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THE LAPLACIAN MATRIX OF WEIGHTED THRESHOLD GRAPHS*

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Abstract. Threshold graphs are generated from one node by repeatedly adding a node that links to all existing nodes or adding a node without links. In the weighted threshold graph, we add a new node in step i, which is linked to all existing nodes by a link of weight w_i. In this work, we consider the set $\mathcal{A}_N$ that contains all Laplacian matrices of weighted threshold graphs of order N. We show that $\mathcal{A}_N$ forms a commutative algebra. Using this, we find a common basis of eigenvectors for the matrices in $\mathcal{A}_N$. It follows that the eigenvalues of each matrix in $\mathcal{A}_N$ can be represented as a linear transformation of the link weights. In addition, we prove that, if there are just three or fewer different weights, two weighted threshold graphs with the same Laplacian spectrum must be isomorphic.

Key words. Threshold graphs, Laplacian matrix, Commutative algebra, Cospectral graphs.

AMS subject classifications. 05C22, 05C50, 05C75.

1. Introduction.

The adjacency matrix A of an unweighted graph G with nodes $\{1,\ldots,N\}$ is a matrix $N\times N$ with elements a_{ij}, where $a_{ij}=1$ if there is a link between node i and node j, otherwise $a_{ij}=0$. We use notation and notions from [1]. In a weighted graph, each link $\{i,j\}$ is assigned a weight $a_{ij}\in\mathbb{R}$, which can be an arbitrary real value, including both positive and negative numbers. We identify a non-link with a weight-zero link. Then the elements of the (weighted) adjacency matrix A are the link weights a_{ij}. The Laplacian matrix of a (weighted) graph is defined by $Q=\Delta-A$, where $\Delta=\text{diag}(d_1,d_2,...,d_N)$ and $d_i=\sum_{j=1}^{N}a_{ij}$ is the (weighted) degree of node i. The Laplacian matrix necessarily has a zero eigenvalue because its row sum is zero.

A threshold graph is a graph obtained from one node by repeatedly adding an isolated node or a dominant node, where an isolated node is not connected to any other node in a graph and a dominant node is a node that links to all other nodes in a graph. The concept of threshold graphs is introduced in [2]. A comprehensive review of threshold graphs is provided in [3]. Formulas for the Laplacian spectrum and the number of spanning trees in a threshold graph are given in [4]. The application of threshold graphs in building real-world networks is discussed in [5, 6, 7].

In the weighted threshold graph, we start with one node and add a new node in step i $(i=2,\ldots,N)$, which is linked to all existing nodes by a link of weight $w_i\in\mathbb{R}$. We label the nodes as $1,2,3,\ldots,N$ according to the order in which the nodes are added. Then, the $N\times N$ adjacency matrix A of a weighted threshold graph equals

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$$A = \begin{bmatrix} 0 & w_2 & w_3 & \cdots & \cdots & w_N \\ w_2 & 0 & w_3 & \cdots & \cdots & w_N \\ w_3 & w_3 & 0 & \cdots & \cdots & w_N \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & w_N \\ w_N & w_N & w_N & \cdots & w_N & 0 \end{bmatrix}. \tag{1.1}$$

The vector $W = (w_2, w_3, \ldots, w_N)$ determines the weighted threshold graph. We call W the weight vector and write G_W for the corresponding weighted threshold graph.

The row sums of the adjacency matrix A in Eq. (1.1) give the degrees of the nodes in G_W, that is,

$$d_i = (i-1)w_i + \sum_{j=i+1}^{N} w_j, \text{ for } 1 \leq i \leq N. \tag{1.2}$$

The Laplacian matrix $\Delta - A$ of G_W is written as:

$$Q_W = \begin{bmatrix} d_1 & -w_2 & -w_3 & \cdots & \cdots & -w_N \\ -w_2 & d_2 & -w_3 & \cdots & \cdots & -w_N \\ -w_3 & -w_3 & d_3 & \cdots & \cdots & -w_N \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & -w_N \\ -w_N & -w_N & -w_N & \cdots & -w_N & d_N \end{bmatrix}. \tag{1.3}$$

Figure 2 illustrates the threshold graph G_W coded by weight vector $W = (1, 0, -\sqrt{2}, 0,\ 2)$ and its Laplacian matrix Q_W. The graph G_W is constructed by sequentially adding node i, along with its associated weight w_i for $1 \leq i \leq 6$ (see Fig. 1).

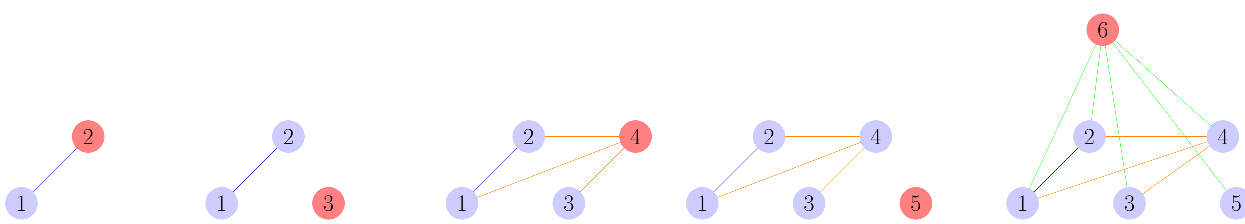

FIGURE 1. *Construction of G_W coded by weight vector $W = (1, 0, -\sqrt{2}, 0,\ 2)$. At each step, a node i (highlighted in red) is added with links of weight w_i, which connect it to all previous nodes $j < i$. Line colors represent link weights. Blue, orange, and green lines have weights 1, $-\sqrt{2}$, and 2, respectively.*

We define $\mathcal{A}_N$ to be the set of Laplacian matrices of weighted threshold graphs of order N. The main purpose of this paper is to investigate the structure of $\mathcal{A}_N$ and the eigensystem of the matrices in $\mathcal{A}_N$. In Section 2, we establish that $\mathcal{A}_N$ constitutes a commutative algebra of dimension $N-1$ and provide a basis. In Section 3, we derive eigenvectors and eigenvalues for the matrices in $\mathcal{A}_N$. In particular, we prove that the spectrum of $Q_W \in \mathcal{A}_N$ can be written as a linear transformation of its weight vector W. In Section 4, we show that, in case of at most three values for the weights, two weighted threshold graphs are isomorphic if their Laplacian matrices are cospectral, that is, they have the same eigenvalues.

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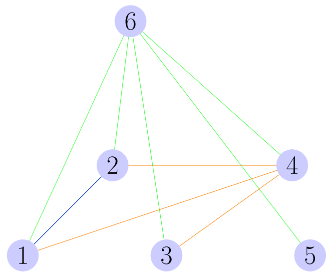

$$Q_W = \begin{bmatrix} 3-\sqrt{2} & -1 & 0 & \sqrt{2} & 0 & -2 \\ -1 & 3-\sqrt{2} & 0 & \sqrt{2} & 0 & -2 \\ 0 & 0 & 2-\sqrt{2} & \sqrt{2} & 0 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & 2-3\sqrt{2} & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ -2 & -2 & -2 & -2 & -2 & 10 \end{bmatrix}$$

FIGURE 2. *The threshold graph G_W coded by weight vector $W = (1, 0, -\sqrt{2}, 0,\ 2)$ and its Laplacian matrix Q_W.*

2. Structure of $\mathcal{A}_N$.

2.1. A basis. Consider the threshold graph G_{W_i} with N nodes and weight vector $W_i = (0, \ldots, 0, w_i = 1, 0, \ldots, 0)$, the corresponding Laplacian matrix $Q_i := Q_{W_i}$ is given by

$$(Q_i)_{mn} = \begin{cases} -1 & 1 \le n < m = i, \text{ or } 1 \le m < n = i, \\ 1 & 1 \le m = n < i, \\ i-1 & m = n = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

For any threshold graph G_W coded by $W = (w_2, w_3, \ldots, w_N)$, its Laplacian matrix Q_W can be represented by

$$Q_W = \sum_{i=2}^{N} w_i Q_i, \tag{2.5}$$

which shows that $\mathcal{A}_N = \operatorname{span}\{Q_2, \ldots, Q_N\}$. Clearly, the matrices $Q_2, \ldots, Q_N$ are independent, which implies that $\{Q_2, \ldots, Q_N\}$ is a basis for $\mathcal{A}_N$ which is an $(N-1)$-dimensional linear subspace of $\mathbb{R}^{N\times N}$, the vector space of real $N \times N$ matrices.

The Lapalcian matrix in Fig. 2 can be decomposed as:

$$Q = \textstyle\sum_{i=2}^{5} w_i Q_i = 1 \times \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + 0 \times \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\sqrt{2} \times \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + 0 \times \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + 1 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

2.2. The algebra $\mathcal{A}_N$. In this subsection, we establish that $\mathcal{A}_N$ is an $(N-1)$-dimensional commutative subalgebra of $\mathbb{R}^{N\times N}$. We start with some properties of the matrices Q_i defined in Eq. (2.4).

LEMMA 2.1.

$$Q_i Q_j = Q_j Q_i = Q_i \text{ for } 2 \le i < j \le N.$$

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Proof. When $i < j$, every row (or column) of Q_i is orthogonal with every column (or row) of $Q_j - I_N$. Therefore $Q_i(Q_j - I_N) = (Q_j - I_N)Q_i = O$, and hence $Q_iQ_j = Q_jQ_i = Q_i$. ☐

LEMMA 2.2.

$$Q_i^p = i^{p-1}Q_i - \frac{i^{p-1}-1}{i-1}\sum_{j=2}^{i-1} Q_j \quad \textit{for } 1 \le p,\ 2 \le i \le N. \tag{2.6}$$

Proof. If $p = 1$, then Eq. (2.6) is obvious. If $p = 2$, we have

$$\left(Q_i^2\right)_{mn} = \sum_{k=1}^{N} (Q_i)_{mk} (Q_i)_{kn} = \begin{cases} 1 & 1 \le m \ne n < i, \\ 2 & 1 \le m = n < i, \\ i(i-1) & m = n = i, \\ -i & 1 \le n < m = i \text{ or } 1 \le m < n = i, \\ 0 & \text{otherwise.} \end{cases}$$

For $(iQ_i - \sum_{j=2}^{i-1} Q_j)_{mn}$, we find the same value. Therefore, $Q_i^2 = iQ_i - \sum_{j=2}^{i-1} Q_j$.
Next, suppose that Eq. (2.6) holds. Using Lemma 2.1, we obtain

$$\begin{aligned} Q_i^{p+1} = Q_i^p Q_i &= \left(i^{p-1}Q_i - \frac{i^{p-1}-1}{i-1}\sum_{j=2}^{i-1} Q_j \right) Q_i = i^{p-1}Q_i^2 - \frac{i^{p-1}-1}{i-1}\sum_{j=2}^{i-1} Q_j \\ &= i^{p-1}\left(iQ_i - \sum_{j=2}^{i-1} Q_j \right) - \frac{i^{p-1}-1}{i-1}\sum_{j=2}^{i-1} Q_j = i^p Q_i - \frac{i^p-1}{i-1}\sum_{j=2}^{i-1} Q_j. \end{aligned}$$

Thus, Eq. (2.6) holds for $p \ge 3$ by induction. ☐

Recall that $Q_W \in \mathcal{A}_N$ with weight vector $W = (w_2, \dots, w_N)$ can be written as:

$$Q_W = \sum_{i=2}^{N} w_i Q_i. \tag{2.7}$$

Lemma 2.1 shows that any two matrices Q_i and Q_j commute, that is, $Q_iQ_j = Q_jQ_i$. Moreover, Lemma 2.1 and 2.2 show that $Q_iQ_j \in \mathcal{A}_N$ for $2 \le i, j \le N$. Using Eq. (2.7), it follows that any two matrices in $\mathcal{A}_N$ commute and that $\mathcal{A}_N$ is closed under multiplication. Thus, we have established that

THEOREM 2.3. *$\mathcal{A}_N$ is an $(N-1)$-dimensional commutative subalgebra of $\mathbb{R}^{N\times N}$.*

Take $Q_W, Q_{W'} \in \mathcal{A}_N$ with weight vectors $W = (w_2, \dots, w_N)$ and $W' = (w_2', \dots, w_N')$. Then $Q_W + Q_{W'} \in \mathcal{A}_N$ with weight vector $W + W'$, and similarly $aQ_W \in \mathcal{A}_N$ with weight vector aW for every $a \in \mathbb{R}$. Also, the product $Q_WQ_{W'} \in \mathcal{A}_N$, but more work is required to obtain the weight vector of $Q_WQ_{W'}$ from W and W'.

PROPOSITION 2.4. $Q_WQ_{W'} = \sum_{i=2}^{N} \left(iw_iw_i' - \sum_{j=i+1}^{N} \left(w_jw_j' - w_iw_j' - w_jw_i' \right) \right) Q_i.$

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Proof. Using Lemma 2.1, we start from Eq. (2.7),

$$Q_W Q_{W'} = \sum_{i=2}^{N} w_i w_i' Q_i^2 + \sum_{i=2}^{N-1}\sum_{j=i+1}^{N} \left(w_i w_j' + w_j w_i'\right) Q_i. \tag{2.8}$$

We invoke Lemma 2.2 and obtain

$$\sum_{i=2}^{N} w_i w_i' Q_i^2 = \sum_{i=2}^{N} i w_i w_i' Q_i - \sum_{i=2}^{N}\sum_{j=2}^{i-1} w_i w_i' Q_j. \tag{2.9}$$

Reversing the i and j sums in (2.9) yields

$$\sum_{i=2}^{N}\sum_{j=2}^{i-1} w_i w_i' Q_j = \sum_{j=2}^{N-1}\sum_{i=j+1}^{N} w_i w_i' Q_j = \sum_{i=2}^{N-1}\sum_{j=i+1}^{N} w_j w_j' Q_i. \tag{2.10}$$

Substituting (2.10) and (2.9) into (2.8) proves Proposition 2.4. ◻

3. An eigensystem. It is known that a set of mutually commuting symmetric matrices has a common basis of eigenvectors [8]. In this section, we derive these common eigenvectors for the matrices in $\mathcal{A}_N$. As a result, the eigenvalues of each $Q_W \in \mathcal{A}_N$ can be obtained by a linear transformation of its weight vector W. We have the following lemma on the eigenvalues and eigenvectors for each matrix Q_i defined in Eq. (2.4).

Lemma 3.1. *Let v_1 be the all-one vector* $\mathbf{1}$ *and for $2 \le j \le N$, we define the $N \times 1$ vector v_j as:*

$$\left(v_j\right)_k = \begin{cases} 1 & k < j, \\ 1-j & k = j, \\ 0 & k > j. \end{cases} \tag{3.11}$$

Then, $\{v_1, \cdots, v_N\}$ is an orthogonal basis of eigenvectors for each Q_i. The corresponding eigenvalues $Q_i v_j = \left(\mu_i\right)_j v_j$ are

$$\left(\mu_i\right)_j = \begin{cases} 1 & 2 \le j < i, \\ i & 2 \le j = i, \\ 0 & j > i \text{ or } j = 1. \end{cases} \tag{3.12}$$

and the characteristic polynomial is $P_{Q_i}(\lambda) = (\lambda - 1)^{i-2}(\lambda - i)\lambda^{N-i+1}$.

Proof. The vector $Q_i v_1$ equals the all-zero vector. Any two distinct vectors from $\{v_1, \cdots, v_N\}$ are orthogonal, that is, $v_l^T v_m = \delta_{lm}$ for $1 \le l, m \le N$. Next, we consider three cases depending on the relationship between i and j for each matrix Q_i.

Case 1: $2 \le j < i$. Each component $\left(Q_i v_j\right)_m$ of the vector $Q_i v_j$ equals

$$\begin{aligned}\left(Q_i v_j\right)_m &= \sum_{k=1}^{N} (Q_i)_{mk} (v_j)_k = \sum_{k=1}^{j} (Q_i)_{mk} (v_j)_k \\ &= \begin{cases} (Q_i)_{mm}(v_j)_m = (v_j)_m & m \le j, \\ \sum_{k=1}^{j} (Q_i)_{ik}(v_j)_k = -\sum_{k=1}^{N}(v_j)_k = 0 & m = i, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

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Thus, we deduce $Q_i v_j = v_j$ for each vector v_j with $2 \leq j < i$.
Case 2. $2 \leq j = i$. Each component $(Q_i v_i)_m$ of the vector $Q_i v_i$ equals

$$\begin{aligned}(Q_i v_i)_m &= \sum_{k=1}^{i} (Q_i)_{mk} (v_i)_k \\ &= \begin{cases} (Q_i)_{mm}(v_i)_m + (Q_i)_{mi}(v_i)_i = i = i(v_i)_m & m < i, \\ \sum_{k=1}^{i-1} (Q_i)_{ik}(v_i)_k + (Q_i)_{ii}(v_i)_i = i(1-i) = i(v_i)_i & m = i, \\ 0 & m > i. \end{cases}\end{aligned}$$

which indicates $Q_i v_i = i v_i$ for the vector v_i.
Case 3: $j > i$. Each component $(Q_i v_j)_m$ of the vector $Q_i v_j$ can be written as $(Q_i v_j)_m = \sum_{k=1}^{N} (Q_i)_{mk} (v_j)_k = \sum_{k=1}^{i} (Q_i)_{mk} (v_j)_k = \sum_{k=1}^{i} (Q_i)_{mk} = \sum_{k=1}^{N} (Q_i) = 0$ for $1 \leq m \leq N$. Consequently, we have $Q_i v_j = 0 v_j$ for each vector v_j with $j > i$. ◻

Now we compute the eigenvalues and eigenvectors of $Q_W \in \mathcal{A}_N$.

THEOREM 3.2. *Let G_W be a weighted threshold graph on N nodes with weight vector $W = (w_2, \ldots, w_N)$. The spectrum of the Laplacian Q_W of G_W is $\{0, \mu_2, \ldots, \mu_N\}$ with*

$$\mu_i = i w_i + \sum_{j=i+1}^{N} w_j. \tag{3.13}$$

This can be written in a matrix form as

$$\mu^T = (\mu_2, \ldots, \mu_N)^T = U W^T, \tag{3.14}$$

where U is an $(N-1) \times (N-1)$ upper triangular matrix defined by

$$U_{ij} = \begin{cases} 1 & 1 \leq i < j \leq N-1, \\ i+1 & 1 \leq i = j \leq N-1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.15}$$

Proof. Let v_j be a common eigenvector for $Q_2, \ldots, Q_N$ defined in Lemma 3.1. Then

$$Q_W v_j = \left(\sum_{i=2}^{N} w_i Q_i \right) v_j = j w_j v_j + \sum_{i=j+1}^{N} w_i v_j = \left(j w_j + \sum_{k=j+1}^{N} w_k \right) v_j,$$

for $j = 2, \ldots, N$. This proves Eq. (3.13), and Eq. (3.14) follows straightforwardly. ◻

For the graph coded by weight vector $W = (1, 0, -\sqrt{2}, 0,\ 2)$ in Fig. 2, Theorem 3.2 directly gives

$$\mu^T = U W^T = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4-\sqrt{2} \\ 2-\sqrt{2} \\ 2-4\sqrt{2} \\ 2 \\ 12 \end{bmatrix},$$

and the Laplacian spectrum is $\{0,\ 4-\sqrt{2},\ 2-\sqrt{2},\ 2-4\sqrt{2},\ 2,\ 12\}$.

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Independently, Anđelić and Stanic [9] also found a closed formula for the Laplacian eigenvalues of weighted threshold graphs. They define the weighted Ferrers diagram and then prove that the Laplacian eigenvalues of weighted threshold graphs are associated with the column sums of the corresponding weighted Ferrers diagrams. For more details, the readers are referred to their paper.

Theorem 3.2 indicates that weighted threshold graphs with integral weights have integral spectrum. Since U is an upper triangular matrix with nonzero diagonal, U is invertible, and it is straightforward to compute its inverse U^{-1}

$$U^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{12} & -\frac{1}{20} & \cdots & -\frac{1}{(N-1)N} \\ & \frac{1}{3} & -\frac{1}{12} & -\frac{1}{20} & \cdots & -\frac{1}{(N-1)N} \\ & & \frac{1}{4} & -\frac{1}{20} & \cdots & -\frac{1}{(N-1)N} \\ & & & \frac{1}{5} & \cdots & -\frac{1}{(N-1)N} \\ & & 0 & & \ddots & \vdots \\ & & & & & \frac{1}{N} \end{bmatrix},$$

and obtain $(w_2, \dots, w_N)^T = U^{-1}(\mu_2, \dots, \mu_N)^T$. As a result, we have the following corollary.

Corollary 3.3. *For any given real vector $\mu = (0, \mu_2, \dots, \mu_N)$, there exists a weighted threshold graph of order N, whose Laplacian spectrum is $\{0, \mu_2, \dots, \mu_N\}$.*

Corollary 3.3 establishes that the set of vectors $\{\mathbf{1}, v_j\}$ forms a totally ordered Soules basis [10, 11], which is a specially constructed orthonormal basis of eigenvectors for symmetric matrices, particularly Laplacian matrices of graphs. Given a prescribed spectrum μ and a Soules basis $\{r_n\}$, the matrix $Q = \sum_{i=1}^{N} \mu_i r_n r_n^T$ is the Laplacian matrix of an undirected, weighted graph. More importantly, if we ignore the actual link weights and look only at which links are present or absent, then Corollary 3.3 reveals that the underlying unweighted graphs arising from the Soules basis $\{\mathbf{1}, v_j\}$ are not arbitrary, but threshold.

A different ordering of $\mu_2, \dots, \mu_N$ gives a different weight vector, and therefore, a different weighted threshold graph with the same Laplacian spectrum.

The degree expression in Eq. (1.2) yields the following corollary.

Corollary 3.4. *Let G_W be a weighted threshold graph of order N, coded by a weight vector $W = (w_2, \dots, w_N)$. The Laplacian spectrum of G_W is given by $\{\mu_1 = 0, \mu_2, \dots, \mu_N\}$ with*

$$\mu_i = d_i + w_i, \text{ for } 2 \le 2 \le N,$$

where d_i denotes the weighted degree for the node i.

4. Cospectrality. Two matrices are called cospectral if they have the same eigenvalues. In the previous section, we observed that different weighted threshold graphs can have cospectral Laplacian matrices. We will show that the observation is not the case if the number of distinct weights is limited to three. We start with an observation about the Laplacian matrix of arbitrary weighted graphs (see [1], **Art.** 125).

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PROPOSITION 4.1. *Consider two weighted graphs of order N with cospectral Laplacian matrices Q and Q'. Then $aQ + b(J_N - NI_N)$ and $aQ' + b(J_N - NI_N)$ are also cospectral Laplacian matrices for any real numbers a and b, where J_N is the $N \times N$ all-one matrix and I_N is the identity matrix of order N.*

Proof. Let $v_1, \ldots, v_N$ be an orthogonal basis of eigenvectors for Laplacian Q corresponding to eigenvalues $\mu_1, \mu_2, \ldots, \mu_N$, such that $\mu_1 = 0$ and $v_1 = \mathbf{1}$ is the all-ones vector. Then $v_1, \ldots, v_N$ is also a set of eigenvectors for J_N with eigenvalues $N, 0, \ldots, 0$. Therefore, $aQ + b(J_N - NI_N)$ has eigenvalues $0, a\mu_2 - bN, \ldots, a\mu_N - bN$. Since Q' has the same spectrum as Q, we find by the same argument that $aQ' + b(J_N - NI_N)$ also has eigenvalues $0, a\mu_2 - bN, \ldots, a\mu_N - bN$. ∎

THEOREM 4.2. *Consider two threshold graphs G_W and $G_{W'}$ with N nodes and weight vectors W and W'. Suppose that all weights are taken from $\{x_1, x_2, x_3\}$. If G_W and $G_{W'}$ have cospectral Laplacian matrices, then G_W and $G_{W'}$ are isomorphic.*

Proof. For convenience. we first apply Proposition 4.1 and assume without loss of generality that $-1 = x_1 \leq x_2 \leq x_3 = 1$. Suppose G_W and $G_{W'}$ both have spectrum $\{0, \mu_2, \ldots, \mu_N\}$, and define $\mu_{max} = \max\{\mu_2, \ldots, \mu_N\}$ and $\mu_{min} = \min\{\mu_2, \ldots, \mu_N\}$. We apply Eq. (3.15). The matrix U has all row sums equal to N, no negative entries, and only positive entries in the last column. This implies that $U\mathbf{1} = N\mathbf{1}$, and $Uv < N\mathbf{1}$ for every vector $v < \mathbf{1}$ (recall that $\mathbf{1}$ is the all-one vector). Suppose $w_N = 1$. Then clearly $W^T \leq \mathbf{1}$, $\mu^T = UW^T \leq N\mathbf{1}$, and $\mu_N = N$. This implies that $\mu_{max} = N$. Next, suppose $w_N < 1$, then $W^T < \mathbf{1}$ and, therefore, $UW^T < N\mathbf{1}$; hence, $\mu_{max} < N$. Thus, we conclude that $w_N = 1$ if and only if $\mu_{max} = N$. Similarly, $w_N = -1$ if and only if $\mu_{min} = -N$. Therefore, $w_N = x_2$ if and only if $-N < \mu_{min}$ and $\mu_{max} < N$. The same holds for $G_{W'}$ and, hence, $w_N = w'_N$. Now, the result follows by deleting node N and applying induction. ∎

If the weights take more than three distinct values, then two weighted threshold graphs with cospectral Laplacian matrices need not be isomorphic as shown by the following two matrices (both have spectrum $\{0, 0, 6\}$):

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

If $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, then the considered weighted threshold graphs are signed graphs. Signed threshold graphs are studied in [12], where the name 'net-Laplacian' instead of 'Laplacian' is used. (because the Laplacian matrix of a signed graph has a different meaning). Two signed threshold graphs with cospectral net-Laplacian matrices must be isomorphic [12]. Our theorem generalizes the result in [12] to three possible weights with arbitrary real value.

Theorem 4.2 does not imply that in case of three weights, a weighted threshold graph is determined by its Laplacian spectrum. There could be a weighted graph with the same weights and the same Laplacian spectrum, which is not a threshold graph. However, we do not expect that this will happen. It is known [4] that unweighted threshold graphs are determined by their Laplacian spectrum. This means that any unweighted graph with the same Laplacian spectrum as a threshold graph G must be isomorphic to G. Proposition 4.1 shows that in the case of only two possible weights, a weighted threshold graph is also determined by its Laplacian spectrum. In particular, if all weights are ± 1 the matrix is known as the Seidel Laplacian, and thus we conclude that threshold graphs are determined by the spectrum of the Seidel Laplacian matrix.

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