Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

A FINITENESS RESULT FOR COMPLEX HADAMARD MATRICES IN MAXIMAL ABELIAN SELF-ADJOINT SUBALGEBRAS OF $\mathrm{M_N}(\mathbb{C})$*

SHULER HOPKINS† AND REMUS NICOARĂ‡

Abstract. Let $\mathcal{A} = U\mathrm{D_n}(\mathbb{C})U^*$ be a Maximal Abelian Self-Adjoint subalgebra (MASA) of $\mathrm{M_n}(\mathbb{C})$, where $\mathrm{D_n}(\mathbb{C})$ denotes the diagonal matrices and $U \in \mathrm{M_n}(\mathbb{C})$ is a unitary matrix. Assume that U is *full superregular*, i.e., all the minors of U are nonzero. We show that $\mathcal{A}$ contains at most finitely many complex Hadamard matrices, up to equivalence given by multiplication by complex units. In particular, since almost every unitary is full superregular (with respect to the Haar distribution), it follows that almost every MASA of $\mathrm{M_n}(\mathbb{C})$ contains only finitely many complex nonequivalent Hadamard matrices.

Key words. Hadamard matrices, Full superregular matrices, MASA, β-twistulant matrices.

AMS subject classifications. 15B34

1. Introduction.

A complex Hadamard matrix is a matrix $H \in \mathrm{M_n}(\mathbb{C})$ having all entries of absolute value 1 and all rows mutually orthogonal (with respect to the complex inner product). Equivalently, H is Hadamard if $\frac{1}{\sqrt{n}}H$ is a unitary matrix with all entries of the same absolute value $\frac{1}{\sqrt{n}}$. For example, the discrete Fourier transform $F_n = (\omega^{k\cdot l})_{k,l\in\mathbb{Z}_n}$, with $\omega = e^{2\pi \mathrm{i}/n}$, is a complex Hadamard matrix.

In recent years, complex Hadamard matrices have found significant applications in various topics of mathematics and physics, including Quantum Information Theory (see [2, 3, 15, 16, 19]), Operator Algebras (see [8, 9, 10, 11, 12]), Cyclic Roots of Unity (see [7]), and Fuglede's Conjecture in Harmonic Analysis (see [17]). A general classification of complex Hadamard matrices is not available. A catalogue of most known complex Hadamard matrices can be found in [16]. The complete classification is only known for $n \leq 5$ (see [6]) and for self-adjoint matrices of order 6 (see [4]).

In this paper, we look at complex Hadamard matrices belonging to a fixed Maximal Abelian Self-Adjoint subalgebra (MASA) of $\mathrm{M_n}(\mathbb{C})$. Denote by $\mathrm{D_n}(\mathbb{C})$ the algebra of $n\times n$ diagonal matrices with complex entries. Then, any MASA $\mathcal{A}$ of $\mathrm{M_n}(\mathbb{C})$ is of the form $\mathcal{A} = U\mathrm{D_n}(\mathbb{C})U^*$, where $U \in \mathrm{M_n}(\mathbb{C})$ is a unitary matrix. The main result of this paper is the following:

THEOREM 1.1. *Let $U \in \mathrm{M_n}(\mathbb{C})$ be a unitary matrix that is* full superregular *(i.e., all minors of U are nonzero). Then, the algebra $\mathcal{A} = U\mathrm{D_n}(\mathbb{C})U^*$ contains at most finitely many complex Hadamard matrices, up to equivalence given by multiplication by complex units.*

To prove this result, we first embed the *real* algebraic variety of complex Hadamard matrices that belong to $\mathcal{A} = U\mathrm{D_n}(\mathbb{C})U^*$ in a *complex* algebraic variety of $\mathbb{C}^N$, with $N = 2n$. Since any compact complex algebraic variety of $\mathbb{C}^N$ is finite, it is sufficient to show that our complex variety is compact. Since it is clear that our algebraic variety is closed, it remains to be shown that it cannot be unbounded. We achieve this by contradiction; a limit argument leads to new relations that contradict the superregularity of U. The

*Received by the editors on July 8, 2025. Accepted for publication on January 20, 2026. Handling Editor: Dario Fasino.
Corresponding Author: Remus Nicoara.
†Mathematics Department, Sewanee: The University of the South, Sewanee, Tennessee 37383, USA (sghopkin@sewanee.edu).
‡Mathematics Department, University of Tennessee, Knoxville, Tennessee 37996, USA (rnicoara@utk.edu)

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

contradiction is obtained by first proving a more general version of *Tao's uncertainty principle* for cyclic groups of prime order [18] (see also Haagerup's equivalent formulation of this principle from [7]).

We note that almost every unitary matrix with respect to the Haar distribution has all minors nonzero (see for instance [1]); hence, it is full superregular and satisfies the hypothesis of our theorem.

The terminology *full superregular* comes from Information Theory, as full superregular matrices can be used to generate maximal distance separable block codes [13]. In particular, it is known that Cauchy matrices are full superregular. Additionally, real orthogonal Cauchy-like matrices are another concrete class of full superregular matrices. These matrices were completely classified in [5], also in connection to information theory.

In the last section of this paper we present a concrete class of 'nice' MASAs of $\mathrm{M_n}(\mathbb{C})$ that satisfy the hypothesis of Theorem 1.1, and which often contain complex Hadamard matrices: for n prime and $|\beta| = 1$ we consider the algebra of all β-*circulant* (or β-*twistulant*) $n \times n$ matrices.

In particular, for $\beta = 1$ this MASA is the algebra of circulant matrices. Thus, as a consequence of our result we also obtain that there exist only finitely many circulant complex Hadamard matrices of prime dimension (up to multiplication by complex units). This is a theorem of Haagerup from [7], and it is the result that inspired this paper.

2. Proof of the main result. In this section we prove Theorem 1.1. Let $U \in \mathrm{M_n}(\mathbb{C})$ be a unitary full superregular matrix. We want to show that the algebra $\mathcal{A} = U\mathrm{D_n}(\mathbb{C})U^*$ contains at most finitely many complex Hadamard matrices, up to multiplication by complex units.

For convenience, we note that $\mathrm{D_n}(\mathbb{C}) = \sqrt{n}\mathrm{D_n}(\mathbb{C})$ so we can write $\mathcal{A} = \sqrt{n}U\mathrm{D_n}(\mathbb{C})U^*$. Thus we want to prove that there exist only finitely many complex Hadamard matrices of the form $\sqrt{n}UDU^*$ (up to complex units multiplication), with D a diagonal matrix.

For any $x = (x_0, \dots, x_{n-1}) \in \mathbb{C}^n$ denote by D_x the diagonal matrix with entry x_k on position (k, k) for all $k \in \mathbb{Z}_n$, and denote by $\hat{x}_{i,j}$ the $(i, j)^{\text{th}}$ entry of the matrix $\sqrt{n}UD_xU^*$.

Note that if $\sqrt{n}UD_xU^*$ is a Hadamard matrix then in particular UD_xU^* is unitary, so $|x_k| = 1$ for all $k \in \mathbb{Z}_n$. By replacing the matrix $\sqrt{n}UD_xU^*$ by the matrix $\overline{x_0}\sqrt{n}UD_xU^* = \sqrt{n}U(\overline{x_0}D_x)U^*$, or equivalently by replacing D_x by $\overline{x_0}D_x$, it suffices to work with diagonal matrices D_x with $x_0 = 1$ (up to equivalence via complex units multiplication).

The next proposition is crucial to the proof, as it allows to work with a *complex* algebraic manifold instead of the *real* algebraic manifold of complex Hadamard matrices in $\mathcal{A}$.

PROPOSITION 2.1. *If $\sqrt{n}UD_xU^*$ is a Hadamard matrix with $x_0 = 1$, and if we let $y = (y_0, y_1, ..., y_{n-1})$ with $y_k = \overline{x_k}$, then (x, y) is a solution to the set of equations*

$$x_0 = y_0 = 1,\ x_k y_k = 1,\ \hat{x}_{k,0}\hat{y}_{0,k} = 1 \textit{ for all } k \in \mathbb{Z}_n.$$

Proof. We note that $x_0 = y_0 = 1$ follows from the hypothesis of this proposition. Since $\sqrt{n}UD_xU^*$ is Hadamard, in particular UD_xU^* is unitary. This implies that D_x is unitary, hence $|x_k| = 1$ for all k. Thus, for all $k \in \mathbb{Z}_n$,

$$x_k y_k = x_k \overline{x_k} = x_k \frac{1}{x_k} = 1.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

Now, since $\sqrt{n}UD_xU^*$ is Hadamard each entry has modulus 1, and thus $|\hat{x}_{k,0}| = 1$ for all $k \in \mathbb{Z}_n$. Note also that, since $(UD_xU^*)^* = UD_{\overline{x}}U^* = UD_yU^*$, we have $\hat{y}_{0,k} = \overline{\hat{x}_{k,0}}$. Hence, for all $k \in \mathbb{Z}_n$, we have

$$\hat{x}_{k,0}\hat{y}_{0,k} = \hat{x}_{k,0}\overline{\hat{x}_{k,0}} = 1.$$

We now introduce some notations that will be used in the next proposition. For $K, L \subset \mathbb{Z}_n$, denote by $(U)_{K\times L}$ the submatrix of U obtained by keeping the elements at the intersection of the rows of U indexed by K with the columns of U indexed by L. Also, for $x \in \mathbb{C}^n$ define its support, $supp(x)$, to be the set of all $k \in \mathbb{Z}_n$ such that $x_k \neq 0$.

The following statement is in the vein of Tao's uncertainty principle from [18].

Proposition 2.2. *Suppose U is an $n\times n$ unitary matrix, which is full superregular (i.e.,* $\det(U)_{K\times L} \neq 0$ *for any subsets $K, L \subset \mathbb{Z}_n$ with $|K| = |L| \geq 1$). For every nonzero $x \in \mathbb{C}^n$ and for every $t \in \mathbb{Z}_n$, we have:*

$$|supp(x)| + |supp(\hat{x}_t)| \geq n + 1,$$

where $\hat{x}_t = (\hat{x}_{k,t})_{k\in\mathbb{Z}_n} \in \mathbb{C}^n$ is the t^{th} column of $\sqrt{n}UD_xU^*$.

Proof. Suppose by contradiction that for some nonzero $x \in \mathbb{C}^n$ and some $t \in \mathbb{Z}_n$, we have

$$|supp(x)| + |supp(\hat{x}_t)| \leq n.$$

Set $L = supp(x)$, then $|L| \geq 1$ (since $x \neq 0$) and

$$|\mathbb{Z}_n \setminus supp(\hat{x}_t)| = n - |supp(\hat{x}_t)| \geq |supp(x)| = |L|.$$

So, we may choose $K \subset \mathbb{Z}_n \setminus supp(\hat{x}_t)$ such that $|K| = |L|$. Since $K \subset \mathbb{Z}_n \setminus supp(\hat{x}_t)$, we have for all $k \in K$

$$0 = \hat{x}_{k,t} = \sqrt{n}\sum_{i\in\mathbb{Z}_n} u_{k,i}x_iu^*_{i,t} = \sqrt{n}\sum_{i\in L} u_{k,i}x_iu^*_{i,t}.$$

Let $T \subset \mathbb{Z}_n$ such that $t \in T$ and $|T| = |L| = |K|$. Note that the (k, t) entry of $\sqrt{n}UD_xU^*$ is

$$(\sqrt{n}(U)_{K\times L}(D_x)_{L\times L}(U^*)_{L\times T})_{k,t} = \sqrt{n}\sum_{i\in L} u_{k,i}x_iu^*_{i,t}.$$

As this holds for all $k \in K$, the t^{th} column of $\sqrt{n}(U)_{K\times L}(D_x)_{L\times L}(U^*)_{L\times T}$ consists only of zeros. This implies

$$0 = \det\left(\sqrt{n}(U)_{K\times L}(D_x)_{L\times L}(U^*)_{L\times T}\right) = \sqrt{n}\det(U)_{K\times L}\cdot\det(D_x)_{L\times L}\cdot\det(U^*)_{L\times T}.$$

But this is a contradiction since $\det(U)_{K\times L} \neq 0$, $\det(U^*)_{L\times T} \neq 0$, and $\det(D_x)_{L\times L} = \prod_{l\in L} x_l \neq 0$ since $L = supp(x)$. ∎

Proposition 2.3. *Suppose U is an $n \times n$ unitary matrix, which is full superregular. For every nonzero $x \in \mathbb{C}^n$ and for every $s \in \mathbb{Z}_n$, we have:*

$$|supp(x)| + |supp(\hat{x}^s)| \geq n + 1,$$

where $\hat{x}^s = (\hat{x}_{s,l})_{l\in\mathbb{Z}_n}$ is the s^{th} row of $\sqrt{n}UD_xU^*$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

Proof. This follows from the previous proposition applied to $\overline{x}$, by using that $|supp(x)| = |supp(\overline{x})|$ and $|supp(\hat{x}^s)| = |supp(\hat{\overline{x}}_s)|$. The last equality follows by noticing that the s^{th} row of $\sqrt{n}UD_xU^*$ is the transposed conjugate of the s^{th} column of $\sqrt{n}UD_{\overline{x}}U^*$; hence, their supports have the same cardinality. ∎

We are now ready to prove Theorem 1.1. Note that, by using Proposition 2.1, the following statement will imply Theorem 1.1.

Theorem 2.4. *If U is a full superregular unitary matrix, then there are only finitely many complex solutions to the following set of equations*

$$x_0 = y_0 = 1, \qquad x_k y_k = 1, \qquad \hat{x}_{k,0}\hat{y}_{0,k} = 1 \qquad \textit{for all } k \in \mathbb{Z}_n.$$

Proof. Suppose, by contradiction, that there are infinitely many $z = (x, y) \in \mathbb{C}^n \times \mathbb{C}^n$ that satisfy the system of equations. Since this system yields a complex algebraic variety, and compact complex algebraic varieties in $\mathbb{C}^N$ are finite (see for instance Theorem 14.3.i in [14]; here $N = 2n$), it follows that the set of solutions to the system is not compact. This set is clearly closed; thus, it must be unbounded. Let $(z^{(m)})_{m\geq 1} = ((x^{(m)}, y^{(m)}))_{m\geq 1}$ be a sequence such that

$$\lim_{m\to\infty} \|z^{(m)}\|_2 = \infty.$$

Here, we use the notation $\|z^{(m)}\|_2$ for the complex Euclidean norm of the element $z^{(m)} \in \mathbb{C}^{2n}$. Next, note that

$$\|x^{(m)}\|_2^2\|y^{(m)}\|_2^2 = \left(1 + \sum_{i=1}^{n-1} |x_i^{(m)}|^2\right)\left(1 + \sum_{i=1}^{n-1} |y_i^{(m)}|^2\right) \geq \|z^{(m)}\|_2^2 - 1,$$

which implies $\|x^{(m)}\|_2\|y^{(m)}\|_2 \to \infty$. Now we set

$$r^{(m)} = \frac{x^{(m)}}{\|x^{(m)}\|_2}, \qquad s^{(m)} = \frac{y^{(m)}}{\|y^{(m)}\|_2}.$$

Since $\|s^{(m)}\|_2 = \|r^{(m)}\|_2 = 1$ for all m, the sequence $(r^{(m)}, s^{(m)})_{m\geq 1}$ is bounded; hence, it has a convergent subsequence. By replacing the original sequence by its convergent subsequence, we may assume that the following limits exist:

$$r = \lim_{m\to\infty} r^{(m)}, \qquad s = \lim_{m\to\infty} s^{(m)}.$$

It follows that for all $k \in \mathbb{Z}_n$, we have

$$r_k s_k = \lim_{m\to\infty} r_k^{(m)} s_k^{(m)} = \lim_{m\to\infty} \frac{x_k^{(m)} y_k^{(m)}}{\|x^{(m)}\|_2\|y^{(m)}\|_2} = \lim_{m\to\infty} \frac{1}{\|x^{(m)}\|_2\|y^{(m)}\|_2} = 0,$$

$$\hat{r}_{k,0}\hat{s}_{0,k} = \lim_{m\to\infty} \hat{r}_{k,0}^{(m)} \hat{s}_{0,k}^{(m)} = \lim_{m\to\infty} \frac{\hat{x}_{k,0}^{(m)} \hat{y}_{0,k}^{(m)}}{\|x^{(m)}\|_2\|y^{(m)}\|_2} = \lim_{m\to\infty} \frac{1}{\|x^{(m)}\|_2\|y^{(m)}\|_2} = 0.$$

This implies that

$$supp(r) \cap supp(s) = \emptyset \ \text{ hence } |supp(r)| + |supp(s)| \leq n,$$
$$supp(\hat{r}_0) \cap supp(\hat{s}^0) = \emptyset \ \text{ hence } |supp(\hat{r}_0)| + |supp(\hat{s}^0)| \leq n.$$

Combining this with Proposition 2.2 and Proposition 2.3, we obtain

$$2n + 2 \leq |supp(r)| + |supp(s)| + |supp(\hat{r}_0)| + |supp(\hat{s}^0)| \leq 2n.$$

This contradiction ends the proof. ∎

Thus, Theorem 1.1 follows by combining Proposition 2.1 with Theorem 2.1.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

3. A class of examples: β-twistulant matrices. Let $\beta \neq 0$ be a complex number, and let n be prime. In this section, we show that the $n \times n$ β-twistulant matrices form a MASA $\mathcal{A}_\beta \subset \mathrm{M_n}(\mathbb{C})$ and $\mathcal{A}_\beta = U\mathrm{D_n}(\mathbb{C})U^*$ for some U superregular. Note that this satisfies the hypothesis of Theorem 1.1.

We start by recalling the definition of a β-twistulant matrix. An $n \times n$ β-twistulant matrix is any matrix of the form

$$T_\beta(a) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \beta a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \beta a_{n-2} & \beta a_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta a_2 & \beta a_3 & \beta a_4 & \dots & a_0 & a_1 \\ \beta a_1 & \beta a_2 & \beta a_3 & \dots & \beta a_{n-1} & a_0 \end{pmatrix},$$

for some $(a_0, a_1, ..., a_{n-1}) \in \mathbb{C}^n$.

It is easy to check that

$$\mathcal{A}_\beta = \{T_\beta(a) : a \in \mathbb{C}^n\},$$

is a MASA of $\mathrm{M_n}(\mathbb{C})$. In what follows, we will be particularly interested in the case $|\beta| = 1$, since otherwise this MASA cannot contain any complex Hadamard matrices. Note that this MASA can be rich in complex Hadamard matrices even for $\beta = 1$ (i.e., circulant complex Hadamard matrices).

One can diagonalize the elements of $\mathcal{A}_\beta$ as follows: Let γ be some n-root of β (i.e., $\gamma^n = \beta$) and denote $\omega = e^{2\pi \mathrm{i}/n}$. Consider the matrix

$$U = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \gamma & \gamma\omega & \gamma\omega^2 & \dots & \gamma\omega^{n-1} \\ \gamma^2 & \gamma^2\omega^2 & \gamma^2\omega^4 & \dots & \gamma^2\omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma^{n-1} & \gamma^{n-1}\omega^{n-1} & \gamma^{n-1}\omega^{2(n-1)} & \dots & \gamma^{n-1}\omega^{(n-1)^2} \end{pmatrix}.$$

It is easy to check that $U\mathrm{D_n}U^* = \mathcal{A}_\beta$, where $\mathrm{D_n}$ denotes the MASA of $n \times n$ diagonal matrices with complex entries. Note also that $U = D_\gamma \cdot F_n$, where $F_n = (\omega^{k \cdot l})_{k,l \in \mathbb{Z}_n}$ denotes the Fourier matrix, and D_γ =diag$(1, \gamma, \gamma^2, ..., \gamma^{n-1})$ is a diagonal matrix.

We note that the discrete Fourier transform F_n is superregular for n prime. Indeed, it is a classical result of Chebotarëv that all minors of this matrix are nonzero (see [18] for an elementary proof).

From here it also follows that all minors of U are nonzero, since any such minor is obtained from a minor of F_n by multiplying each of its rows by a fixed power of γ. Thus, U is superregular when n is prime, and Theorem 1.1 applies to β-twistulant matrices of prime dimension.

In particular, for $\beta = 1$, we have $\mathcal{A}_1 = F_n\mathrm{D_n}(\mathbb{C})F_n^*$ is the algebra of circulant matrices:

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \dots & x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_0 \end{pmatrix} : x_0, x_1, ...x_{n-1} \in \mathbb{C} \right\}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 21-26, January 2026.

Thus, as a consequence of our result, we also obtain that there exist only finitely many circulant complex Hadamard matrices of prime dimension (up to multiplication by complex units). This is a theorem of Haagerup from [7], and it is the result that inspired this paper. In fact, applying Theorem 1.1 to $\mathcal{A}_\beta$ with $\beta \neq 1$ is equivalent to applying it to $\mathcal{A}_1$, since $\mathcal{A}_\beta = D_\gamma \mathcal{A}_1 D_\gamma^*$.

REFERENCES

[1] G. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*. Cambridge University Press, Cambridge, 2010.

[2] T. Banica. *Complex Hadamard Matrices and Applications. viXra*, (2019). https://api.semanticscholar.org/CorpusID:204575805

[3] N. Barros e Sá and I. Bengtsson. Families of complex Hadamard matrices. *Linear Algebra Appl.*, 438:2929–2957, 2013. https://doi.org/10.1016/j.laa.2012.10.029.

[4] K. Beauchamp and R. Nicoara. Orthogonal maximal Abelian $*$-subalgebras of the 6×6 matrices. *Linear Algebra Appl.*, 428:1833–1853, 2008. https://doi.org/10.1016/j.laa.2007.10.023.

[5] D. Fasino. Orthogonal cauchy-like matrices. *Numer. Algorithms*, 92:619–637, 2023.

[6] U. Haagerup. Orthogonal maximal Abelian $*$-subalgebras of the $n \times n$ matrices and cyclic n-roots. In: *Operator algebras and quantum field theory (Rome, 1996)*. International Press, Cambridge, MA, 296–322, 1997. https://doi.org/10.1215/s0012-7094-97-09004-9.

[7] U. Haagerup. Cyclic p-roots of prime lengths p and related complex hadamard matrices, arXiv:0803.2629v1 [math.AC], 2008.

[8] V. Jones and V. Sunder. *Introduction to Subfactors*. London Mathematical Society Lecture Note Series, Cambridge University Press, 1994.

[9] V.F.R. Jones. Planar algebras. *New Zealand J. Math.*, 52:1–107, 2021.

[10] R. Nicoară. Subfactors and Hadamard matrices. *J. Operator Theory*, 64:453–468, 2010.

[11] S. Popa. Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras. *J. Operator Theory*, 9:253–268, 1983.

[12] S. Popa. Classification of subfactors: The reduction to commuting squares. *Invent. Math.*, 101:19–43, 1990. https://doi.org/10.1007/BF01231494.

[13] R. Roth and A. Lempel. On MDS codes via cauchy matrices. *IEEE Trans. Inform. Theory*, 35:1314–1319, 1989. https://doi.org/10.1109/18.45291.

[14] W. Rudin. Function theory in the unit ball of $\mathbb{C}^n$. *Grundlehren Math. Wiss.*, 241, 1980.

[15] F. Szöllősi. Complex hadamard matrices of order 6: A four-parameter family. *J. London Math. Soc.*, 85:616–632, 2012.

[16] W. Tadej and K. Życzkowski. A concise guide to complex Hadamard matrices. *Open Syst. Inf. Dyn.*, 13:133–177, 2006. https://doi.org/10.1007/s11080-006-8220-2.

[17] T. Tao. Fuglede's conjecture is false in 5 and higher dimensions. *Math. Res. Lett.*, 11:251–258, 2004. https://doi.org/10.4310/MRL.2004.v11.n2.a8.

[18] T. Tao. An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.*, 12:121–127, 2005. https://doi.org/10.4310/MRL.2005.v12.n1.a11.

[19] R.F. Werner. All teleportation and dense coding schemes. *J. Phys. A*, 34:7081–7094, 2001. https://doi.org/10.1088/0305-4470/34/35/332.