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THE SPANNING K-HYPERTREE AND ADJACENCY SPECTRAL RADIUS IN HYPERGRAPHS*

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Abstract. The spanning k-tree of a connected graph is a spanning tree with a maximum degree not exceeding k. Hypergraphs provide a natural framework for modeling higher-order interactions. It is natural to extend the concept of spanning k-trees from graphs to hypergraphs. Tensor-based spectral models for hypergraphs are inherently higher-order and nonlinear, while Banerjee [Linear Algebra and its Applications, 614 (2021) 82-110] proposed a novel adjacency matrix $A(H)$ for general hypergraphs, whose matrix-based representation preserves typical graph-like spectral features, including the correspondence between row sums and vertex degrees. For the existence of spanning k-trees in graphs, authors typically use quotient matrices as tools to achieve expected results. Due to the complex structure of the entropy matrix in hypergraphs. In this paper, by means of a different approach, we establish a relationship between adjacency spectral condition on $A(H)$ and the existence of spanning k-hypertree for hypergraph H and verify a sharp lower bound on adjacency spectral radius to guarantee the existence of spanning k-hypertrees in uniform hypergraphs.

Key words. Spectral radius, Spanning k-hypertree, Hypergraph.

AMS subject classifications. 05C50, 05C40.

1. Introduction.

In 2005, L. Qi [1] and independently L. Lim [2] introduced the concept of eigenvalues of a real supersymmetric tensor (hypermatrix). Since then, spectral theory for tensors has been developing rapidly. But in spectral hypergraph theory, the computation of eigenvalues using tensors (hypermatrices) is computationally intractable, as it belongs to the class of NP-hard problems [3]. Consequently, researchers have been actively seeking alternative approaches and simplified methods to investigate hypergraphs.

Let hypergraph $H = (V, E)$ with vertex set $V = V(H)$ and edge set $E = E(G)$. A hypergraph is *r-uniform* if each edge is an r-set (a set of r-elements). In 2021, Banerjee [4] introduced a new adjacency matrix for hypergraphs. The adjacency matrix possesses three desirable properties. First, it is well compatible with ordinary graphs: for any 2-uniform hypergraph, this matrix strictly reduces to the standard adjacency matrix of the corresponding simple graph. Second, this matrix is symmetric, which facilitates the spectral characteristic analysis. Third, this matrix characterizes the pairwise interactions of hypergraphs through binary associations. Although it cannot fully capture the complete higher-order structural information of hypergraphs, it still retains the core structural characteristics of hypergraphs.

DEFINITION 1.1. *[4] Let $E_{ij} = \{e \in E : i, j \in e\}$. The adjacency matrix $A(H) = (a_{ij})$ of a hypergraph $H = (V, E)$ is defined as*

$$a_{ij} = \sum_{e \in E_{ij}} \frac{1}{|e| - 1}.$$

For an r-uniform hypergraph, the adjacency matrix becomes $a_{ij} = d_{ij} \cdot \frac{1}{r-1}$, where d_{ij} is the codegree of vertices i and j. The codegree d_{ij} of vertices i and j is the number of edges that contain the vertices i and

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j both, i.e., $d_{ij} = |E_{ij}|$. Now, the degree, d_i, of a vertex $i \in V$, which is the number of edges that contain i can be expressed as $d_i = \sum_{j=1}^{n} a_{ij}$.

The adjacent matrix $A(H)$ (in brief of A) of H is the $n \times n$ real symmetric matrix, whose (i, j)-entry is a_{ij}. The eigenvalues of $A(H)$ are the eigenvalues of H. Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues in nonincreasing order. We call $\lambda(A) = \lambda_1(A)$ (in brief of λ) the spectral radius in H.

As is well known, hypergraph structures are considerably more complex than graph structures, yet they demonstrate unique advantages and broad prospects in practical applications. In [4], A. Banerjee explored various structural properties of general hypergraphs using the adjacency spectral radius for hypergraphs. Recently, Yu et al. [5] obtained some results on how the spectral radius changes under certain graphic structural perturbations, and they also determined the supertrees with the maximum, the second maximum, and the minimum spectral radius, respectively, among all r-uniform ($r \geq 3$) supertrees with a fixed number of vertices.

As an extension of graph acyclicity to hypergraphs, *beta-acyclicity* (β-acyclicity) naturally extends the acyclic property from graphs to hypergraph structures. A *beta cycle* (or β-cycle) in a hypergraph H is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_k, e_k)$ with $k \geq 3$ satisfying the following conditions:

1. the vertices $v_1, \ldots, v_k$ are all distinct;
2. the hyperedges $e_1, \ldots, e_k$ are all distinct;
3. for each i, the hyperedge e_i contains exactly the pair $\{v_i, v_{i+1}\}$ among the vertices of the cycle, where $v_{k+1} = v_1$; and
4. no hyperedge $e \in E \setminus \{e_1, \ldots, e_k\}$ contains two non-consecutive vertices of the cycle.

A hypergraph that contains no beta cycle is called *beta-acyclic*. A connected, beta-acyclic hypergraph is referred to as a *hypertree*.

A spanning k-tree of a connected graph is a spanning tree with maximum degree at most k [6]. Research on the spanning k-tree is great significance in various fields of graph theory, including network design, combinatorial optimization, and the study of graph connectivity and robustness, and so on. Therefore, it is natural to extend the concept of spanning k-trees from graphs to hypergraphs [7]. In graphs, a spanning k-tree is an acyclic subgraph that connects all vertices, with the number of branches at each vertex bounded above by k [8]. As a generalization of Hamiltonian paths, it characterizes whether a graph can maintain global connectivity while distributing the branching load across its vertices and preventing excessive branching at a single vertex. Inspired by this, we introduce the definition of a spanning k-hypertree to capture an analogous property for hypergraphs: whether a hypergraph can preserve global connectivity and spread the branching load of a hypertree over all vertices, so as to avoid over-branching at any individual vertex.

Definition 1.2. *[7] For an integer $k \geq 2$, a spanning k-hypertree T is a hypergraph where T is a spanning hypertree (beta-acyclic and connected) and the maximum degree $d_T(v)$ for each vertex $v \in V(T)$ does not exceed k.*

Example 1.3. *As illustrated in Figure 1, the hypergraph H is connected since there exists a path between every pair of its vertices. Consider the vertex-hyperedge sequence $(v_1, e_1, v_2, e_2, v_3, e_3)$. If e_4 is absent, then this sequence forms a β-cycle $(v_1, e_1, v_2, e_2, v_3, e_3)$ in H. If e_4 is included, H becomes β-acyclic because the fourth condition in the definition of a β-cycle fails to hold. In this case, H is a spanning 2-hypertree, as the degree of each vertex is at most 2.*

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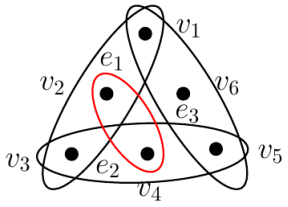


FIGURE 1. *Hypergraph H with edges $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_3, v_4, v_5\}$, $e_3 = \{v_5, v_6, v_1\}$, where $v_1 = v_7$.*

In recent years, spectral-based conditions have also been investigated for the existence of spanning k-trees. For example, Fan et al. [9] provided some sufficient conditions, in terms of (adjacency and signless Laplacian) spectral radius, for the existence of a k-tree in a connected graph. Wu et al. [10] obtained a distance spectral radius condition in a connected graph G to guarantee the existence of a spanning k-tree in G. Liu et al. [11] proved an upper bound for distance signless Laplacian spectral radius in a connected graph to guarantee the existence of a spanning k-tree. Earlier works established various structural sufficient conditions for the existence of spanning k-trees in graphs. Motivated by these works, we turn to spectral sufficient conditions for hypergraphs in the present paper.

In [9, 10, 11], authors typically use quotient matrices as tools to achieve expected results. Due to the complex structure of the entropy matrix in hypergraphs, we note that A. Banerjee provided a degree condition based on the adjacency spectral radius in [4]. Inspired by this, we attempt to establish a relationship between adjacency spectral condition and the existence of spanning k-hypertree for hypergraphs.

This paper is organized as follows. In Section 2, we introduce some necessary notations and definitions for hypergraphs. In Section 3, we give some results on how the spectral radius $\lambda(H)$ changes under certain hypergraph structural perturbations. We also verify a sharp lower bound for $\lambda(H)$ to guarantee the existence of spanning k-hypertrees in r-uniform hypergraphs.

2. Preliminary. Below, we introduce some necessary notations and definitions for hypergraphs.

A hypergraph H denoted by $H = (V, E = (e_i)_{i\in I})$ on a finite set V is a family $(e_i)_{i\in I}$ (I is a finite set of indexes) of subsets of V called hyperedges (or edges). Sometimes V is denoted by $V(H)$, and E is denoted by $E(H)$, respectively. The *order* of the hypergraph $H = (V, E)$ is the cardinality of V, i.e., $|V| = n$; its *size* is cardinality of E, i.e., $|E| = m$. Given two hypergraphs $H = (V, E)$ and $H' = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$, then H' is said to be a subhypergraph of H. If $V' = V$ and $E' \subseteq E$, then H' is said to be a spanning subhypergraph of H. A vertex $v \in V$ is said to be *incident* with an edge $e \in E$, if e is a subset of V containing v. Two distinct vertices u and v of H that are *adjacent* if some edges of H contain both, denoted by $u \sim v$.

For a vertex $v \in V$, we denote by E_v the set of edges containing v in H. Similarly, E_{uv} denotes the set of edges containing both u and v. The cardinality $|E_v|$ is called the degree of v, denoted by $d_H(v)$. The minimum and maximum degrees of H are defined as $\delta(H) = \min\{d_H(v) \mid v \in V\}$ and $\Delta(H) = \max\{d_H(v) \mid v \in V\}$, respectively. Let $E = \{E_1, E_2, \cdots, E_m\}$, the hypergraph H is *simple*, if $E_i \subseteq E_j$ implies that $i = j$ for any i, j with $1 \leq i, j \leq m$. In the present paper, we only consider simple hypergraphs.

A *walk* W in a hypergraph H is a finite alternating sequence $W = (v_0, e_1, v_1, e_2, ..., e_k, v_k)$, where v_i is a vertex for $i = 0, 1, 2, ..., k$. A walk W is a *path* if all the vertices v_i for $i = 0, 1, ..., k$, and all the edges e_j for

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$j = 1, 2, ..., k$ in W are distinct. A hypergraph is said to be *connected* if there exists a path between every pair of vertices of the hypergraph and disconnected otherwise. A *connected component* of a hypergraph is a maximal subset of the vertex set for which there exists a path between any two vertices of this maximal subset in the hypergraph. We let $c(H - S)$ be *the number of connected components* of $V(H) - V(S)$. (Unexplained terminology is referred to in [12, 13].)

In addition, we provide the following definition, which facilitates the statement of the main result.

DEFINITION 2.1. *[12] A hypergraph H of order n is* complete *if $H = (V, E = P(V) \setminus \{\emptyset\})$ (where $P(X)$ denote the powerset of X), denoted by K_n. Let $H = (V, E)$ be an r-uniform hypergraph of order n, if E consists of all r-subsets of V, then H is an* r-uniform complete hypergraph*, denoted by K_n^r. It is important to note that when $n < r$, K_n^r refers to a hypergraph containing n isolated vertices (as shown in Figure 2).*

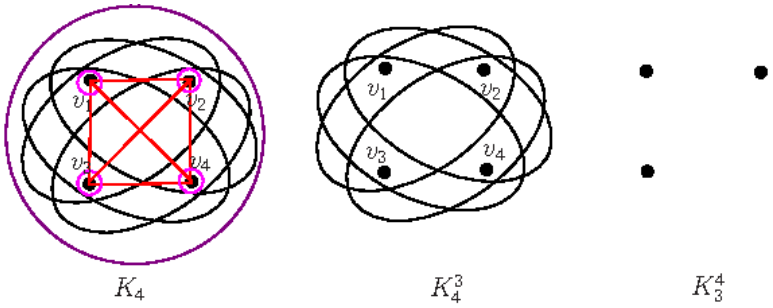


FIGURE 2. *Examples of hypergraph*

For a vertex $v \in V(H)$, $H - v$ is obtained by deleting the vertex v in H, i.e., removing v from its incident edges in H, but do not delete the incident edges, until there is no vertex in it. (For example, for $v \in e_v$ and $|e| = r$, after v is deleted we obtain an edge e' and $|e'| = r - 1$.) $H - e$ is obtained by deleting edge e in H, i.e., removing edge e and its vertices from H. For a vertex subset $S \subseteq V$, let $H - S$ be a hypergraph, where the vertices and edges in S are deleted as above. For two r-uniform hypergraphs H_1 and H_2, we denote the *disjoint union* of H_1 and H_2 by $H_1 \cup H_2$, which is the hypergraph with vertex set $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and edge set $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$. The *join* $H_1 \vee H_2$ of two r-uniform hypergraphs H_1 and H_2 is an r-uniform hypergraph on the vertex set $V(H_1) \cup V(H_2)$. The edge set of $H_1 \vee H_2$ consists of all original edges of H_1 and H_2, together with all r-element subsets of $V(H_1) \cup V(H_2)$ that intersect both $V(H_1)$ and $V(H_2)$. In other words, we keep all edges from H_1 and H_2 and add every possible cross edge connecting the two vertex subsets. Formally, $E(H_1 \vee H_2) = E(H_1) \cup E(H_2) \cup \big\{e \subseteq V(H_1) \cup V(H_2) \,\big|\, |e| = r,\ e \cap V(H_1) \neq \emptyset,\ e \cap V(H_2) \neq \emptyset\big\}$.

3. Main results. In this section, for the r-uniform hypergraph H, we give some results on the bounds of $\lambda(H)$. We also verify a sharp lower bound for $\lambda(H)$ to guarantee the existence of spanning k-hypertree in H.

We adopt the definitions of beta-acyclicity and spanning k-hypertree given in Definition 1.2 throughout the following discussions.

LEMMA 3.1. *[14] Let G be a weighted graph that is simple, connected, in which the edge weights are positive numbers (i.e.,1×1 matrices). Then,*

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$$\lambda(G) \leq \max_{i\sim j}\{\sqrt{w_i w_j}\},$$

where w_i is the sum of the weights of the edges that are incident to vertex i. Moreover, equality holds if and only if G is a regular graph or G is a bipartite semiregular graph.

Since the hypergraph H is corresponding weighted graph $G[A]$ (constructed from the adjacency matrix A of H)[4], so we can also estimate the upper bound for spectral radius $\lambda(H)$ of H as the following Corollary 3.2.

COROLLARY 3.2. *[4] Let H be a connected hypergraph, in which the edge weights are positive numbers. Then,*

$$\lambda(H) \leq \max_{i\sim j}\{\sqrt{d_i d_j}\},$$

where d_i is the degree of vertex i. Moreover, this equality holds if and only if H is a regular hypergraph

For any $S \subseteq V(H)$ of a connected hypergraph, $c(H-S)$ is the number of components in $H-S$.

LEMMA 3.3. *[7] Suppose that $k \geq 2$ and H is a connected hypergraph such that*

$$c(H-S) \leq (k-2)|S| + 2,$$

for all $S \subseteq V(H)$. Then, H has a spanning k-hypertree.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. Next, we introduce the partial order relation between two matrices, define $A \preceq B$ if $a_{ij} \leq b_{ij}$ for all i and j, and define $A \prec B$ if $A \preceq B$ and $A \neq B$.

LEMMA 3.4. *[15] Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices with the spectral radius $\lambda(A)$ and $\lambda(B)$, respectively. If $0 \preceq A \preceq B$, then $\lambda(A) \leq \lambda(B)$. Furthermore, if B is irreducible and $0 \preceq A \prec B$, then $\lambda(A) < \lambda(B)$.*

LEMMA 3.5. *For the r-uniform complete hypergraph K_n^r with n vertices, the spectral radius (i.e., the largest eigenvalue) of its adjacency matrix is*

$$\lambda(K_n^r) = \binom{n-1}{r-1}.$$

Proof. For the complete r-uniform hypergraph K_n^r, the adjacency matrix $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} \frac{d_{ij}}{r-1} & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where d_{ij} is the codegree of vertices i and j. Since K_n^r contains all possible r-element hyperedges, for any $i \neq j$,

$$d_{ij} = \binom{n-2}{r-2}.$$

Thus, the off-diagonal entries become

$$a_{ij} = \frac{\binom{n-2}{r-2}}{r-1} = \frac{(n-2)!}{(r-1)!(n-r)!} = \frac{\binom{n-1}{r-1}}{n-1}.$$

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The adjacency matrix can be expressed as,

$$A(K_n^r) = \frac{\binom{n-1}{r-1}}{n-1}(\mathbf{J} - \mathbf{I}),$$

where $\mathbf{J}$ is the all-ones matrix and $\mathbf{I}$ is the identity matrix.

The matrix $\mathbf{J} - \mathbf{I}$ has eigenvalues,

- $n-1$ (simple)
- -1 (with multiplicity $n-1$)

Consequently, A has eigenvalues

$$\lambda_1 = \frac{\binom{n-1}{r-1}}{n-1} \cdot (n-1) = \binom{n-1}{r-1},$$

and

$$\lambda_k = -\frac{\binom{n-1}{r-1}}{n-1} \quad \text{for } k = 2, \ldots, n.$$

The spectral radius is the maximum absolute eigenvalue

$$\lambda(K_n^r) = \lambda_1 = \binom{n-1}{r-1}.$$

This completes the proof. □

A hypergraph H is *isomorphic* to a hypergraph H^* if there is a bijection $\sigma : V(H) \to V(H^*)$ such that $\{v_1, v_2, \ldots, v_r\} \in E(H)$ if and only if $\{\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_r)\} \in E(H^*)$. The bijection σ is called an *isomorphism* between H and H^*. If $H^* = H$, then σ is called an *automorphism* of H. Let $\mathbf{x}$ be a vector defined on $V(H)$ and σ be an automorphism of H. Two vertices u and v are *equivalent* in H, written $u \sim v$, if there exists an automorphism σ of H such that $\sigma(u) = v$. Denote by $\mathbf{x}_\sigma$ the vector satisfying $\mathbf{x}_\sigma(u) = \mathbf{x}(\sigma(u))$ for each $u \in V(H)$.

To characterize the extremal structure of hypergraphs in our Theorem 3.7, it is necessary to compare the adjacency spectral properties of specially constructed hypergraphs with identical vertex number. The parameters $n_1, n_2, \ldots, n_t$ are introduced to describe the flexible partition of vertices in the disjoint union of cliques, which enables us to characterize the general structural form of potential extremal hypergraphs. Based on the hypergraph join and union operations, we further construct two classes of candidate extremal hypergraphs for spectral comparison. Next, we provide two hypergraphs $H = K_s^r \vee (K_{n_1}^r \cup K_{n_2}^r \cup \cdots \cup K_{n_t}^r)$ and $H' = K_s^r \vee \left(K_{n-s-t+1}^r \cup (t-1)K_1\right)$ (see Figure 3). By comparing their adjacency spectral radius, we can determine the optimal vertex partition and identify the exact extremal hypergraph corresponding to our main theorem.

The following Lemma 3.6, which compares the adjacency spectral radius of two hypergraphs, investigates how the spectral radius changes when vertices are redistributed among the t components.

LEMMA 3.6. *Let $n = \sum_{i=1}^{t} n_i + s$ and $s \geq r-1$. If $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$, then*

$$\lambda(K_s^r \vee (K_{n_1}^r \cup K_{n_2}^r \cup \cdots \cup K_{n_t}^r)) < \lambda(K_s^r \vee (K_{n-s-t+1}^r \cup (t-1)K_1)).$$

Proof. Since we are discussing connected hypergraphs, for a hypergraph $H' = K_s^r \vee (K_{n-s-t+1}^r \cup (t-1)K_1))$ to be connected, the condition $s \geq r-1$ must hold; otherwise, isolated vertices may arise, resulting

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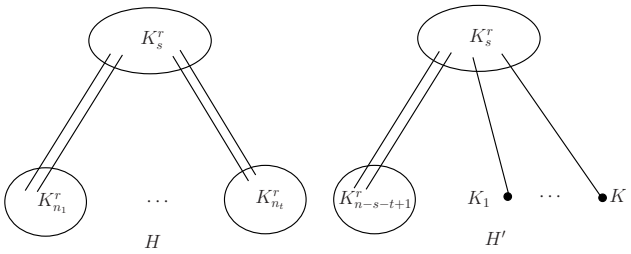


FIGURE 3. $H = K_s^r \vee (K_{n_1}^r \cup K_{n_2}^r \cup \cdots \cup K_{n_t}^r)$ *and* $H' = K_s^r \vee (K_{n-s-t+1}^r) \cup (t-1)K_1)$.

in a disconnected hypergraph. Let $H = K_s^r \vee (K_{n_1}^r \cup K_{n_2}^r \cup \cdots \cup K_{n_t}^r)$, and let X denote the Perron vector of $A(H)$. Let σ be an automorphism of H, then $X_\sigma = X$. Further, for two vertices u and v in $V(H)$, if $u \sim v$, there must be $x_u = x_v$. By the structural symmetry of H, we may assume $x(v) = x_i$ for all $v \in V(K_{n_i}^r)$, where $1 \le i \le t$, and $x(u) = y_1$ for all $u \in V(K_s^r)$. We use Rayleigh quotient

$$\lambda(H) = \max_{X \in R^n, X \neq 0} \frac{X^T A(H) X}{X^T X},$$

then $A(H)X = \lambda(H)X$.

Fact 1. Let $X = (x_1, x_2, \ldots, x_n)^T$ denote the Perron vector of $A(H)$ normalized so that $X^T X = 1$, where $H = K_s^r \vee \left(K_{n_1}^r \cup K_{n_2}^r \cup \cdots \cup K_{n_t}^r\right)$. Then, $x_1 \ge x_j$ for $2 \le j \le t$.

proof of Fact 1. We note that $K_s^r \vee K_{n_1}^r = K_{n_1+s}^r$. For an r-uniform complete hypergraph $K_{n_1+s}^r$, we have $d(v_1) = \binom{n_1+s-1}{r-1}$ for $v_1 \in V(K_{n_1+s}^r)$, and in the complete r-uniform hypergraph, two vertices belong together to $\frac{r-1}{n_1+s-1} \cdot \binom{n_1+s-1}{r-1} = \binom{n_1+s-2}{r-2}$ hyperedges. Applying the Definition 1.1, we obtain

$$a_{v_1 u} = \frac{1}{r-1}\binom{s+n_1-2}{r-2}; \quad v_1, u \in V(K_{n_1+s}^r) \text{ and } \ v_1 \neq u.$$

For a vertex $u \in V(K_{n_1}^r)$, the Perron equation $\lambda x_1 = \sum_{w \in V} a_{uw} x_w$ separates into two contributions: from vertices within $V(K_{n_1}^r)$ and from vertices in $V(K_s^r)$. So we get

$$\lambda x_1 = \frac{1}{r-1} \cdot \binom{n_1+s-2}{r-2} \cdot \left[(n_1-1)x_1 + s y_1\right],$$

and

$$\left(\lambda - \frac{1}{r-1} \cdot \binom{n_1+s-2}{r-2} \cdot (n_1-1)\right) x_1 = \frac{1}{r-1} \cdot \binom{n_1+s-2}{r-2} \cdot s y_1 > 0, \tag{3.1}$$

which imples that

$$\lambda > \frac{1}{r-1} \cdot \binom{n_1+s-2}{r-2} \cdot (n_1-1).$$

Again, for $2 \le j \le t$, we obtain

$$\begin{aligned}\left(\lambda - \frac{1}{r-1} \cdot \binom{n_j+s-2}{r-2} \cdot (n_j-1)\right) x_j &= \frac{1}{r-1} \cdot \binom{n_j+s-2}{r-2} \cdot s y_1 \\ &\le \frac{1}{r-1} \cdot \binom{n_1+s-2}{r-2} \cdot s y_1.\end{aligned}$$

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Combining (3.1), we have

$$\left(\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_1-1)\right)x_1\geq\left(\lambda-\frac{1}{r-1}\cdot\binom{n_j+s-2}{r-2}\cdot(n_j-1)\right)x_j.$$

The left-hand side of the inequality can be rewritten as

$$\left(\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_1-1)\right)x_1=\left[\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1+n_1-n_j)\right]x_1.$$

Because $n_1\geq n_j$, the right-hand side of the inequality yields

$$\left(\lambda-\frac{1}{r-1}\cdot\binom{n_j+s-2}{r-2}\cdot(n_j-1)\right)x_j\geq\left(\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1)\right)x_j.$$

Furthermore, we get

$$\left[\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1+n_1-n_j)\right]x_1\geq\left(\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1)\right)x_j,$$

then

$$\left[\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1)\right](x_1-x_j)\geq\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_1-n_j)x_1.$$

Obviously, the right-hand side of the inequality

$$\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_1-n_j)x_1\geq 0,$$

we get

$$\left[\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1)\right](x_1-x_j)\geq 0.$$

Since $n_1\geq n_j$ and

$$\lambda>\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_1-1)\geq\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1),$$

so,

$$\lambda-\frac{1}{r-1}\cdot\binom{n_1+s-2}{r-2}\cdot(n_j-1)\geq 0,$$

we must have $x_1\geq x_j$ for $2\leq j\leq t$. Fact 1 holds.

Let $H'=K_s^r\vee\left(K_{n-s-t+1}^r\cup(t-1)K_1\right)$. Combining the inequality $x_1\geq x_j$ $(2\leq j\leq t)$ established by Fact 1, we obtain

$$\binom{n_1+n_j-1}{r}-\binom{n_1}{r}-\binom{n_j-1}{r}>0.$$

From this inequality with $X^TX=1$, it follows that

$$\begin{aligned}\lambda(H')-\lambda(H)&\geq X^T(A(H')-A(H))X\\&=2\Big\{\sum_{j=2}^{t}\left[\binom{n_1+n_j-1}{r}-\binom{n_1}{r}-\binom{n_j-1}{r}x_1x_j-\binom{n_j-1}{r-1}x_j^2\right]\\&\quad+\sum_{j=2}^{t-1}\sum_{j=i+1}^{t}\left[\binom{n_i+n_j-2}{r}-\binom{n_i-1}{r}-\binom{n_j-1}{r}\right]x_1x_j\Big\}\\&>0.\end{aligned}$$

Lemma 3.6 holds. □

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Remark: From Lemma 3.6, we observe that the extremal configuration is attained when vertices are concentrated into one component.

Next, we further consider the spectral condition for the existence of spanning k-hypertrees in a connected r-uniform hypergraph H. By Lemma 3.3, there exists some nonempty subset S of $V(H)$ such that $c(H-S) \geq (k-2)|S|+3$. Let $|S| = s$. We are now ready to prove Theorem 3.7, our main result.

THEOREM 3.7. *Let H be a connected r-uniform hypergraph of order n with $k \geq 3$. If*

$$\lambda(H) \geq \lambda(H_{s,k}),$$

where $H_{s,k}$ is defined as

$$H_{s,k} = \begin{cases} K_1 \vee \big(K^r_{n-r} \cup (r-1)K_1\big), & s = 1,\ k = r-1, \\ K_1 \vee \big(K^r_{n-2} \cup K_1\big), & s = 1,\ k \geq r,\ n \geq 2r+1, \\ K_1 \vee \big(K^r_{n-r} \cup (r-1)K_1\big), & 2 \leq s \leq r-2,\ n \geq 2t, \\ K_{r-1} \vee \Big(K^r_{n-(k-1)(r-1)-2} \cup \big((k-2)(r-1)+2\big)K_1\Big), & \begin{array}{l} s \geq r-1, \\ n \geq \max\{2r(k-2)+5,\, r(k+r)\} \end{array} \end{cases},$$

then H contains a spanning k-hypertree, unless $H \cong H_{s,k}$.

Proof. We shall prove the contrapositive of the Theorem 3.7, we only need to prove that: Let H be a connected r-uniform hypergraph of order n with $k \geq 3$ and $r \geq 3$. If H containing no spanning k-hypertrees, then $\lambda(H) \leq \lambda(H_{s,k})$.

By Lemma 3.3, there exists some nonempty subset S of $V(H)$ such that $c(H-S) \geq (k-2)|S|+3$. Let $|S| = s$ and $t = (k-2)s+3$.

If there exists a nonempty subset $S \subseteq V(H)$ such that $c(H-S) \geq (k-2)|S|+3$, then the vertex set of H can be partitioned into S and t parts corresponding to the t components of $H-S$, where $t = (k-2)|S|+3$. In the extremal case that maximizes the spectral radius, each component is a complete r-uniform hypergraph $K^r_{n_i}$, and S is completely joined to all components, yielding the extremal hypergraph

$$H^1_s = K^r_s \vee \big(K^r_{n_1} \cup K^r_{n_2} \cup \cdots \cup K^r_{n_t}\big),$$

for some positive integers $n_1 \geq n_2 \geq \cdots \geq n_t$ with $\sum_{i=1}^t n_i = n-s$. For a hypergraph H satisfying the given condition, it can contain at most a subset of the hyperedges of H^1_s, hence H is a spanning subhypergraph of H^1_s. By Lemma 3.4, we obtain

$$\lambda(H) \leq \lambda(H^1_s), \tag{3.2}$$

where the equality holds if and only if $H \cong H^1_s$.

Take $H^2_s = K^r_s \vee (K^r_{n-s-t+1} \cup (t-1)K_1)$. Note that $t = (k-2)s+3$. By Lemma 3.6, we have

$$\lambda(H^1_s) \leq \lambda(H^2_s), \tag{3.3}$$

with equality holding if and only if $(n_1, \cdots, n_t) = (n-s-t+1, 1, \cdots, 1)$. Since $|S| = s$ with $S \subseteq V(H)$, we consider four cases for the parameter s: $s = 1$, $2 \leq s \leq r-2$, $s = r-1$, and $s \geq r$. We shall treat each case separately and derive corresponding spectral radius inequalities for every range.

Case 1. $s = 1$.

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If $s = 1$, then $t = (k-2)+3 = k+1$ and $H_s^2 = K_1 \vee (K_{n-k-1}^r \cup kK_1)$. Since H_s^2 is connected, then $t \geq r$, i.e., $k \geq r-1$. So, we proceed with a detailed analysis of the following two subcases.

Subcase 1.1. $k = r-1$.

If $t = k+1 = r$, then $H_s^2 = K_1 \vee (K_{n-r}^r \cup (r-1)K_1)$. Combining with (3.3), we conclude that

$$\lambda(H) \leq \lambda(K_1 \vee (K_{n-r}^r \cup (r-1)K_1)),$$

where the equality holds if and only if $H \cong K_1 \vee (K_{n-r}^r \cup (r-1)K_1)$, as desired.

Subcase 1.2. $k \geq r$.

For $t = k+1 \geq r+1$, by Corollary 3.2, we get

$$\lambda(H_s^2) \leq \max_{i \sim j}\{\sqrt{d_i d_j}\} \leq \Delta(H_s^2),$$

In the case $s = 1$, we consider the construction $H_s^2 = K_1 \vee \left(K_{n-t}^r \cup (t-1)K_1\right)$. There are three distinct types of vertices in this hypergraph: the single vertex of K_1, the vertices of K_{n-t}^r, and the $t-1$ singleton vertices. It turns out that the vertex in K_1 attains the maximum degree $\Delta(H_2^s)$. Counting the r-edges incident to this vertex according to the join definition yields.

$$\Delta(H_s^2) = \binom{n-t}{r-1} + \binom{t-1}{r-1}. \tag{3.4}$$

Since $n \geq s+t = t+1$, then $r+1 \leq t \leq n-1$. Let

$$f(t) = \binom{n-t}{r-1} + \binom{t-1}{r-1}.$$

For $r \geq 3$, the function

$$f(t) = \binom{n-t}{r-1} + \binom{t-1}{r-1}, \qquad r+1 \leq t \leq n-1,$$

satisfies the following monotonicity properties:

- On the interval $\left[r+1, \left\lfloor \frac{n+1}{2} \right\rfloor\right]$, $\binom{n-t}{r-1} \geq \binom{t-1}{r-1}$, then $f(t)$ is strictly decreasing.
- On the interval $\left[\left\lceil \frac{n+1}{2} \right\rceil, n-1\right]$, $\binom{n-t}{r-1} \leq \binom{t-1}{r-1}$, then $f(t)$ is strictly increasing.

Consequently, $f(t)$ attains its minimum at the point(s) closest to $\frac{n+1}{2}$ and attains its maximum at one of the endpoints $t = r+1$ or $t = n-1$. Hence,

$$\begin{aligned} f(n-1) - f(r+1) &= \binom{n-2}{r-1} - \binom{n-r-1}{r-1} - \binom{r}{r-1} \\ &= \binom{n-2}{r-1} - \binom{n-r-1}{r-1} - r, \end{aligned}$$

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According to Pascal's identity, $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$. We have

$$\begin{aligned}\binom{n-r}{r-1} > \binom{n-r-1}{r-1} + \binom{n-r-1}{r-2} &\geq \binom{n-r-1}{r-1} + \binom{n-r-1}{3-2} \quad (\text{since } \ r \geq 3)\\ &= \binom{n-r-1}{r-1} + (n-r-1)\\ &\geq \binom{n-r-1}{r-1} + r. \end{aligned} \tag{3.5}$$

Since $n \geq 2r+1$, the final inequality is valid. Thus,

$$\binom{n-2}{r-1} > \binom{n-r}{r-1} > \binom{n-r-1}{r-1} + r.$$

Therefore,

$$f(n-1) - f(r+1) > 0,$$

and

$$f_{\max} = f(n-1) = \binom{n-2}{r-1}.$$

Since $K_1 \vee (K_{n-2}^r \cup K_1)$ contains $K_{n-1}^r \cup K_1$ as a proper spanning subhypergraph (we present an example with $n = 5$ and $r = 3$, as shown in Figure 4), by Lemma 3.6, we have

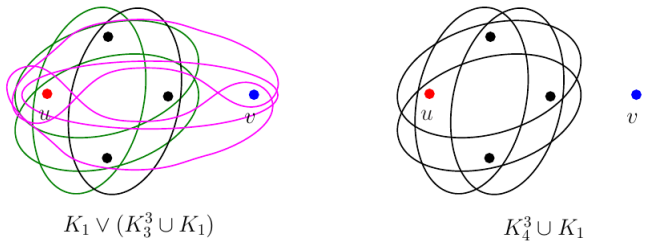


FIGURE 4. $K_1 \vee (K_3^3 \cup K_1)$ *contains* $K_4^3 \cup K_1$ *as a proper spanning subhypergraph.*

$$\lambda(K_1 \vee (K_{n-2}^r \cup K_1)) > \lambda(K_{n-1}^r \cup K_1). \tag{3.6}$$

Also by Lemma 3.5, $\lambda(K_{n-1}^r \cup K_1) = \binom{n-2}{r-1}$. Combining with (3.2) (3.3) and (3.6), we conclude that

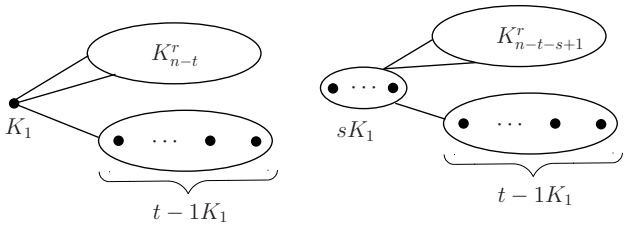
$$\lambda(H) \leq \lambda(H_s^1) \leq \lambda(H_s^2) < \binom{n-2}{r-1} = \lambda(K_{n-1}^r \cup K_1) < \lambda(K_1 \vee (K_{n-2}^r \cup K_1)).$$

This suggests that $K_1 \vee (K_{n-2}^r \cup K_1)$ contains no spanning k-hypertrees, and so the result follows.

Case 2. $2 \leq s \leq r-2$.

Recall that $H_s^2 = K_s^r \vee (K_{n-s-t+1}^r \cup (t-1)K_1)$, if $2 \leq s \leq r-2$, then we have $K_s^r = sK_1$. Based on the Figure 5 and the analysis of hypergraph structure, we observe that when adding a vertex to sK_1 while

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$K_1 \vee (K^r_{n-t} \cup (t-1)K_1)$ $\quad$ $K_s \vee (K^r_{n-t-s+1} \cup (t-1)K_1)$

FIGURE 5. $H^2_s = K_1 \vee (K^r_{n-t} \cup (t-1)K_1)$ *and* $H^2_s = K_s \vee (K^r_{n-t-s+1} \cup (t-1)K_1)$.

removing a vertex from $K^r_{n-t-s+1}$, the total number of edges experiences a net decrease since $n \geq 2t$ (with $\binom{t-1}{r-1}$ edges added and $\binom{n-t-1}{r-1}$ edges removed). According to Lemma 3.4, this operation leads to a decrease on the spectral radius. So we have

$$\lambda(K_s \vee (K^r_{n-t-s+1} \cup (t-1)K_1)) < \lambda(K_1 \vee (K^r_{n-t} \cup (t-1)K_1)).$$

Combining with Case 1, we obtain

$$\lambda(K_1 \vee (K^r_{n-t} \cup (t-1)K_1)) < \lambda(K_1 \vee (K^r_{n-r} \cup (r-1)K_1)),$$

as desired.

Case 3. $s = r-1$.

If $s = r-1$, then $H^2_s = K_{r-1} \vee (K^r_{n-t-r+2} \cup (t-1)K_1)$ where $t = (k-2)(r-1)+3$. Combining with (3.3), we conclude that

$$\lambda(H) \leq \lambda(K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1)),$$

where the equality holds if and only if $H \cong K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1$, as desired.

Case 4. $s \geq r$.

For $s \geq r$, by Corollary 3.2, we get

$$\lambda(H^2_s) \leq \max_{i \sim j}\{\sqrt{d_i d_j}\} \leq \Delta(H^2_s),$$

where

$$\begin{aligned} \Delta(H^2_s) &= \binom{n-t}{r-1} + (t-1) \cdot \binom{s-1}{r-2} \\ &= \binom{n-(k-2)s-3}{r-1} + ((k-2)s+2) \cdot \binom{s-1}{r-2}. \end{aligned} \tag{3.7}$$

Since $n \geq s+t = (k-1)s+3$, then $r \leq s \leq \frac{n-s}{k-1}$. Let

$$f(s) = \binom{n-(k-2)s-3}{r-1} + ((k-2)s+2) \cdot \binom{s-1}{r-2}.$$

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The growth rate of $\binom{n-(k-2)s-3}{r-1}$ is greater than the decreasing rate of $((k-2)s+2)\cdot\binom{s-1}{r-2}$, the expression (3.7) is maximized when $s=r$. So, we have $f(r)\geq f(s)$. Furthermore,

$$\begin{aligned}\lambda(H_s^2) &\leq \binom{n-(k-2)s-3}{r-1}+((k-2)s+2)\cdot\binom{s-1}{r-2}\\ &\leq \binom{n-(k-2)r-3}{r-1}+((k-2)r+2)\cdot(r-1).\end{aligned}$$

We proceed with a detailed analysis of the following two subcases:

Subcase 3.1. $k>r$.

We partition the m vertices into two disjoint subsets with sizes $m-s$ and s, respectively. The total number of ways to select t vertices from all m vertices is $\binom{m}{t}$. This total number is strictly larger than the sum of the following three mutually exclusive cases:

1. choosing all t vertices from the subset of size $m-s$, which gives $\binom{m-s}{t}$;
2. choosing all t vertices from the subset of size s, which gives $\binom{s}{t}$;
3. choosing 1 vertex from the subset of size $m-s$ and $t-1$ vertices from the subset of size s, which gives $(m-s)\binom{s}{t-1}$.

Thus, we obtain the strict inequality

$$\binom{m}{t}>\binom{m-s}{t}+\binom{s}{t}+(m-s)\binom{s}{t-1}. \tag{3.8}$$

According to Inequality (3.8) above, we obtain

$$\begin{aligned}\binom{n-(k-2)(r-1)-3}{r-1} &> \binom{n-(k-2)r-3}{r-1}+\binom{k-2}{r-1}+\binom{n-(k-2)r-3}{1}\cdot\binom{k-2}{r-2}\\ &> \binom{n-(k-2)r-3}{r-1}+(n-(k-2)r-3)\cdot\binom{k-2}{r-2}\\ &\geq \binom{n-(k-2)r-3}{r-1}+(n-(k-2)r-3)\cdot(r-1)\\ &> \binom{n-(k-2)r-3}{r-1}+((k-2)r+2)\cdot(r-1).\end{aligned}$$

Since $n>2r(k-2)+5$, thus the final inequality is valid.

Subcase 3.2 $k\leq r$.

According to Inequality (3.8) above, we obtain

$$\begin{aligned}\binom{n-(k-2)(r-1)-3}{r-1} &> \binom{n-(k-2)r-3}{r-1}+\binom{k-2}{r-1}+\binom{n-(k-2)r-3}{r-2}\cdot\binom{k-2}{1}\\ &\geq \binom{n-(k-2)r-3}{r-1}+(n-(k-2)r-3)\cdot(k-2)\\ &\geq \binom{n-(k-2)r-3}{r-1}+((k-2)r+2)\cdot(r-1).\end{aligned}$$

Since $n>r(k+r)$, thus the final inequality is valid due to the condition $n-(k-2)r-3>(k-2)r+2$. Then,

$$\lambda(H_s^2)<\binom{n-(k-2)(r-1)-3}{r-1}.$$

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Since $K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1)$ contains $K^r_{n-(k-2)(r-1)-2} \cup ((k-2)(r-1)+2)K_1$ as a proper spanning subhypergraph, by Lemma 3.4, we have

$$\begin{aligned} &\lambda(K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1)) \\ &> \lambda(K^r_{n-(k-2)(r-1)-2} \cup ((k-2)(r-1)+2)K_1). \end{aligned} \tag{3.9}$$

Also by Lemma 3.5, $\lambda(K^r_{n-(k-2)(r-1)-2} \cup ((k-2)(r-1)+2)K_1) = \binom{n-(k-2)(r-1)-3}{r-1}$. Combining with (3.2) (3.3) and (3.8), we conclude that

$$\lambda(H) \leq \lambda(H^1_s) \leq \lambda(H^2_s) < \lambda(K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1)).$$

This suggests that $K_{r-1} \vee (K^r_{n-(k-1)(r-1)-2} \cup ((k-2)(r-1)+2)K_1)$ contains no spanning k-hypertrees. The four cases above exhaust all admissible values of the parameter s. Combining the comparisons obtained in each case, we arrive at the desired extremal hypergraph configuration, and so the result follows. □

Remark: By means of adjacency spectral analysis and hypergraph structural characterization, we derive some sufficient conditions for the existence of spanning k-hypertrees in hypergraphs. Distinct from the weighted graph interpretation, this results fundamentally rely on hypergraph-specific structural properties, including hypergraph join operations, spanning k-hypertrees, and extremal obstruction families. This work extends the adjacency spectral criteria for spanning k-trees from graphs to hypergraphs and further broadens the application scope of spectral theory in the existence analysis of spanning hypertrees.

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The author does not have any conflicts of interest.

Data availability

No data were used for the research described in the article.

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