1}^m w_j(A_j+\mu I)^{-1}\right]^{-1} = \mathcal{L}_\mu(\omega;\mathbb{A})+\mu I.$$

□

When A and B are $n\times n$ lower triangular matrices with positive diagonal entries, the iteration process of arithmetic and harmonic means is

$$X_0 = A,\quad Y_0 = B,\quad X_{k+1} = \frac{X_k+Y_k}{2},\quad Y_{k+1} = \left(\frac{X_k^{-1}+Y_k^{-1}}{2}\right)^{-1}.$$

It has been proved in [5] that

$$\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = A\#B.$$

We extend this result to the convergence of the iteration process of t-weighed arithmetic and harmonic means.

THEOREM 5.2. *Let $A, B \in \mathrm{CP}_n$. Given $t\in[0,1]$, the sequences $\{X_k \mid k\geq 0\}$ and $\{Y_k \mid k \geq 0\}$, constructed by*

$$\begin{aligned}&X_0 = A,\quad Y_0 = B,\\ &X_{k+1} = (1-t)X_k + tY_k,\quad Y_{k+1} = \left((1-t)X_k^{-1}+tY_k^{-1}\right)^{-1},\end{aligned}\tag{5.37}$$

converge to the same limit, that is, $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k$.

When $t=1/2$, we have $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = A\#B$. Moreover, when A and B have the same diagonal, the sequences $\{X_k \mid k\geq 0\}$ and $\{Y_k \mid k\geq 0\}$ converge in no more than $\lceil \log_2 n\rceil$ steps.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 42, pp. 294-313, March 2026.

Proof. Let $Z_k = X_k Y_k^{-1}$. (5.37) implies that

$$X_{k+1}Y_k^{-1} = (1-t)Z_k + tI, \tag{5.38}$$

$$Y_{k+1}Y_k^{-1} = [(1-t)Z_k^{-1} + tI]^{-1}, \tag{5.39}$$

$$\begin{aligned} Z_{k+1} &= [(1-t)Z_k + tI][(1-t)Z_k^{-1} + tI] \\ &= [(1-t)^2 + t^2]I + t(1-t)(Z_k + Z_k^{-1}), \end{aligned} \tag{5.40}$$

$$Z_{k+1} - I = t(1-t)(Z_k + Z_k^{-1} - 2I) = t(1-t)Z_k^{-1}(Z_k - I)^2. \tag{5.41}$$

Write $Z_k = z_k I + N_k$ for $z_k > 0$ and $N_k \in \mathcal{N}^{(1)}$. (5.41) implies that

$$z_{k+1} - 1 = t(1-t)(z_k + z_k^{-1} - 2), \tag{5.42}$$

$$= t(1-t)z_k^{-1}(z_k - 1)^2. \tag{5.43}$$

(5.43) implies that $z_{k+1} - 1 \geq 0$ for $k = 0, 1, 2, \ldots$. So for $k \geq 1$, we have $z_k \geq 1$ and by (5.42),

$$0 \leq z_{k+1} - 1 \leq t(1-t)(2z_k - 2) \leq \frac{1}{2}(z_k - 1).$$

Hence, $\lim_{k\to\infty} z_k = 1$.

Let $Z_k^{-1} = z_k^{-1} I + N_k'$ for $N_k' \in \mathcal{N}^{(1)}$. There is m such that for every $k > m$, we have $|1 - z_k^{-1}| < 1/2$. Fix such a k. There is a scalar $c \neq 0$ such that $\|cN_k'\|_2 < \frac{1}{2}$. By Lemma 2.1 (3), there is a nonsingular matrix P such that $P^{-1}N_k'P = cN_k'$. Then, (5.41) implies that

$$\begin{aligned} P^{-1}Z_{k+1}P - I &= t(1-t)(I - P^{-1}Z_k^{-1}P)(P^{-1}Z_kP - I) \\ &= t(1-t)[(1 - z_k^{-1})I - cN_k'](P^{-1}Z_kP - I). \end{aligned}$$

Therefore,

$$\|P^{-1}Z_{k+1}P - I\|_2 \leq t(1-t)[|1 - z_k^{-1}| + \|cN_k'\|_2]\|P^{-1}Z_kP - I\|_2 \leq \frac{1}{4}\|P^{-1}Z_kP - I\|_2.$$

Replace k by sequential numbers and repeat the argument, we can get that $\|P^{-1}Z_{k+i+1}P - I\|_2 \leq \frac{1}{4}\|P^{-1}Z_{k+i}P - I\|_2$ for all $i = 0, 1, 2, \ldots$. Therefore, $\lim_{k\to\infty} P^{-1}Z_{k+i}P = I$ so that $\lim_{k\to\infty} Z_{k+i} = I$. We obtain $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k$.

The limit for $t = 1/2$ has been proved in [5]. Now, suppose that A and B have the same diagonal. Then, each $Z_k \in \mathrm{UP}_n$. (5.41) becomes

$$N_{k+1} = t(1-t)Z_k^{-1}N_k^2. \tag{5.44}$$

We get $N_0 \in \mathcal{N}^{(1)}$, and if $N_k \in \mathcal{N}^{(p)}$, then (5.44) and (4.24) imply that $N_{k+1} \in \mathcal{N}^{(2p)}$. Therefore, when $k \geq \lceil \log_2 n \rceil$, we have $N_k = 0$, which leads to $X_k = Y_k = X_{k+1} = Y_{k+1} = \cdots$. ∎

Remark 5.3. *When* $t \in \{0, 1/2, 1\}$, *the process* (5.37) *produces sequences* $\{X_k \mid k \in \mathbb{N}\}$ *and* $\{Y_k \mid k \in \mathbb{N}\}$ *that converge to* $A\#_tB$. *However, this is not the case for other* $t \in (0, 1)$.

Theorem 5.4. *Let* $\mathbb{A} = (A_1, \ldots, A_m) \in \mathrm{CP}_n^m$ *and* $\omega = (w_1, \ldots, w_m)$ *be a positive probability vector in* $\mathbb{R}^m$. *Given* $t \in [0, 1]$, *the sequences* $\{X_k \mid k \geq 0\}$ *and* $\{Y_k \mid k \geq 0\}$ *given by*

$$X_0 = \sum_{j=1}^m w_j A_j, \quad Y_0 = \mathcal{R}_\mu(\omega; \mathbb{A}),$$

$$X_{k+1} = (1-t)X_k + tY_k, \quad Y_{k+1} = \mathcal{R}_\mu(1-t, t; X_k, Y_k),$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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both converge to the same limit $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k$.

When $t = 1/2$*, we have*

$$\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = \mathcal{L}_\mu(\omega; \mathbb{A}).$$

Moreover, if $(A_1, \ldots, A_m) \in \mathrm{UP}_n^m$*, then the sequences* $\{X_k \mid k \geq 0\}$ *and* $\{Y_k \mid k \geq 0\}$ *both converge in no more than* $\lceil \log_2 n \rceil$ *steps.*

Proof. Let $\tilde{X}_k = X_k + \mu I$ and $\tilde{Y}_k = Y_k + \mu I$ for all $k \geq 0$. The iteration of $\{(X_k, Y_k) \mid k \geq 0\}$ can be rewritten in terms of $\{(\tilde{X}_k, \tilde{Y}_k) \mid k \geq 0\}$ as

$$\tilde{X}_0 = \sum_{j=1}^{n} w_j(A_j + \mu I), \quad \tilde{Y}_0 = \left[\sum_{j=1}^{m} w_j(A_j + \mu I)^{-1}\right]^{-1},$$
$$\tilde{X}_{k+1} = (1-t)\tilde{X}_k + t\tilde{Y}_k, \quad \tilde{Y}_{k+1} = \left((1-t)\tilde{X}_k^{-1} + t\tilde{Y}_k^{-1}\right)^{-1} \quad \text{for} \quad k \geq 0.$$

By Theorem 5.2, the sequences $\{\tilde{X}_k \mid k \in \mathbb{N}\}$ and $\{\tilde{Y}_k \mid k \in \mathbb{N}\}$ both converge to the same limit, that is, $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k$.

When $t = 1/2$, $\{\tilde{X}_k \mid k \in \mathbb{N}\}$ and $\{\tilde{Y}_k \mid k \in \mathbb{N}\}$ both converge to

$$\tilde{X}_0 \# \tilde{Y}_0 = \left[\sum_{j=1}^{m} w_j(A_j + \mu I)\right] \# \left[\sum_{j=1}^{m} w_j(A_j + \mu I)^{-1}\right]^{-1}.$$

Hence, we get $\lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = \mathcal{L}_\mu(\omega; \mathbb{A})$.

Finally, if $(A_1, \ldots, A_m) \in \mathrm{UP}_n^m$, then $\tilde{X}_0$ and $\tilde{Y}_0$ have the same diagonal and Theorem 5.2 implies that $\{\tilde{X}_k \mid k \geq 0\}$ and $\{\tilde{Y}_k \mid k \geq 0\}$ both converge in no more than $\lceil \log_2 n \rceil$ steps. The proof is completed. □

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