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11 CAN BE REDUCED TO 10*

ABRAHAM BERMAN† AND ELIAHU LEVY†

Abstract. Laffey and Šmigoc proved that for every 2×2 doubly nonnegative integer matrix A, icpr(A) ≤ 11. In this paper, it is shown that 11 can be replaced by 10 and that for many small matrices, even by 9.

Key words. Complete positivity, Complete positivity over the integers, Sums of perfect squares, Integer CP rank.

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1. Introduction.

Laffey and Šmigoc proved, [4], that every 2×2 doubly nonnegative (DNN) integer matrix can be decomposed as BB^T, where B is a 2×11 nonnegative integer matrix. They pointed out that 11 cannot be replaced by 8 and asked if it can be replaced by 9 or 10. We prove that 11 can be replaced by 10 and that for many small matrices, even by 9. The interest in the problem stems from the theory of completely positive matrices so we start the paper with a very short introduction to this theory. All the numbers (vectors and matrices) in the paper are nonnegative and after section 2, they are nonnegative integers. The main tool used in the paper are the theorems of Lagrange and Legendre on expressing a number as sum of perfect squares, so we cite these important theorems. The proofs of the new results involve many special cases, so we divide them to description of the proof idea and tables that depict the special cases.

2. Complete positivity.

Definition 2.1 (Completely positive matrix and the CP rank). *A matrix A is completely positive if it can be decomposed as* $A = BB^T$ *where B is a nonnegative matrix. The smallest number of columns in such matrix B is called the CP rank of A.*

References on properties and applications of complete positivity and on bounds on the CP rank are [1] and [7].

Definition 2.2 (Doubly nonnegative matrix). *A positive semidefinite matrix that is also elementwise nonnegative is called doubly nonnegative (DNN).*

Obviously, completely positive matrices are doubly nonnegative, but the necessary condition is not sufficient.

Definition 2.3 (Matrix realization). *Let G be a simple graph with vertices 1, 2, ..., n. A symmetric matrix A is a matrix realization of G if for* $i \neq j, a_{ij} \neq 0$ *if and only if i and j are adjacent (there is no restriction on the diagonal entries of A).*

Definition 2.4 (Completely positive graph). *A simple graph G is completely positive if every doubly nonnegative matrix realization of G is completely positive.*

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†Mathematics Department, Technion, Israel Institute of Technology, Haifa 32000, Israel (berman@technion.ac.il, elevy44@gmail.com).

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Theorem 2.5 ([2]). *A graph G is completely positive if and only if it does not contain an odd cycle of length* > 4.

Remark 2.6. *An equivalent characterization is, [6], that G is the line graph of a perfect graph.*

Corollary 2.7 ([5]). *Graphs with less than 5 vertices are completely positive.*

3. Integer completely positive matrices.

Definition 3.1 (Completely positive matrix over the integers and the integer CP rank). *A matrix A is completely positive over the integers if it can be decomposed as* $A = BB^T$, *where B is an integer nonnegative matrix. The smallest number of columns in such matrix B is called the integer CP rank of A and is denoted by icpr(A).*

The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ *is an example of a 3×3 integer completely positive matrix that is not completely positive over the integers, but for* $n = 2$, *Laffey and Smigoc [3] proved that every 2×2 integer completely positive matrix (that by Corollary 2.7 is the same as every 2×2 integer DNN matrix) is completely positive over the integers. In 2019, they proved:*

Theorem 3.2. *For every 2×2 DNN matrix A, icpr(A)* ≤ 11.

The matrix $\begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix}$ *shows that 11 cannot be replaced by 8. They asked if it can be replaced by 9 or 10.*

We prove that it can be replaced by 10 and show that for many small matrices, even by 9. A key step in the proof of Theorem 3.2 was:

Theorem 3.3. *If* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ *is DNN and* $b > c$ *(or* $b > a$*), then there exists a DNN matrix A' =* $\begin{bmatrix} a' & b' \\ b' & c' \end{bmatrix}$, *where* $a' \geq b'$ *and* $c' \geq b'$, *and A' and A have the same icpr.*

4. Sums of perfect squares.

The central tool in the paper is the theorems of Lagrange and Legendre.

Theorem 4.1 (Lagrange's theorem). *Every natural number can be written as sum of 4 perfect squares.*

Theorem 4.2 (Legendre's theorem). *A natural number can be written as sum of 3 perfect squares if and only if it is not of the form* $4^k(8m + 7)$.

Corollary 4.3. *The numbers 8k+1, 8k+2, 8k+3, 8k+5, 8k+6 can be written as sum of 3 perfect squares.*

Definition 4.4 (Good numbers and bad numbers). *We say that a number x is good if x (mod 8)* $\in$ $\{1, 2, 3, 5, 6\}$ *and bad if x (mod 8)* $\in \{0, 4, 7\}$.

Definition 4.5 (Bad, good, and very good triplets). *A triplet (e, f, g) is good if at least two of the numbers (e-f) (mod 8), f , (g-f) (mod 8) are good. It is very good if all three are good and it is bad if at least two numbers are bad.*

Remark 4.6. *In terms of integer CP rank, Theorem 4.1 says that for a 1×1 matrix, the icpr is bounded by 4 and for the numbers in Corollary 2 (the good numbers) it is less than or equal to 3.*

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5. Conjectures and theorems.

CONJECTURE 5.1. *Every 2×2 DNN matrix A can be decomposed as* $A = BB^T$ *where* $B \in R^{2\times 9}$.

To bound the CP rank, we observe that *icpr(A+B)≤ icpr(A)+ icpr(B).*

Let A = $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be DNN .

Without the loss of generality, we can assume that c ≥ a. By Theorem 3.3, we can assume that c≥ a≥ b≥0. If b=0 , icpr(A) = icpr (a) + icpr(c) ≤8, so we can assume that b>0 .

If a=b , then A= $\begin{bmatrix} 0 & 0 \\ 0 & c-b \end{bmatrix} + \begin{bmatrix} b & b \\ b & b \end{bmatrix}$ and icpr(A) ≤ 8 so we can assume that c ≥a > b> 0.

If $b^2 \le c$, then A=$\begin{bmatrix} a-1 & 0 \\ 0 & c-b^2 \end{bmatrix} + \begin{bmatrix} 1 & b \\ b & b^2 \end{bmatrix}$ and icpr(A) ≤ 9 so we can assume that $b^2 > c \ge a > b > 0$.

To prove the conjecture,we suggest two additional conjectures:

CONJECTURE 5.2. *For any two natural numbers* $a > b$, *there exist vectors* $u = (u_1u_2u_3u_4u_5)$ *and* $v = (v_1v_2v_3v_4v_5)$ *such that* $uu^T = a$, *and* $vu^T = b$ *and* $vv^T \le b$.

CONJECTURE 5.3. *For any two natural numbers* $a > b$, *there exist vectors* $u = (u_1u_2u_3u_4u_5)$ *and* $v = (v_1v_2v_3v_4v_5)$ *such that* $v \le u$, $uu^T = a$, *and* $vu^T = b$.

REMARK 5.4. *Obviously, Conjecture 3 implies Conjecture 2. Conjecture 2, in turn, implies Conjecture 1 with* $\begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & 0 & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & v_5 & w_1 & w_2 & w_3 & w_4 \end{bmatrix}$ *where* $c - \sum_{i=1}^{5} v_i^2 = \sum_{i=1}^{4} w_i^2$.

To prove the first new result, we need the following definitions and lemmas:

DEFINITION 5.5 (Spanning vector). *A vector* $u = (u_k \dots u_2u_1)$ *is spanning if for every* $b \le uu^T$, *there exists a vector v such that* $v \le u$ *and* $vu^T = b$.

DEFINITION 5.6 (Step vector). *A vector* $u = (u_k \dots u_2u_1)$ *is a step vector if* u_1*=1 and* $(u_{i+1} - u_i) \in \{0, 1\}$, *i=1,...,k-1.*

LEMMA 5.7. *Step vectors are spanning.*

This is a special case of:

LEMMA 5.8. *If* $u = (u_k \dots u_2u_1)$ *is a spanning vector (SV) and* $u_{k+1)} \le uu^T + 1$, *then* $(u_{k+1} \dots u_2u_1)$ *is also spanning.*

Proof. Every number between 1 and $\sum_{i=1}^{k+1} u_i^2$ can be written as $pu_{k+1} + q$, $0 \le p \le u_{k+1}$, $0 \le q \le \sum_{i=1}^{k} u_i^2$ ∎

THEOREM 5.9. *If* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ *is DNN and* $a \le 64$, *then A can be decomposed as* $A = BB^T$, *where* $B \in R^{2\times 9}$.

Proof. We show that Conjecture 5.3 holds for a ≤ 64. In Table 1, we show that for every number between 1 and 64 , except 33, there is a SV. In Table 2, we construct vectors u and v , for a=33 and $b < 33$. ∎

REMARK 5.10. *Conjecture 5.1 holds for every number that has an SV. Unfortunately, the number of such numbers is finite.*

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THEOREM 5.11. *Every 2×2 DNN matrix A can be decomposed as* $A = BB^T$ *where* $B \in R^{2\times 10}$.

Proof. Let A= $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $b^2 > c \geq a > b$. We want to show that icpr(A) ≤ 10. If $a < 65$, then, by Theorem 5.9, icpr(A) ≤ 9.

If $a > 64$ then $b > 8$ and $c > 64$. Consider the remainders α, β, γ when a, b, and c are divided by 8.

$$A = \begin{bmatrix} 8t+\alpha & 8r+\beta \\ 8r+\beta & 8s+\gamma \end{bmatrix}) = (8r+\beta)J + \begin{bmatrix} 8(t-r)+\alpha-\beta & 0 \\ 0 & 8(s-r)+\gamma-\beta \end{bmatrix},$$

where J= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

If (α, β, γ) is very good then, by Corollary 4.3, $icpr(A) \leq 9$.

If it is good then, by the same corollary , $icpr(A) \leq 10$. If (α, β, γ) is bad we subtract from A a rank one matrix $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, where x, y, xy are less than or equal to 7 . Since $a > 64$, $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ - $\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \geq 0$.

Denoting $\alpha' = \alpha - x^2 (\text{mod } 8)$, $\gamma' = \gamma - y^2$ (mod 8) , $\beta' = \beta - xy$ (mod 8), we get

$$A = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + (8r+\beta-xy)J + \begin{bmatrix} 8(t-r)+\alpha-\beta-x^2 & 0 \\ 0 & 8(s-r)+\gamma-\beta-y^2 \end{bmatrix} =$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + (8r+\beta')J + \begin{bmatrix} 8(t-r)+\alpha'-\beta' & 0 \\ 0 & 8(s-r)+\gamma'-\beta' \end{bmatrix}.$$

To complete the proof, we show in Table 3 that for every bad triplet (α, β, γ) there is a pair (x, y) such that $(\alpha', \beta', \gamma')$ is very good.

The pairs used are (1,2) (1,3) (1,4) (1,5) (1,6) (1,7) (2,1) (2,3) so $1 \leq x, xy, y \leq 7$. □

6. TABLES.

TABLE 1. For every number up to 64, except 33, there is an SV that can be obtained by expressing n-1 as sum of 4 perfect squares.

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TABLE 1
Proof of Theorem 5.9 for $n \neq 33$

a	SV	a	SV	a	SV	a	SV
64	73211	48	53321	32	52111	16	32111
63	7321	47	54211	31	5211	15	3211
62	72221	46	44321	30	4321	14	321
61	73111	45	6221	29	51111	13	31111
60	7311	44	54111	28	43111	12	3111
59	72211	43	62111	27	4311	11	22111
58	7221	42	6211	26	42211	10	2211
57	6421	41	44221	25	4221	9	221
56	63311	40	53211	24	33211	8	21111
55	54321	39	5321	23	3321	7	2111
54	63221	38	52221	22	4211	6	211
53	44421	37	53111	21	421	5	21
52	54311	36	5311	20	41111	4	1111
51	63211	35	52211	19	4111	3	111
50	54221	34	43221	18	3221	2	11
49	62221	33	******	17	22221	1	1

TABLE 2. For $a = 33$, there is no SV. We must use all three representations of 33 as sums of up to 5 squares, $33 = 4^2 + 4^2 + 1 = 5^2 + 2^2 + 2^2 = 4^2 + 3^2 + 2^2 + 2^2$. In the table, we denote $u = (44100)$ by 441, $u = (52200)$ by 522 and $u = (43220)$ by 4322 and construct (shortened) vectors v for $1 \leq b \leq 32$.

TABLE 2
Proof of Theorem 5.9 for n=33

Set1	Repr1	Set2	Repr2	Set3	Repr3
b	**441**	**b**	**522**	**b**	**4322**
1	001	2	001	3	0100
4	010	6	021	18	4001
5	011	7	101	30	4222
8	020	10	200		
9	021	11	121		
12	030	14	202		
13	031	15	300		
16	040	19	311		
17	041	22	401		
20	140	23	322		
21	141	26	421		
24	240	27	501		
25	241	31	521		
28	340				
29	341				
32	440				

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TABLE 3. We consider the bad triplets (α, β, γ). The symmetry allows us to consider only the cases $\gamma \leq \alpha$. We list them according to the pairs (x, y) used in the proof of Theorem 7. The columns are organized in decreasing order, first of β, then α, and then γ.

Table 3
Triplets for proof of Theorem 5.11

$x = 1, y = 2$	
α	7777743355516665554444107773344210
β	7777777755554444444444443333300000
γ	7643233254114304304310007323243000
$x = 1, y = 3$	
α	66655244777335551107777443
β	66666655444441111110000000
γ	65252241430305101007310100
$x = 1, y = 4$	
α	7776666621
β	7777777722
γ	5106541011
$x = 1, y = 5$	
α	3322777
β	4443000
γ	2136526
$x = 1, y = 6$	
α	536655
β	770000
γ	304040
$x = 1, y = 7$	
α	6666627
β	7722220
γ	3262124
$x = 2, y = 1$	
α	34
β	70
γ	12
$x = 2, y = 3$	
α	4
β	4
γ	2

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