Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

STABLE POLYNOMIALS VIA UNDIRECTED COLORED GRAPHS*

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Abstract. This paper initiates a systematic study of connections between undirected colored graphs and associated two-variable stable polynomials obtained via Cauchy transform-type formulas. Examples of such stable polynomials have played crucial roles in other recent studies, though their general properties have remained rather opaque. Using linear algebra techniques, this paper characterizes when these stable polynomials have boundary zeros and studies the finer behavior, called contact order, of their zero sets near the guaranteed boundary zeros.

Key words. Stable polynomials, Colored graphs, Rational inner functions, Contact order.

AMS subject classifications. Primary 32A08; Secondary 05C50; 32A40; 47A48.

1. Introduction.

1.1. Key idea. A two-variable polynomial p is said to be stable on a domain Ω if it has no zeros on Ω. In this paper, we will generally let Ω be the bidisk $\mathbb{D}^2$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For example, $q(z_1, z_2) = 2 - z_1 - z_2$ is stable on $\mathbb{D}^2$, though stable polynomials can be much more complicated. Their structure and properties are still being explored, see [10, 11, 13, 23, 24, 32], and they (and related objects) have appeared recently as key tools in a number of fields including probability theory, control theory, electrical engineering, combinatorics, graph theory, physics, and dynamical systems, see [8, 14, 15, 25, 26, 27, 28, 33].

In this paper, motivated by work of Pascoe in [30], we initiate a study of connections between stable polynomials and undirected, colored graphs. Specifically, we will start with a graph and use its adjacency matrix to construct a (generally quite nontrivial) two-variable stable polynomial; this construction method had yielded crucial examples in recent papers [11, 12, 13], though the properties of general polynomials obtained via this construction have not been previously studied.

We illustrate this construction with an example: starting with the 4-vertex colored graph given in Fig. 1, we can extract its adjacency matrix A and a matrix Y representing the coloring of blue on vertices $1-3$ and of red on vertex 4 (a blue coloring is also indicated by circle-shaped vertices and a red coloring is indicated by square-shaped vertices). Define the associated function

$$f(z_1, z_2) := \Big(A - z_1 Y - z_2(I - Y)\Big)^{-1}_{11} = \frac{z_1 + z_2 - z_1^2 z_2}{z_1^3 z_2 - 2z_1^2 - 2z_1 z_2}, \tag{1.1}$$

where we took the $(1,1)$ entry of the inverse of the matrix $A - z_1 Y - z_2(I - Y)$. Because A is self-adjoint and $0 \le Y \le I$ (where 0 and I are the zero and identity matrices, respectively), the function f maps $\mathbb{H}^2$ to $\mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$ is the upper-half plane. After composing f with conformal maps between $\mathbb{H}$ and $\mathbb{D}$ and extracting the denominator, one can obtain

*Received by the editors on June 14, 2025. Accepted for publication on March 3, 2026. Handling Editor: Ilya Spitkovsky. Corresponding Author: Kelly Bickel.

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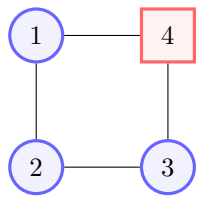
Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

FIGURE 1. *A graph with adjacency matrix A and coloring matrix Y.*

$$p(z_1, z_2) = 4 - z_2 - z_1 z_2 - 3z_1^2 z_2 + z_1^3 z_2, \tag{1.2}$$

which is a stable polynomial on $\mathbb{D}^2$. In general, this method yields exotic stable polynomials whose properties reflect the properties of the original graph. This construction and associated investigations fit into a larger body of ongoing research on connections between graph theory, linear algebra, and complex analysis, see, for example, [1, 6, 7, 20].

1.2. Background and general setup. The construction above is rooted in representation theory for two-variable holomorphic functions. In one variable, if $f : \mathbb{H} \to \overline{\mathbb{H}}$ is holomorphic and $\liminf_{y\to\infty} |yf(iy)| < \infty$, then Nevanlinna's representation theorem says that f is the Cauchy transform of a finite positive Borel measure μ on $\mathbb{R}$, namely:

$$f(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = \left\langle (A - zI)^{-1} 1, 1 \right\rangle_{L^2(\mu)},$$

where A is multiplication by the independent variable on $L^2(\mu)$ and 1 is the constant function.

In the two-variable setting, Agler, Tully-Doyle, and Young in [5, Theorem 1.6] obtained a useful generalization of Nevanlinna's representation theorem. They proved that for any holomorphic $f : \mathbb{H}^2 \to \overline{\mathbb{H}}$ with $\liminf_{y\to\infty} |f(iy, iy)y| < \infty$, there is a Hilbert space $\mathcal{H}$, a densely-defined, self-adjoint operator A on $\mathcal{H}$, a positive operator Y on $\mathcal{H}$ with $0 \le Y \le I$, and a vector $\alpha \in \mathcal{H}$ so that

$$f(z_1, z_2) = \left\langle (A - z_1 Y - z_2(I - Y))^{-1} \alpha, \alpha \right\rangle_{\mathcal{H}},$$

for $z \in \mathbb{H}^2$. A careful reading of Agler's earlier work [2] implies that if f is rational and real almost everywhere on $\mathbb{R}^2$, then we can take the Hilbert space $\mathcal{H}$ to be finite-dimensional, so that A can be viewed as an $n \times n$ self-adjoint matrix and Y as a positive semidefinite matrix. Then, we can write

$$f(z_1, z_2) = \alpha^\star (A - z_Y)^{-1} \alpha, \tag{1.3}$$

where $\alpha^\star$ is the conjugate transpose of α and z_Y is shorthand for $z_1 Y + z_2(I - Y)$. This is exactly the form of f in (1.1) with $\alpha = e_1$, the first standard basis vector of $\mathbb{R}^n$. To move to $\mathbb{D}^2$, one can use the following conformal maps from $\mathbb{D}$ to $\mathbb{H}$ and back:

$$\beta : \mathbb{H} \to \mathbb{D}, \text{ with } \beta(z) = \frac{1 + zi}{1 - zi} \quad \text{and} \quad \beta^{-1} : \mathbb{D} \to \mathbb{H}, \text{ with } \beta^{-1}(z) = i\left(\frac{1 - z}{1 + z}\right). \tag{1.4}$$

If f is defined as in (1.3) with $\alpha \in \mathbb{R}^n$, then $\phi := \beta \circ f \circ \beta^{-1}$ is a rational function holomorphic on $\mathbb{D}^2$ with $|\phi| = 1$ almost everywhere on $\mathbb{T}^2$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Such functions ϕ are called *rational inner functions* and have denominators that are stable polynomials on $\mathbb{D}^2$. Moreover, given a stable polynomial p on $\mathbb{D}^2$ with bidegree (m, n), its reflection polynomial $\tilde{p}$ is defined to be

$$\tilde{p}(z_1, z_2) = z_1^m z_2^n \overline{p\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)}, \tag{1.5}$$

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A publication of the International Linear Algebra Society
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and any rational inner function ϕ with denominator p is of the form

$$\phi(z_1, z_2) = \lambda z_1^k z_2^\ell \frac{\tilde{p}(z_1, z_2)}{p(z_1, z_2)}, \tag{1.6}$$

where $\lambda \in \mathbb{T}$ and $k, \ell \in \mathbb{N}$, see [31]. If ϕ is written so that its numerator and denominator have no common factors, then its stable polynomial denominator will also have no zeros on the faces of the bidisk $(\mathbb{D} \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{D})$, see Lemma 10.1 in [24]. Throughout this paper, when we refer to the denominator of a rational inner function, we will assume that any common factors between numerator and denominator have already been canceled.

Rational inner functions on $\mathbb{D}^2$ are the two-variable analogues of important one-variable functions called finite Blaschke products and play crucial roles in two-variable interpolation, approximation, and matrix monotonicity questions, see [3, 4, 31]. The behavior of rational inner functions near boundary singularities (the boundary zeros of their stable polynomial denominators) can be complicated and a number of properties (including derivative integrability, non-tangential polynomial approximations, and unimodular level set structures) have been studied in [10, 11, 13].

In this paper, we construct functions f as in (1.3) and associated rational inner functions ϕ by starting with an n-vertex, undirected, simple graph G with vertices $v_1, \ldots, v_n$, letting A denote its adjacency matrix, and letting Y be a given coloring matrix, defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between } v_i \text{ and } v_j \\ 0 & \text{if there is no edge between } v_i \text{ and } v_j \end{cases} \qquad \text{and} \qquad Y = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & t \end{bmatrix},$$

where $t \in [0, 1)$. Here, the matrix Y should be interpreted as coloring vertices v_1 through v_{n-1} with blue and vertex n with a color ranging from red ($t = 0$) to shades of purple ($0 < t < 1$). Other "colorings" could be studied using different choices of Y. We let $\alpha = e_1$ as before, which somewhat focuses our study on the first vertex of G.

For these choices of A, Y, α, we denote f in (1.3) by f_A^t and the associated function $\beta \circ f_A^t \circ \beta^{-1}$ by $\phi_A^t = \frac{q_A^t}{p_A^t}$ (with no common terms between numerator and denominator polynomials q_A^t and p_A^t). With the choice of $\alpha = e_1$, observe that

$$f_A^t(z_1, z_2) = \alpha^\star (A - z_1 Y - z_2(I - Y))^{-1} \alpha = (A - z_Y)^{-1}_{11}, \tag{1.7}$$

the $(1, 1)$-entry of the matrix inverse of $A - z_Y$. If $t = 0$, we generally drop the t-dependence and write f_A, ϕ_A, q_A, and p_A. We often denote the graph G with adjacency matrix A by G_A.

This construction was used to good effect in [10, 11, 12, 30], where it yielded key examples that illustrated various stable polynomial and rational inner function behavior. In this paper, we begin a study of the properties of the initial graph G_A and the resulting rational inner functions ϕ_A^t and their stable polynomial denominators p_A^t, which explains many of the behaviors observed in [10, 11, 12]. The interested reader should also consult the recent paper [1] by Adlin, Thai, Tiscarenoby, and Tully-Doyle, which explores the structure of the rational functions in (1.7) using a number of creative graph-theoretic and linear algebra techniques.

Other stable polynomial constructions appear in the literature. For example, in [16], Geronimo and Woerdeman give matrix positivity conditions that allow one to check whether a given polynomial is stable

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A publication of the International Linear Algebra Society
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and also provide an algorithm for producing stable p with some specified zeros. Meanwhile, in [13, Theorem 2.19], the authors construct examples of stable polynomials with particular Puiseux expansions at a specified boundary zero. Recently, in [32], Sola constructs stable polynomials on the polydisk with certain behaviors at boundary zeros using iterates of rational inner functions. In all constructions, the key point of concern is whether the polynomial has zeros on $\mathbb{T}^2$ and how the polynomial behaves near such boundary zeros.

1.3. Main results and paper outline. In this paper, we assume that we are starting with an undirected, simple graph G_A with vertices $v_1, \ldots, v_n$ and then investigate the connections between the graph G_A and the associated stable polynomials p_A and p_A^t for $t \in (0,1)$. Specifically, in Section 2, we address the first natural question one should ask when given a class of stable polynomials on $\mathbb{D}^2$:

Question 1: When does a stable polynomial in the class have a boundary zero?

As encoded in the following result, the basic structure of the graph G_A immediately tells us whether p_A depends on both variables and whether it has a zero on $\mathbb{T}^2$:

THEOREM 1.1. *Let G_A be an n-vertex graph and let p_A be the associated stable polynomial. Then the following hold:*

i. If v_1 is isolated, then p_A is constant.
ii. If v_1 is neither isolated nor path connected to v_n, then $p_A \in \mathbb{C}[z_1]$ is not constant and has no zeros on $\mathbb{T}^2$.
iii. If v_1 and v_n are path connected, then $p_A \in \mathbb{C}[z_1, z_2]$ depends on z_2 and $p_A(-1,1) = 0$.

Subsection 2.1 includes some straightforward but useful lemmas. Then, Theorem 1.1 is proved in Subsection 2.2 as a sequence of shorter results.

Graphs illustrating the three cases in Theorem 1.1 are given in Fig. 2. As an example of Theorem 1.1, one can check that the polynomial p from (1.2) satisfies $p(-1,1) = 0$. Additionally, note that any p_A must satisfy exactly one of these conditions: p_A is constant, $p_A \in \mathbb{C}[z_1]$ is not constant, and $p_A \in \mathbb{C}[z_1, z_2]$ has some z_2 dependence. Thus, Theorem 1.1 gives these double implications: p_A is constant if and only if v_1 is isolated, p_A has only z_1 dependence if and only if v_1 is not isolated and not path connected to v_n, and p_A has z_2 dependence if and only if v_1 and v_n are path connected.

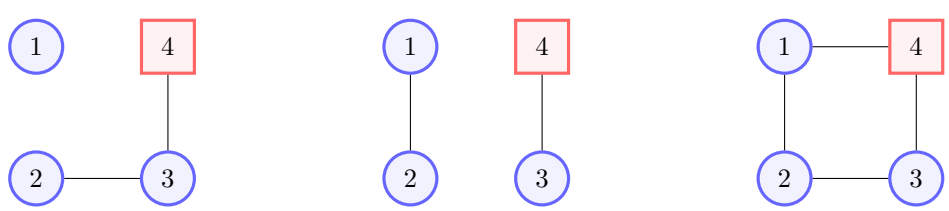

FIGURE 2. *Examples of the three situations in Theorem 1.1 with $n = 4$.*

Because p_A is the denominator of a rational inner function, if it has a zero on $\mathbb{T}^2$, then it must have both z_1 and z_2 dependence. Thus, Theorem 1.1 implies that if p_A has any zero on $\mathbb{T}^2$, then it must possess some z_2 dependence and so be zero at $(-1,1)$. Many polynomials in the class under consideration can have additional boundary zeros; for example, the polynomial in (1.2) also satisfies $p(1,1) = 0$. Classes of special special graphs and their polynomials were studied by the second author in [20], where such additional boundary zeros were identified.

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Although Theorem 1.1 is stated when $t = 0$ (when the last vertex is colored red), change-of-variables arguments give the following:

Corollary 1.2. *Let G_A be an n-vertex graph and let p_A^t be the associated stable polynomial with $t \in (0,1)$. If v_1 and v_n are path connected, then $p_A^t(-1,-1) = 0$. If v_1 and v_n are not path connected, then p_A^t has no boundary zeros.*

For example, using the graph in Fig. 1 with $t \in (0,1)$ gives the stable polynomial

$$\begin{aligned} p_A^t(z_1,z_2) = 4 + 4z_1 - 5tz_1 - tz_1^2 - 3tz_1^3 + tz_1^4 - z_2 + 5tz_2 - 2z_1z_2 + tz_1z_2 \\ - 4z_1^2z_2 + 3tz_1^2z_2 - 2z_1^3z_2 - tz_1^3z_2 + z_1^4z_2, \end{aligned}$$

which satisfies $p_A^t(-1,-1) = 0$. Corollary 1.2 is proved in Subsection 2.3. In that subsection, we also track how general boundary zeros of p_A and p_A^t are related, see Proposition 2.9.

The second natural question one should ask when given a class of stable polynomials with boundary zeros is:

Question 2: How do the polynomials' zero sets behave near the guaranteed boundary zero(s)?

For stable polynomials on $\mathbb{D}^2$, their zero set behavior is often described using a positive, even integer called *contact order*, which measures how the function's zero set approaches $\mathbb{T}^2$. As proved in [10, 11, 13], the contact order K governs the behavior of the associated rational inner function ϕ's unimodular level sets, derivative integrability, and local non-tangential polynomial approximations. While most stable polynomials (for example, the one in (1.2)) have contact order $K = 2$ at their boundary zeros, this is not always true.

In Section 3, we study the contact order of stable polynomials p_A^t constructed from graphs. Subsection 3.1 gives important information about defining, studying, and computing contact order. Meanwhile, Subsection 3.2 notes that for *most* boundary zeros of p_A, the associated zero of p_A^t has the same contact order; this appears as Proposition 3.3.

In Subsection 3.3, we consider polynomials constructed from *path graphs*, which are graphs where each vertex is only connected to the following vertex. For example, the graph in Fig. 1 would be a path graph if we removed the edge between v_1 and v_4. We prove the following:

Theorem 1.3. *Let G_A be an n-vertex path graph. Then, p_A has contact order $K = 2(n-1)$ at its boundary zero $(-1,1)$.*

Additionally, as part of the proof, we obtain a concrete formula for f_A. This result also aligns with the work of Pascoe in [30]. Specifically, the final example in that paper shows that rational functions (1.7) constructed with path graphs and $t \in (0,1)$ have particular non-tangential regularity at infinity. Later work in [10, 13] showed that this non-tangential regularity measures the contact order (in this case, $K = 2n$) of the associated stable polynomial at $(-1,-1)$. So, a $t \neq 0$ version of Theorem 1.3 follows from earlier work of Pascoe and collaborators. Our Theorem 1.3 shows that a similar result (with a drop in contact order) holds when $t = 0$ and aligns with what appeared in examples in [10, 13].

For path graphs, $n-1$ is also the length of the shortest path connecting v_1 and v_n. We conjecture that this quantity more generally controls the contact order of p_A^t at the guaranteed boundary zero $(-1,1)$ for $t = 0$ and $(-1,-1)$ for $t \in (0,1)$.

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Conjecture 1.4. *Let G_A be an n-vertex graph and let ℓ denote the length of the shortest path connecting vertices v_1 and v_n in G_A. If $t = 0$, then p_A has contact order 2ℓ at $(-1,1)$ and if $t \in (0,1)$, then p_A^t has contact order $2\ell + 2$ at $(-1,-1)$.*

To investigate this conjecture, in Subsection 3.4, we study how the contact order of p_A at $(-1,1)$ changes when we append additional vertices and edges to the graph G_A. The results are summarized in the following theorem:

Theorem 1.5. *Let G_A be an n-vertex graph such that p_A has contact order K at $(-1,1)$.*

i. Create $G_{\widehat{A}}$ from G_A by adding a vertex v_{n+1} that is only connected directly to v_n. Then, $p_{\widehat{A}}$ has contact order $K+2$ at $(-1,1)$.
ii. Create $G_{\widetilde{A}}$ from G_A by adding a vertex v_0 that is only connected directly to v_1 (and then renumbering to $v_1, \ldots, v_{n+1}$). Then, $p_{\widetilde{A}}$ has contact order $K+2$ at $(-1,1)$.
iii. Assume that v_n is only connected to the rest of G_A by an edge to v_{n-1}. For $m \geq n+1$, create an m-vertex graph $G_{\check{A}}$ from G_A by attaching an $(m-n)$-vertex graph G' to G_A via an edge to v_{n-1} and then relabeling v_n to v_m. Then, $p_{\check{A}}$ has contact order K at $(-1,1)$.

Note that all three situations in Theorem 1.5 support Conjecture 1.4. In the first two situations, we extend the length of the shortest path connecting the first and last vertices by 1; thus, it makes sense that the contact order increases by 2. In the last situation, the length of the shortest path connecting the first and last vertices does not change, so we would expect the contact order to remain the same. Theorem 1.5 is proved as three separate results (Theorem 3.5, Theorem 3.6, and Theorem 3.7), where Theorem 3.7 also allows the option of constructing $G_{\check{A}}$ by attaching G' to both v_{n-1} and v_n.

It is worth noting that since the completion of this work, the $t = 0$ part of Conjecture 1.4 has been resolved by Adlin, Thai, Tiscarenoby, and Tully-Doyle using very interesting and efficient methods in the new paper [1]. The $t \neq 0$ case appears to still be open.

Finally, Section 4 contains a short list of open questions related to stable polynomials and graphs. Of particular interest to the authors are investigations that go in the other direction: given a stable polynomial or Cauchy transform function associated to a colored graph, what can one say about the original graph? To the best of our knowledge, little in that direction is currently known.

2. Existence of boundary zeros.

2.1. Key lemmas. In this section, we record some facts about inverses, determinants, and the functions f_A^t defined in (1.7) that will be used later.

Lemma 2.1. *If A is a square matrix and $\overline{A}$ is the matrix obtained from A by removing its first row and column, then the $(1,1)$ entry of A^{-1} is given by*

$$(A^{-1})_{11} = \frac{\det \overline{A}}{\det A}.$$

This is an immediate consequence of the adjugate formula for A^{-1}. We will also require the following.

Lemma 2.2. *Let M be a square matrix, partitioned as*

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

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where A and D are square matrices and A^{-1} and D^{-1} exist. Then,

$$\det(M) = \det(A)\det(D - CA^{-1}B) = \det(D)\det(A - BD^{-1}C). \tag{2.8}$$

This is well known and can be found for example in (50) in Example 47 of [22]. We lastly record the following fact about the functions f_A^t from (1.7):

LEMMA 2.3. *Let G_A be an n-vertex graph with $n \geq 3$. Fix $1 < i, j < n$ and let $G_{A'}$ be the graph obtained from G_A by switching v_i and v_j. Then, for $t \in [0, 1)$, $f_A^t = f_{A'}^t$.*

Proof. Let U be the unitary matrix obtained from the $n \times n$ identity matrix I by switching its ith and j^{th} columns and note that $U^{-1} = U$. Then, the adjacency matrices satisfy $A' = UAU$, and the formula for z_Y gives $z_Y = Uz_YU$. Thus,

$$(A' - z_Y)^{-1} = (UAU - z_Y)^{-1} = U(A - Uz_YU)^{-1}U = U(A - z_Y)^{-1}U.$$

As the application of U does not alter the $(1, 1)$ entry of $(A - z_Y)^{-1}$, (1.7) implies that $f_A^t = f_{A'}^t$. ∎

2.2. Proof of Theorem 1.1. This section provides the proof of Theorem 1.1. We first establish parts (i) and (ii) as the following proposition. The functions $\beta, \beta^{-1}, f_A, \phi_A$, and p_A are defined as in the introduction with $t = 0$.

PROPOSITION 2.4. *If v_1 and v_n are not path connected, then $p_A \in \mathbb{C}[z_1]$. Furthermore, if v_1 is isolated, then p_A is constant, and if v_1 is not isolated, then p_A is not constant.*

Proof. By Lemma 2.3, we can reorder the vertices of G_A so that $v_1, \ldots, v_m$ are exactly the vertices in the path-connected component of G_A containing v_1. Then, A can be partitioned as

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & \star \end{array}\right], \quad \text{so} \quad (A - z_Y)^{-1} = \left[\begin{array}{c|c} (A_1 - z_1 I)^{-1} & 0 \\ \hline 0 & \star \end{array}\right],$$

where A_1 is an $m \times m$ matrix, 0 and I denote the zero and identity matrices of the appropriate size, and $\star$ denotes matrices immaterial to our calculations. From this, it is clear that the $(1, 1)$ entry of $(A - z_Y)^{-1}$, and hence f_A, depends only on z_1, so $p_A \in \mathbb{C}[z_1]$. Furthermore if $m = 1$, then $f_A = -\frac{1}{z_1}$, which implies $\phi_A = -z_1$ and p_A is constant. Moreover, the rational inner function form from (1.6) implies that if f_A only depends on z_1 and p_A is constant, then $\phi_A = \alpha z_1^k$, for some $\alpha \in \mathbb{T}$, $k \in \mathbb{N}$. That implies

$$f_A(z_1) = \beta^{-1} \circ \phi_A \circ \beta = -i + \frac{2i}{1 + \alpha(-1 + \frac{2i}{i+z_1})^k},$$

and if we expand this around ∞ for z_1 sufficiently large, we get

$$f_A(z_1) = \left(-i + \frac{2i}{\alpha(-1)^k + 1}\right) - \frac{4(-1)^k \alpha k}{(1 + (-1)^k \alpha)^2} \frac{1}{z_1} + \text{higher order terms.} \tag{2.9}$$

Meanwhile, we can write f_A using a Neumann series (for sufficiently large z_1) as

$$\begin{aligned} f_A(z_1) &= (A_1 - z_1 I)_{11}^{-1} = -\frac{1}{z_1}(I - \tfrac{1}{z_1}A_1)_{11}^{-1} \\ &= \sum_{n=0}^{\infty} -\frac{1}{z_1^{n+1}}(A_1^n)_{11} = 0 - \frac{1}{z_1} + \text{higher order terms.} \end{aligned}$$

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Setting the first coefficient in (2.9) to 0 and the second to -1 implies that $\alpha = -1$ and $k = 1$. Thus, $f_A = -\frac{1}{z_1}$ is the only function in our paradigm yielding a constant p_A.

Now, assume v_1 is not isolated. Thus, v_1 is connected to some other vertex v_j and so there are paths of length 2 that begin and end at v_1. Because $(A^2)_{11}$ is the number of paths of length 2 that begin and end at v_1, we know $(A^2)_{11} \neq 0$. Thus, by the above expansion of f_A, if v_1 is not isolated, then $f_A \neq -\frac{1}{z_1}$, which means p_A is not constant. ∎

To prove part (iii) of Theorem 1.1, we require the following lemma, which gives us an easy way to use f_A to identify when p_A has a boundary zero at $(-1, 1)$.

Lemma 2.5. *Define h_A by $h_A(z_1, z_2) = f_A(-\frac{1}{z_1}, z_2)$ and write $h_A = \frac{q_1}{q_2}$, where q_1, q_2 are polynomials with no common factors. If $q_1(0,0) = 0 = q_2(0,0)$, then $p_A(-1,1) = 0$.*

Proof. Note that for almost every $(z_1, z_2) \in \mathbb{C}^2$, ϕ_A satisfies

$$\begin{aligned}\phi_A(z_1, z_2) &= \beta \circ h_A\left(\frac{-1}{\beta^{-1}(z_1)}, \beta^{-1}(z_2)\right)\\ &= \frac{q_2\left(-\frac{1}{\beta^{-1}(z_1)}, \beta^{-1}(z_2)\right) + iq_1\left(-\frac{1}{\beta^{-1}(z_1)}, \beta^{-1}(z_2)\right)}{q_2\left(-\frac{1}{\beta^{-1}(z_1)}, \beta^{-1}(z_2)\right) - iq_1\left(-\frac{1}{\beta^{-1}(z_1)}, \beta^{-1}(z_2)\right)}.\end{aligned}$$

Multiply through by the lowest powers of $(z_1 - 1)$ and $(z_2 + 1)$ to turn both the numerator and denominator into polynomials and denote those polynomials by r_A, s_A. Then,

$$\phi_A(z_1, z_2) = \frac{q_A(z_1, z_2)}{p_A(z_1, z_2)} = \frac{r_A(z_1, z_2)}{s_A(z_1, z_2)},$$

for almost every $(z_1, z_2) \in \mathbb{C}^2$. By way of contradiction, assume $p_A(-1, 1) \neq 0$. By our assumptions, $r_A(-1, 1) = 0 = s_A(-1, 1)$, which implies that r_A, s_A must share a common, non-constant factor. This implies that q_1, q_2 share a common curve of zeros as well and, hence, a common factor, which gives the contradiction. ∎

Now we can prove part (iii) of Theorem 1.1.

Proposition 2.6. *If v_1 and v_n are path connected, then $p_A \in \mathbb{C}[z_1, z_2]$ depends on z_2 and $p_A(-1, 1) = 0$.*

Proof. Assume that v_1 and v_n are path connected. Define the submatrices A_1, A_2, A_3 and vectors $\alpha, \gamma, \zeta, \eta, b$ of the appropriate size, so that

$$A = \left[\begin{array}{c|c} 0 & \alpha^T \\ \hline \alpha & A_1 \end{array}\right] = \left[\begin{array}{c|c} A_2 & \gamma \\ \hline \gamma^T & 0 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & \zeta^T & b \\ \hline \zeta & A_3 & \eta \\ \hline b & \eta^T & 0 \end{array}\right].$$

Then using Lemma 2.1 and Lemma 2.2, we obtain

$$f_A(z_1, z_2) = \frac{\det(A_1 - z_Y)}{\det(A - z_Y)} = \frac{\det(A_3 - z_1 I)(-z_2 - \eta^T(A_3 - z_1 I)^{-1}\eta)}{\det(A_2 - z_1 I)(-z_2 - \gamma^T(A_2 - z_1 I)^{-1}\gamma)}.$$

With an eye toward using Lemma 2.5, replace z_1 with $-\frac{1}{z_1}$ and multiply by $\frac{z_1^{n-1}}{z_1^{n-1}}$ to obtain

$$h_A(z_1, z_2) := f_A(-\tfrac{1}{z_1}, z_2) = \frac{z_1 \det(z_1 A_3 + I)(-z_2 - z_1\eta^T(z_1 A_3 + I)^{-1}\eta)}{\det(z_1 A_2 + I)(-z_2 - z_1\gamma^T(z_1 A_2 + I)^{-1}\gamma)}. \tag{2.10}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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We claim that the factors containing z_2 in the numerator and denominator in (2.10) cannot cancel out. By way of contradiction, assume that they do. Then, for all z_1,

$$\gamma^T (z_1 A_2 + I)^{-1} \gamma = \eta^T (z_1 A_3 + I)^{-1} \eta.$$

For small z_1, the Neumann series for $-z_1 A_1$ and $-z_1 A_2$ converge and so we can conclude

$$\sum_{m=0}^{\infty} (-z_1)^m \gamma^T A_2^m \gamma = \gamma^T (z_1 A_2 + I)^{-1} \gamma = \eta^T (z_1 A_3 + I)^{-1} \eta = \sum_{m=0}^{\infty} (-z_1)^m \eta^T A_3^m \eta,$$

or more specifically, for all $m \geq 0$,

$$\gamma^T A_2^m \gamma = \eta^T A_3^m \eta. \tag{2.11}$$

Setting $m = 0$, this immediately gives $\gamma^T \gamma = \eta^T \eta$, so $b = 0$. This implies that v_1 and v_n are not connected by a single edge. But since they are path connected, v_1 must be path connected to some other vertex through a path that does not include v_n, such that this intermediate vertex is directly connected to v_n. By Lemma 2.3, we can assume that this intermediate vertex is v_{n-1}.

Let M denote the length of a path between v_1 and v_{n-1} in G_A that does not go through v_n. By well-known properties of adjacency matrices, $k := (A_2)^M_{1,n-1}$ gives the number of paths between v_1 and v_{n-1} of length M in the graph obtained by removing v_n from G_A. Then, we can write

$$A_2^M = \left[\begin{array}{c|cc} c & \epsilon^T & k \\ \hline \epsilon & \multicolumn{2}{c}{\multirow{2}{*}{$B + A_3^M$}} \\ k & & \end{array}\right],$$

where $k > 0$, ϵ is a vector, and B is a matrix, both with real, nonnegative entries. Then, since $k > 0$, $b = 0$, and $\eta_{n-2} = 1$ because v_{n-1} and v_n are path connected, we have

$$\gamma^T A_2^{2M} \gamma = \eta^T \left(\begin{bmatrix} \epsilon \\ k \end{bmatrix} \begin{bmatrix} \epsilon^T & k \end{bmatrix} + (B + A_3^M)^2 \right) \eta > \eta^T A_3^{2M} \eta,$$

which contradicts (2.11). Thus, the factors in the numerator and denominator that involve z_2 in (2.10) do not cancel.

Since those factors vanish at $(0,0)$, we can write $h_A = \frac{q_1}{q_2}$ where q_1 and q_2 are polynomials with no common factors and $q_1(0,0) = 0 = q_2(0,0)$. Thus, by Lemma 2.5, $p_A(-1,1) = 0$. Since $p_A(-1,1) = 0$, it follows immediately that p_A must have both z_1 and z_2-dependence, since if ϕ_A were one variable, it would be a finite Blaschke product, and those do not have singularities on $\mathbb{T}$. ∎

2.3. Boundary zeros for nonzero t. In this subsection, we track how the location of boundary zeros of p_A^t depends on the coloring parameter t. This allows us to use the $t = 0$ case to derive results when $t \in (0,1)$ and, in particular, prove Corollary 1.2.

We require this initial lemma. (Note the superscript n in z_j^n and w_j^n below represents an index and not a power.)

Lemma 2.7. *Let ϕ be a rational inner function on $\mathbb{D}^2$ with stable polynomial denominator p (after any common factors are canceled). Then,*

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A publication of the International Linear Algebra Society
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i. *If $p(\tau_1,\tau_2)=0$ for $(\tau_1,\tau_2)\in\mathbb{T}^2$, then for almost every $\eta\in\mathbb{T}$, there is a sequence $z_n=(z_1^n,z_2^n)\subset\mathbb{T}^2$, with each $z_1^n\neq\tau_1$, each $z_2^n\neq\tau_2$, each $\phi(z_1^n,z_2^n)=\eta$, and $(z_1^n,z_2^n)\to(\tau_1,\tau_2)$.*
ii. *Assume that there are distinct $\eta_1,\eta_2\in\mathbb{T}$ with corresponding sequences $z_n=(z_1^n,z_2^n)\subset\mathbb{T}^2$ and $w_n=(w_1^n,w_2^n)\subset\mathbb{T}^2$ such that (z_1^n,z_2^n) and (w_1^n,w_2^n) both converge to $(\tau_1,\tau_2)\in\mathbb{T}^2$ and $\phi(z_1^n,z_2^n)=\eta_1$ and $\phi(w_1^n,w_2^n)=\eta_2$ for all n. Then $p(\tau_1,\tau_2)=0$.*

Proof. For (i), note that since ϕ is a rational inner function, $p(\tau_1,\tau_2)=0$ implies that ϕ does not have a continuous unimodular extension to a neighborhood of (τ_1,τ_2) in $\mathbb{T}^2$; this would imply that ϕ actually extended analytically to a neighborhood of (τ_1,τ_2) in $\mathbb{C}^2$, see Theorem 1.5 in [9], which in turn would contradict the fact that the numerator and denominator of ϕ have no common factors. Then, the statement about level set sequences follows from an application of Corollary 1.7 in [29] and the description of these level sets (in terms of non-constant analytic functions for almost every η) in Theorem 2.8 in [11]. For (ii), note that the assumptions imply that ϕ is discontinuous at (τ_1,τ_2) and since ϕ is rational, its denominator must vanish at (τ_1,τ_2). ∎

Now we study the correspondence between boundary zeros in the $t=0$ case and in the $t\neq 0$ case. We first study the boundary zero whose existence is often guaranteed by Theorem 1.1.

Proposition 2.8. *Using the notation defined earlier, if $p_A(-1,1)=0$, then $p_A^t(-1,-1)=0$.*

Proof. Assume $p_A(-1,1)=0$. By Lemma 2.7, there exist distinct $\eta_1,\eta_2\in\mathbb{T}$ with $\eta_1,\eta_2\neq -1$ that possess corresponding sequences $z_n=(z_1^n,z_2^n)$ and $w_n=(w_1^n,w_2^n)$ in $\mathbb{T}^2$ such that $z_n,w_n\to(-1,1)$ with $\phi_A(z_1^n,z_2^n)=\eta_1$, $\phi_A(w_1^n,w_2^n)=\eta_2$, and $z_1^n,w_1^n\neq -1$ for each n. Recall the definitions of β,β^{-1} from (1.4) and define sequences $r_n,s_n\subseteq\mathbb{R}^2$ by

$$r_n=(r_1^n,r_2^n)=(\beta^{-1}(z_1^n),\beta^{-1}(z_2^n))\quad\text{and}\quad s_n=(s_1^n,s_2^n)=(\beta^{-1}(w_1^n),\beta^{-1}(w_2^n)),$$

and note that $r_n,s_n\to(\beta^{-1}(-1),\beta^{-1}(1))=(\infty,0)$, in the sense that the sequences with elements $|\beta^{-1}(r_n^1)|$ and $|\beta^{-1}(s_n^1)|$ diverge to infinity. Set $t_1=\beta^{-1}(\eta_1)\in\mathbb{R}$ and $t_2=\beta^{-1}(\eta_2)\in\mathbb{R}$. Since $f_A=\beta^{-1}\circ\phi_A\circ\beta$, we have $f_A(r_n)=t_1$, $f_A(s_n)=t_2$ for all n.

Now we shift these facts to the $t\in(0,1)$ setting. By the definition from (1.7), we have

$$f_A^t(z_1,z_2)=f_A\big(z_1,tz_1+(1-t)z_2\big).$$

Construct a sequence $\hat{r}_n\subseteq\mathbb{R}^2$ from r_n by

$$\hat{r}_1^n=r_1^n,\quad \hat{r}_2^n=\frac{1}{1-t}(r_2^n-tr_1^n),$$

and construct $\hat{s}_n$ from s_n in an identical way. Then, $f_A^t(\hat{r}_n)=f_A(r_n)=t_1$, $f_A^t(\hat{s}_n)=f_A(s_n)=t_2$ for all n, and $\hat{r}_n,\hat{s}_n\to(\infty,\infty)$.

To convert this to the ϕ_A^t setting, consider the sequences $\hat{z}_n=(\beta(\hat{r}_n^1),\beta(\hat{r}_n^2))$ and $\hat{w}_n=(\beta(\hat{s}_n^1),\beta(\hat{s}_n^2))$ in $\mathbb{T}^2$. Then, $\hat{z}_n,\hat{w}_n\to(-1,-1)$ and $\phi_A^t(\hat{z}_1^n,\hat{z}_2^n)=\eta_1$, $\phi_A^t(\hat{w}_1^n,\hat{w}_2^n)=\eta_2$ for all n. An application of Lemma 2.7 gives $p_A^t(-1,-1)=0$, as needed. ∎

We can now prove Corollary 1.2.

Proof. Let vertices v_1 and v_n be path connected. Then, Theorem 1.1 implies that $p_A(-1,1)=0$ and so Proposition 2.8 tells us that $p_A^t(-1,-1)=0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

Conversely, if v_1 and v_n are not path connected, then arguments identical to those at the beginning of Proposition 2.4 imply that f_A^t (and hence ϕ_A^t) only depends on z_1. Then, ϕ_A^t is a finite Blaschke product, which cannot have any boundary singularities, and so its denominator p_A^t cannot have any boundary zeros.□

It is worth noting that one can also use the previous change-of-variables arguments to move between general boundary zeros of p_A and general boundary zeros of p_A^t.

Proposition 2.9. *Using the notation defined earlier, for any $\tau_1, \tau_2 \in \mathbb{T}$ with $\tau_1, \tau_2 \neq -1$, $p_A(\tau_1, \tau_2) = 0$ if and only if $p_A^t(\lambda_1, \lambda_2) = 0$ where $\lambda_1 = \tau_1$ and $\lambda_2 = \beta\left(\frac{\beta^{-1}(\tau_2) - t\beta^{-1}(\tau_1)}{1-t}\right)$.*

Proof. The forward direction is proved in a way analogous to that of Proposition 2.8.

For the backwards direction, the restriction that $\tau_1, \tau_2 \neq -1$ means that the limits of the sequences $\hat{r}_n, \hat{s}_n$ also determine the limits of the r_n, s_n sequences (since none of the sequences diverge to ∞). Then, we can simply reverse the steps in the proof Proposition 2.8. □

3. Contact order.

3.1. Computing contact order. Let ϕ be a rational inner function with stable polynomial denominator p and a boundary singularity at $(1,1)$. Recall the definition of $\tilde{p}$ from (1.5). Then, the contact order of ϕ at $(1,1)$ (or equivalently, the contact order of p at $(1,1)$) is a positive even integer K that measures how the zero set of $\tilde{p}$ approaches $(1,1)$ from within the face $\mathbb{T} \times \mathbb{D}$ of the bidisk. Specifically, K is the number satisfying

$$\inf\{1 - |z_2| : \tilde{p}(z_1, z_2) = 0\} \approx |1 - z_1|^K,$$

for $z_1 \in \mathbb{T}$ and $z_2 \in \mathbb{D}$ both sufficiently close to 1. More generally, if ϕ has a singularity at $(\tau_1, \tau_2) \in \mathbb{T}^2$, then the contact order of ϕ at (τ_1, τ_2) is just the contact order of $\psi(z_1, z_2) := \phi(\tau_1 z_1, \tau_2 z_2)$ at $(1,1)$.

This bidisk definition translates to a definition of contact order on the bi-upper half plane $\mathbb{H}^2$. Specifically, recalling the conformal map $\beta : \mathbb{H} \to \mathbb{D}$ given in (1.4), we can define a related polynomial

$$r(w_1, w_2) = (1 - iw_1)^m (1 - iw_2)^n p\left(\beta(w_1), \beta(w_2)\right). \tag{3.12}$$

Then, the relationship between r, p, and $\tilde{p}$ implies that K is also the number satisfying

$$\inf\{|\mathrm{Im}\,(w_2)| : r(x_1, w_2) = 0\} \approx |x_1|^K, \tag{3.13}$$

for $x_1 \in \mathbb{R}$ $w_2 \in -\mathbb{H}$ both sufficiently close to 0.

In this paper, the graph construction implies that the highest power of z_2 appearing in our rational inner functions is at most 1. When ϕ has a singularity on $\mathbb{T}^2$, the highest power of z_2 in p (and hence in ϕ) cannot be 0 and so must equal 1. In that case, there is a particularly easy way to evaluate contact order on the bi-upper half plane.

Remark 3.1. Let $\deg p = (n, 1)$ and consider f so that $\phi = \beta \circ f \circ \beta^{-1}$. Then, there is a polynomial Q such that

$$\beta \circ f = \phi \circ \beta = \frac{Q}{r},$$

where r is defined in (3.12) and Q and r have no common factors. Clearly, $p(1,1) = 0$ implies that $r(0,0) = 0$. Now find polynomials $r_1, r_2 \in \mathbb{C}[w]$ such that

$$r(w_1, w_2) = r_1(w_1) + w_2 r_2(w_1) = r_2(w_1)\left(w_2 + \frac{r_1(w_1)}{r_2(w_1)}\right).$$

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A publication of the International Linear Algebra Society
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If $r_2(0) = 0$, then since $r_1(0) = 0$, r would vanish along the line $w_1 = 0$, which cannot happen because p can only have a finite number of zeros on $\mathbb{T}^2$. Thus, $r_2(0) \neq 0$, which means that we can expand the function $\frac{r_1}{r_2}$ in a power series centered at $w_1 = 0$. Using (3.13), we can then conclude that the contact order K of ϕ at $(1,1)$ is also the number that satisfies

$$K = \min\left\{k \in \mathbb{N} : \operatorname{Im}\left(\left(\tfrac{r_1}{r_2}\right)^{(k)}(0)\right) \neq 0\right\}.$$

Alternatively, let $\bar{r}_2$ denote the polynomial whose coefficients are exactly the complex complex conjugates of those of r_2. Then, for real inputs x, we have $\bar{r}_2(x) = \overline{r_2(x)}$ and

$$\frac{r_1(x)}{r_2(x)} = \frac{r_1(x)\overline{r_2(x)}}{|r_2(x)|^2},$$

which shows that the contact order K of ϕ at $(1,1)$ is also given by

$$K = \text{ the order of vanishing at 0 of } \operatorname{Im}\left(r_1(x)\overline{r_2(x)}\right).$$

To compute this quantity in practice, we will start with writing $f = \frac{q_1}{q_2}$ for polynomials q_1 and q_2. If q_1 and q_2 have no common factors, then up to a constant factor, $r = q_2 - iq_1$. However if q_1 and q_2 actually have a common factor of the form $m(z_1)$ with $m(0) \neq 0$, then the order of vanishing of $\operatorname{Im}(r_1(x)\overline{r_2(x)})$ at 0 would be unaffected by still letting $r = q_2 - iq_1$. Thus, in what follows, we simply require that q_1 and q_2 not share a common factor of z_1 or a factor involving z_2.

Additionally, in later theorems, we also encounter ϕ with a singularity at $(-1,1)$. As mentioned above, to compute its contact order, we computing the contact order of $\psi(z_1,z_2) := \phi(-z_1,z_2)$ at $(1,1)$. To do this computation, we note that if $\phi \circ \beta = \beta \circ f$, then $\psi \circ \beta = \beta \circ g$, where $g(w_1,w_2) = f(-1/w_1,w_2)$. This means we can conduct the same argument as above to compute contact order, but with $g = \frac{q_1}{q_2}$.

There is also a useful way to define contact order in terms of unimodular level sets of ϕ, see Section 3 in [11], which we encode in the following remark:

Remark 3.2. Let ϕ have a singularity at $(\tau_1,\tau_2) \in \mathbb{T}^2$. Then, for all but a finite number of $\eta \in \mathbb{T}$, the unimodular level set

$$\mathcal{C}_\eta = \text{Closure of } \left\{(z_1,z_2) \in \mathbb{T}^2 : \phi(z_1,z_2) = \eta\right\},$$

near (τ_1,τ_2) can be parameterized by ℓ analytic functions $g_\eta^1,\ldots,g_\eta^\ell$ (where ℓ does not depend on η) in the following way: there is an open neighborhood of (τ_1,τ_2) such that all points of $\mathcal{C}_\eta$ in that set are of the form

$$(z_1, g_\eta^1(z_1)),\ldots,(z_1, g_\eta^\ell(z_1)),$$

and such points are also in $\mathcal{C}_\eta$. Then, contact order is actually a measure of how these unimodular level sets agree near the singularity (τ_1,τ_2) for different η values. Specifically, as shown in [11], for almost every $\eta_1,\eta_2 \in \mathbb{T}$ (called generic values), if we order the parameterizing functions appropriately, then

$$K = \max_{1\le j\le \ell}\left\{\text{Order of vanishing of } |g_{\eta_1}^j(z_1) - g_{\eta_2}^j(z_1)| \text{ at } \tau_1\right\}.$$

In our setting, we generally have $\deg\phi = (n,1)$ for some n, which will imply that the number of branches of a generic C_η will be 1, so $\ell = 1$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

3.2. Contact order and t-values. We can now examine the contact orders of zeros of stable polynomials p_A or p_A^t obtained via our construction. We first establish the following connection between t values and contact order at most related zeros of p_A and p_A^t.

PROPOSITION 3.3. *Let $\tau_1, \tau_2 \in \mathbb{T}\setminus\{-1\}$. By Proposition 2.9, $p_A(\tau_1,\tau_2) = 0$ if and only if $p_A^t(\lambda_1,\lambda_2) = 0$ where*

$$\lambda_1 = \tau_1 \quad \text{and} \quad \lambda_2 = \beta\left(\frac{\beta^{-1}(\tau_2) - t\beta^{-1}(\tau_1)}{1-t}\right).$$

Then, the contact order of p_A at (τ_1,τ_2) equals the contact order of p_A^t at (λ_1,λ_2).

Proof. The fact that $f_A^t(z_1,z_2) = f_A(z_1, tz_1 + (1-t)z_2)$ and the connections between the f_A, ϕ_A and f_A^t, ϕ_A^t functions give

$$\phi_A^t(z_1,z_2) = \phi_A\left(z_1, \beta\big(t\beta^{-1}(z_1) + (1-t)\beta^{-1}(z_2)\big)\right)$$
$$\phi_A(z_1,z_2) = \phi_A^t\left(z_1, \beta\Big(\frac{\beta^{-1}(z_2) - t\beta^{-1}(z_1)}{1-t}\Big)\right),$$

at every $(z_1,z_2) \in \overline{\mathbb{D}^2}$ with $z_1, z_2 \neq -1$.

Assume that $p_A(\tau_1,\tau_2) = 0$, so $p_A^t(\lambda_1,\lambda_2) = 0$. We will use the unimodular level set definition of contact order from Remark 3.2. To that end, for each $\eta \in \mathbb{T}$, define

$$\mathcal{C}_\eta = \text{Closure of } \left\{(z_1,z_2) \in \mathbb{T}^2 : \phi_A(z_1,z_2) = \eta\right\},$$
$$\mathcal{C}_\eta^t = \text{Closure of } \left\{(z_1,z_2) \in \mathbb{T}^2 : \phi_A^t(z_1,z_2) = \eta\right\}.$$

Then, for every $(z_1,z_2) \in \mathbb{T}^2$ near (τ_1,τ_2), we have $(z_1,z_2) \in \mathcal{C}_\eta$ if and only if

$$\left(z_1, \beta\left(\frac{\beta^{-1}(z_2) - t\beta^{-1}(z_1)}{1-t}\right)\right) \in \mathcal{C}_\eta^t.$$

Choose $\eta_1, \eta_2 \in \mathbb{T}$ generic for both ϕ_A and ϕ_A^t. Because the degrees of ϕ_A and ϕ_A^t in z_2 are 1, each $\mathcal{C}_{\eta_k}, \mathcal{C}_{\eta_k}^t$ can be parameterized by a single analytic function $z_2 = g_{\eta_k}(z_1)$ or $z_2 = g_{\eta_k}^t(z_1)$, and the relationship between $\mathcal{C}_\eta$ and $\mathcal{C}_\eta^t$ implies that

$$g_{\eta_k}^t(z_1) = \beta\left(\frac{\beta^{-1}(g_{\eta_k}(z_1)) - t\beta^{-1}(z_1)}{1-t}\right). \tag{3.14}$$

Let K be the contact order of ϕ_A at (τ_1,τ_2). Then,

$$\begin{aligned} K &= \text{Order of vanishing of } |g_{\eta_1}(z_1) - g_{\eta_2}(z_1)| \text{ at } \tau_1 \\ &= \min\left\{L : \frac{d^L}{dz_1^L}\left(g_{\eta_1} - g_{\eta_2}\right)|_{z_1=\tau_1} \neq 0\right\} \\ &= \min\left\{L : g_{\eta_1}^{(L)}(\tau_1) \neq g_{\eta_2}^{(L)}(\tau_1)\right\}. \end{aligned}$$

Let K_t be the contact order of ϕ_A^t at (λ_1,λ_2). Then,

$$\begin{aligned} K_t &= \text{Order of vanishing of } |g_{\eta_1}^t(z_1) - g_{\eta_2}^t(z_1)| \text{ at } \lambda_1 \\ &= \min\left\{L : \left(g_{\eta_1}^t\right)^{(L)}(\lambda_1) \neq \left(g_{\eta_2}^t\right)^{(L)}(\lambda_1)\right\}. \end{aligned}$$

Using (3.14), $\tau_1, \tau_2 \neq -1$, and $\tau_1 = \lambda_1$, we can see that if

$$g_{\eta_1}^{(\ell)}(\tau_1) = g_{\eta_2}^{(\ell)}(\tau_1),$$

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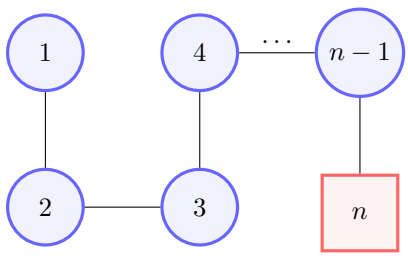
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

for $\ell = 0, \ldots, L-1$, then

$$\left(g^t_{\eta_1}\right)^{(\ell)}(\lambda_1) = \left(g^t_{\eta_2}\right)^{(\ell)}(\lambda_1),$$

for $\ell = 0, \ldots, L-1$ as well. This immediately implies that $K \leq K_t$. A similar argument (using the converse formulas) gives $K_t \leq K$ and completes the proof. ∎

3.3. Contact order and path graphs. In this subsection, we look specifically at stable polynomials constructed from path graphs and prove Theorem 1.3. Note that an n-vertex path graph has adjacency matrix A_n as in Fig. 3.



$$A_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

.

FIGURE 3. *An n-vertex path graph and its adjacency matrix.*

Before proving Theorem 1.3, we prove a related lemma. Below, for nonnegative integers a, b, we let $\binom{a}{b}$ be defined as follows:

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} \text{ if } a \geq b \quad \text{and} \quad \binom{a}{b} = 0 \text{ if } a < b.$$

Then, we have the following computation.

LEMMA 3.4. *Let $a_1, a_2, k \in \mathbb{N}$ with $a_1 > k$ and $a_2 \geq k$. Then*

$$\sum_{m=0}^{k} \binom{a_1 - m}{m}\binom{a_2 - (k-m)}{k-m} = \sum_{m=0}^{k} \binom{a_1 - 1 - m}{m}\binom{a_2 + 1 - (k-m)}{k-m}.$$

Proof. For ease of notation, define

$$\begin{aligned} \mathrm{Sub}(a_1, a_2, k) := &\sum_{m=0}^{k} \binom{a_1 - m}{m}\binom{a_2 - (k-m)}{k-m} \\ &- \sum_{m=0}^{k} \binom{a_1 - 1 - m}{m}\binom{a_2 + 1 - (k-m)}{k-m}. \end{aligned}$$

We will prove $\mathrm{Sub}(a_1, a_2, k) = 0$ for all a_1, a_2, k with $a_1 > k$ and $a_2 \geq k$ by induction on k. For the base case $k = 0$, fix any $a_1, a_2 \in \mathbb{N}$ with $a_1 > 0$ and $a_2 \geq 0$ and observe that

$$\mathrm{Sub}(a_1, a_2, 0) = \binom{a_1}{0}\binom{a_2}{0} - \binom{a_1 - 1}{0}\binom{a_2 + 1}{0} = 1 \cdot 1 - 1 \cdot 1 = 0.$$

For the inductive step, assume that $\mathrm{Sub}(c_1, c_2, k-1) = 0$ for all $c_1, c_2 \in \mathbb{N}$ with $c_1 > k-1$ and $c_2 \geq k-1$. Fix $a_1 > k$ and $a_2 \geq k$. We will show $\mathrm{Sub}(a_1, a_2, k) = 0$. As part of that argument, we use Pascal's formula, which holds for all $j, \ell > 0$ (though if $j < \ell$, then the coefficient is 0):

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A publication of the International Linear Algebra Society
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$$\binom{j}{\ell} = \binom{j-1}{\ell} + \binom{j-1}{\ell-1}. \tag{3.15}$$

Using the formula for $\mathrm{Sub}(a_1, a_2, k)$, extracting one term from each sum, using the Pascal's triangle formula on each sum, and canceling any common terms from the two sums gives:

$$\begin{aligned}
\mathrm{Sub}(a_1, a_2, k) &= \sum_{m=0}^{k} \binom{a_1 - m}{m}\binom{a_2 - (k-m)}{k-m} \\
&\quad - \sum_{m=0}^{k} \binom{a_1 - 1 - m}{m}\binom{a_2 + 1 - (k-m)}{k-m} \\
&= \binom{a_2 - k}{k} + \sum_{m=1}^{k} \left(\binom{a_1 - m - 1}{m} + \binom{a_1 - m - 1}{m-1}\right)\binom{a_2 - (k-m)}{k-m} \\
&\quad - \sum_{m=0}^{k-1} \binom{a_1 - m - 1}{m}\left(\binom{a_2 - (k-m)}{k-m} + \binom{a_2 - (k-m)}{k-m-1}\right) - \binom{a_1 - 1 - k}{k} \\
&= \binom{a_2 - k}{k} - \binom{a_1 - 1 - k}{k} + \binom{a_1 - 1 - k}{k} - \binom{a_2 - k}{k} \\
&\quad + \sum_{m=1}^{k} \binom{a_1 - m - 1}{m-1}\binom{a_2 - (k-m)}{k-m} - \sum_{m=0}^{k-1} \binom{a_1 - m - 1}{m}\binom{a_2 - (k-m)}{k-m-1}.
\end{aligned}$$

Canceling terms and re-indexing the first sum gives:

$$\begin{aligned}
\mathrm{Sub}(a_1, a_2, k) &= \sum_{m=0}^{k-1} \binom{(a_1 - 1) - m - 1}{m}\binom{a_2 - (k-1-m)}{k-1-m} \\
&\quad - \sum_{m=0}^{k-1} \binom{a_1 - 1 - m}{m}\binom{a_2 - 1 - (k-1-m)}{k-1-m} \\
&= -\mathrm{Sub}(a_1 - 1, a_2 - 1, k-1) = 0,
\end{aligned}$$

by the inductive hypothesis. This completes the proof. ∎

Now we can prove Theorem 1.3:

Proof. Since v_1 and v_n are path connected, Theorem 1.1 implies that $p_{A_n}(-1, 1) = 0$. We will use Remark 3.1 to show that the contact order of p_A at $(-1, 1)$ is $2n - 2$. First, write

$$g_{A_n}(z_1, z_2) := f_{A_n}\left(-\tfrac{1}{z_1}, z_2\right) = \frac{q_1}{q_2}(z_1, z_2),$$

where the polynomials q_1, q_2 have no common factors of z_1 or common factors involving z_2 and define $r_1, r_2 \in \mathbb{C}[z]$ so that

$$(q_2 - iq_1)(z_1, z_2) = r_1(z_1) + z_2 r_2(z_1). \tag{3.16}$$

Then, the contact order of p_{A_n} at $(-1, 1)$ is the order of vanishing of $\mathrm{Im}\,(r_1(x)\overline{r_2(x)})$ near 0. To compute that, let $\widehat{z}_Y$ be the diagonal matrix (of appropriate size) with entries $-\frac{1}{z_1}, -\frac{1}{z_1}, \ldots, -\frac{1}{z_1}, z_2$ along the diagonal and observe that for $n > 3$,

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A publication of the International Linear Algebra Society
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$$g_{A_n}(z_1,z_2) = \frac{\det(A_{n-1}-\widehat{z}_Y)}{\det(A_n-\widehat{z}_Y)} = \frac{-z_2\det(A_{n-2}+\frac{1}{z_1}I)-\det(A_{n-3}+\frac{1}{z_1}I)}{-z_2\det(A_{n-1}+\frac{1}{z_1}I)-\det(A_{n-2}+\frac{1}{z_1}I)}, \tag{3.17}$$

where we used the structure of the A_n matrices. This shows that we will need a formula for

$$G(n,z_1) := \det(A_n - z_1 I).$$

Direct computation gives $G(1,z_1) = -z_1$, $G(2,z_1) = z_1^2 - 1$, and the structure of the A_n matrices gives the recursive formula

$$G(n,z_1) = -z_1 G(n-1,z_1) - G(n-2,z_1).$$

We claim that

$$G(n,z_1) = \begin{cases} \displaystyle\sum_{m=0}^{\frac{n-1}{2}} \binom{\frac{n+1}{2}+m}{\frac{n-1}{2}-m}(-1)^{\frac{n+1}{2}-m} z_1^{2m+1}, & \text{for } n \text{ odd,} \\ \displaystyle\sum_{m=0}^{\frac{n}{2}} \binom{\frac{n}{2}+m}{\frac{n}{2}-m}(-1)^{\frac{n}{2}-m} z_1^{2m}, & \text{for } n \text{ even.} \end{cases}$$

Our earlier formulas imply that this holds for $n = 1, 2$, and we prove the result via induction for $n \geq 3$. Specifically, we assume that the formula holds for $n = k-2$ and $n = k-1$, and we show that it holds for $n = k$. Since the cases are almost identical, we show the case when k is odd and leave the case when k is even to the reader.

Since k is odd, we know $k-1$ is even and $k-2$ is odd. Applying the inductive hypothesis, extracting the last term of the first sum, using Pascal's formula (3.15) to combine the two sums, and then adding the final term back in gives

$$\begin{aligned} G(k,z_1) &= -z_1 G(k-1,z_1) - G(k-2,z_1) \\ &= -\sum_{m=0}^{\frac{k-1}{2}} \binom{\frac{k-1}{2}+m}{\frac{k-1}{2}-m}(-1)^{\frac{k-1}{2}-m} z_1^{2m+1} - \sum_{m=0}^{\frac{k-3}{2}} \binom{\frac{k-1}{2}+m}{\frac{k-3}{2}-m}(-1)^{\frac{k-1}{2}-m} z_1^{2m+1} \\ &= \sum_{m=0}^{\frac{k-3}{2}} \binom{\frac{k+1}{2}+m}{\frac{k-1}{2}-m}(-1)^{\frac{k+1}{2}-m} z_1^{2m+1} - z_1^k \\ &= \sum_{m=0}^{\frac{k-1}{2}} \binom{\frac{k+1}{2}+m}{\frac{k-1}{2}-m}(-1)^{\frac{k+1}{2}-m} z_1^{2m+1}, \end{aligned}$$

as needed.

To study g_{A_n}, we consider

$$\begin{aligned} h(n,z_1) := z_1^n G\left(n, \tfrac{1}{z_1}\right) &= \begin{cases} \displaystyle\sum_{m=0}^{\frac{n-1}{2}} \binom{\frac{n+1}{2}+m}{\frac{n-1}{2}-m}(-1)^{\frac{n+1}{2}-m} z_1^{n-2m-1}, & \text{for } n \text{ odd,} \\ \displaystyle\sum_{m=0}^{\frac{n}{2}} \binom{\frac{n}{2}+m}{\frac{n}{2}-m}(-1)^{\frac{n}{2}-m} z_1^{n-2m}, & \text{for } n \text{ even} \end{cases} \\ &= \begin{cases} \displaystyle\sum_{m=0}^{\frac{n-1}{2}} \binom{n-m}{m}(-1)^{m+1} z_1^{2m}, & \text{for } n \text{ odd,} \\ \displaystyle\sum_{m=0}^{\frac{n}{2}} \binom{n-m}{m}(-1)^{m} z_1^{2m} & \text{for } n \text{ even,} \end{cases} \end{aligned}$$

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A publication of the International Linear Algebra Society
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where the second line following by re-indexing the sums (basically replacing m with $\frac{n-1}{2} - m$ for odd cases and m with $\frac{n}{2} - m$ for even cases). Then, from (3.17)

$$\begin{aligned} g_{A_n}(z_1, z_2) &= \frac{-z_2 G(n-2, -\frac{1}{z_1}) - G(n-3, -\frac{1}{z_1})}{-z_2 G(n-1, -\frac{1}{z_1}) - G(n-2, -\frac{1}{z_1})} \\ &= \frac{z_1^{n-1}\big(z_2 G(n-2, -\frac{1}{z_1}) + G(n-3, -\frac{1}{z_1})\big)}{z_1^{n-1}\big(z_2 G(n-1, -\frac{1}{z_1}) + G(n-2, -\frac{1}{z_1})\big)} \\ &= \frac{z_1 z_2 h(n-2, -z_1) + z_1^2 h(n-3, -z_1)}{z_2 h(n-1, -z_1) + z_1 h(n-2, -z_1)} \\ &= \frac{z_1 z_2 h(n-2, z_1) + z_1^2 h(n-3, z_1)}{z_2 h(n-1, z_1) + z_1 h(n-2, z_1)}, \end{aligned}$$

since the powers of z_1 in the formula for $h(n, z_1)$ are always even. The numerator and denominator of the above fraction give us q_1 and q_2; specifically, we can see that they have no common factor involving z_2 since p_{A_n} has a zero at $(-1, 1)$ and no common factor of z_1 since $h(n-1, z_1)$ has a nonzero constant term. Then, we can compute

$$\begin{aligned} (q_2 - iq_1)(z_1, z_2) &= z_2 h(n-1, z_1) + z_1 h(n-2, z_1) \\ &\qquad - i\Big(z_1 z_2 h(n-2, z_1) + z_1^2 h(n-3, z_1)\Big) \\ &= \big(z_1 h(n-2, z_1) - i z_1^2 h(n-3, z_1)\big) + z_2\big(h(n-1, z_1) - i z_1 h(n-2, z_1)\big). \end{aligned}$$

This gives us r_1 and r_2 via (3.16) and so we just need to compute the order of vanishing of

$$\operatorname{Im}\left(r_1(x)\overline{r_2(x)}\right) = x^2 h(n-2, x) h(n-2, x) - x^2 h(n-1, x) h(n-3, x),$$

Note that this requires $n > 3$. The cases $n = 2, 3$ can be easily checked by hand. As before, we assume that n is odd, since the even case can be handled very similarly. Then,

$$\begin{aligned} -\operatorname{Im}\left(r_1(x)\overline{r_2(x)}\right) &= x^2 \sum_{m=0}^{\frac{n-1}{2}} \binom{n-1-m}{m} (-1)^m x^{2m} \sum_{j=0}^{\frac{n-3}{2}} \binom{n-3-j}{j} (-1)^j x^{2j} \\ &\quad - x^2 \sum_{m=0}^{\frac{n-3}{2}} \binom{n-2-m}{m} (-1)^{m+1} x^{2m} \sum_{j=0}^{\frac{n-3}{2}} \binom{n-2-j}{j} (-1)^{j+1} x^{2j} \\ &= x^2 \sum_{k=0}^{n-2} c_k x^{2k}, \end{aligned}$$

for coefficients $c_k \in \mathbb{R}$. We claim that all coefficients are 0, except c_{n-2}. First note that

$$c_{n-2} = \binom{\frac{n-1}{2}}{\frac{n-1}{2}} \binom{\frac{n-3}{2}}{\frac{n-3}{2}} (-1)^{n-2} = (-1)^{n-2} = -1.$$

To show that the other coefficients are zero, we consider two cases. First, if $0 \le k \le \frac{n-3}{2}$, then c_k equals

$$(-1)^k \sum_{m=0}^{k} \left(\binom{n-1-m}{m} \binom{n-3-(k-m)}{k-m} - \binom{n-2-m}{m} \binom{n-2-(k-m)}{k-m} \right),$$

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A publication of the International Linear Algebra Society
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which is 0 by Lemma 3.4. Meanwhile, if $\frac{n-1}{2} \leq k \leq n-3$, then we have some restrictions on the terms that appear in the formula for c_k. Namely, one can check that c_k equals $(-1)^k$ times

$$\sum_{m=k-\frac{n-3}{2}}^{\frac{n-1}{2}} \binom{n-1-m}{m}\binom{n-3-(k-m)}{k-m} - \sum_{m=k-\frac{n-3}{2}}^{\frac{n-3}{2}} \binom{n-2-m}{m}\binom{n-2-(k-m)}{k-m}.$$

However, when m is both outside of the current sum bounds and satisfies $0 \leq m \leq k$, the binomial coefficient formulas in the summands are both well defined and equal zero. Thus, we can also write the formula for c_k as

$$(-1)^k \sum_{m=0}^{k} \left(\binom{n-1-m}{m}\binom{n-3-(k-m)}{k-m} - \binom{n-2-m}{m}\binom{n-2-(k-m)}{k-m} \right),$$

which is again 0 by Lemma 3.4. This means that for n odd,

$$\operatorname{Im}\left(r_1(x)\overline{r_2(x)}\right) = x^{2n-2},$$

and so the contact order of p_{A_n} at $(-1,1)$ is $2n-2$. An analogous argument gives the result for n even. ∎

3.4. Contact order with graph modifications. In this section, we investigate Conjecture 1.4 and prove Theorem 1.5, which gives three cases that support the conjecture. We split the proof into the three different cases below.

Throughout these proofs, we will use the following notation: for an arbitrary $n \times n$ matrix M, we let $\overline{M}$ denote the $(n-1)\times(n-1)$ matrix obtained from M by removing its first row and first column, $\underline{M}$ denote the $(n-1)\times(n-1)$ matrix obtained from M by removing its last row and last column, and $\overline{\underline{M}}$ denote the $(n-2)\times(n-2)$ matrix obtained from M by removing both its first row and first column and its last row and last column.

For the first case in Theorem 1.5, let G_A be an n-vertex graph (as always, undirected and simple) with the vertices v_1 and v_n path connected. We define a new graph $G_{\widehat{A}}$ as the graph we get from G_A by adding a vertex v_{n+1} that is only connected directly to v_n. Specifically, $G_{\widehat{A}}$ has the adjacency matrix

$$\widehat{A} = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & A & & & 0 \\ & & & & 1 \\ \hline 0 & \cdots & 0 & 1 & 0 \end{array}\right]. \tag{3.18}$$

An example of this construction is given in Fig. 4 below.

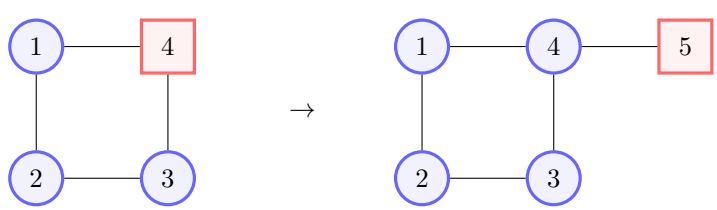

FIGURE 4. *An example of the graph modification G_A to $G_{\widehat{A}}$ for Theorem 3.5.*

Then, we have the following result.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

THEOREM 3.5. *Let G_A be an n-vertex graph with v_1 and v_n path connected. Assume that the contact order of p_A at $(-1,1)$ is K. Then, with $\widehat{A}$ defined in (3.18), the contact order of $p_{\widehat{A}}$ at $(-1,1)$ is $K+2$.*

Proof. We will use Remark 3.1 to obtain tractable formulas for the contact orders of p_A and $p_{\widehat{A}}$ at $(-1,1)$ that will allow us to compare the two quantities directly. To that end, we consider polynomials $q_1, q_2, \hat{q}_1, \hat{q}_2$ that satisfy

$$g_A(z_1,z_2) := f_A\left(-\tfrac{1}{z_1}, z_2\right) = \frac{q_1}{q_2} \quad \text{and} \quad g_{\widehat{A}}(z_1,z_2) := f_{\widehat{A}}\left(-\tfrac{1}{z_1}, z_2\right) = \frac{\hat{q}_1}{\hat{q}_2},$$

and the pairs (q_1,q_2) and $(\hat{q}_1,\hat{q}_2)$ do not have any common factors of z_1 or common factors involving z_2. Then, if we define $r_1, r_2, \hat{r}_1, \hat{r}_2 \in \mathbb{C}[z]$ so that they satisfy

$$(q_2 - iq_1)(z_1,z_2) = r_1(z_1) + z_2 r_2(z_1) \quad \text{and} \quad (\hat{q}_2 - i\hat{q}_1)(z_1,z_2) = \hat{r}_1(z_1) + z_2\hat{r}_2(z_1), \tag{3.19}$$

K is the order of vanishing of $\operatorname{Im}(r_1(x)\overline{r_2(x)})$ near 0, and the contact order of $p_{\widehat{A}}$ at $(-1,1)$ is the order of vanishing of $\operatorname{Im}(\hat{r}_1(x)\overline{\hat{r}_2(x)})$ near 0.

Now we find formulas for $q_1, q_2, \hat{q}_1, \hat{q}_2$. Let $\widehat{z}_Y$ be the diagonal matrix (of appropriate size) with entries $-\frac{1}{z_1}, -\frac{1}{z_1}, \ldots, -\frac{1}{z_1}, z_2$ along the diagonal. Then,

$$g_A(z_1,z_2) = \frac{\det(\overline{A} - \widehat{z}_Y)}{\det(A - \widehat{z}_Y)} = \frac{z_1^{n-1}\left(-z_2\det\left(\underline{\overline{A}} + \frac{1}{z_1}I\right) - s_1\left(\frac{1}{z_1}\right)\right)}{z_1^{n-1}\left(-z_2\det\left(\underline{A} + \frac{1}{z_1}I\right) - s_2\left(\frac{1}{z_1}\right)\right)}, \tag{3.20}$$

where we expanded the determinant using the Laplace expansion method along the last column and last row of $(A - \widehat{z}_Y)$ and $(\overline{A} - \widehat{z}_Y)$ and where $s_1, s_2 \in \mathbb{R}[z]$ satisfy $\deg s_1 \le n-3$ and $\deg s_2 \le n-2$. Then, we can take the numerator and denominator of (3.20) to be q_1, q_2, since the denominator does not vanish identically when $z_1 = 0$ (so there is no common factor of z_1) and vertices v_1 and v_n are path connected (so a factor involving z_2 cannot cancel).

For $g_{\widehat{A}}$, we can use Laplace expansion first along the last row and column to reduce to a formula involving A and then apply Laplace expansion again (in a way analogous to that for g_A) to obtain

$$\begin{aligned} g_{\widehat{A}}(z_1,z_2) &= \frac{\det\left(\overline{\widehat{A}} - \widehat{z}_Y\right)}{\det\left(\widehat{A} - \widehat{z}_Y\right)} = \frac{-z_2\det\left(\overline{A} + \frac{1}{z_1}I\right) - \det\left(\underline{\overline{A}} + \frac{1}{z_1}I\right)}{-z_2\det\left(A + \frac{1}{z_1}I\right) - \det\left(\underline{A} + \frac{1}{z_1}I\right)} \\ &= \frac{z_1^n\left(z_2\left(-\frac{1}{z_1}\det\left(\underline{\overline{A}} + \frac{1}{z_1}I\right) + s_1\left(\frac{1}{z_1}\right)\right) - \det\left(\underline{\overline{A}} + \frac{1}{z_1}I\right)\right)}{z_1^n\left(z_2\left(-\frac{1}{z_1}\det\left(\underline{A} + \frac{1}{z_1}I\right) + s_2\left(\frac{1}{z_1}\right)\right) - \det\left(\underline{A} + \frac{1}{z_1}I\right)\right)}. \end{aligned}$$

As before, we can take the numerator and denominator of the above expression to be $\hat{q}_1, \hat{q}_2$. From this and (3.19), we can easily identify $r_1, r_2, \hat{r}_1$, and $\hat{r}_2$:

$$\begin{aligned} r_1(z_1) &= -z_1^{n-1}\left(s_2\left(\tfrac{1}{z_1}\right) - is_1\left(\tfrac{1}{z_1}\right)\right) \\ r_2(z_1) &= -z_1^{n-1}\left(\det\left(\underline{A} + \tfrac{1}{z_1}I\right) - i\det\left(\underline{\overline{A}} + \tfrac{1}{z_1}I\right)\right) \\ \hat{r}_1(z_1) &= -z_1^{n}\left(\det\left(\underline{A} + \tfrac{1}{z_1}I\right) - i\det\left(\underline{\overline{A}} + \tfrac{1}{z_1}I\right)\right) \\ \hat{r}_2(z_1) &= z_1^{n}\left(-\tfrac{1}{z_1}\det\left(\underline{A} + \tfrac{1}{z_1}I\right) + s_2\left(\tfrac{1}{z_1}\right) - i\left(-\tfrac{1}{z_1}\det\left(\underline{\overline{A}} + \tfrac{1}{z_1}I\right) + s_1\left(\tfrac{1}{z_1}\right)\right)\right). \end{aligned}$$

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A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

By assumption, we know for $x \in \mathbb{R}$ near 0,

$$x^K \approx \operatorname{Im}\left(r_1(x)\overline{r_2(x)}\right) = x^{2n-2}\left(s_2\left(\tfrac{1}{x}\right)\det\left(\underline{\overline{A}} + \tfrac{1}{x}I\right) - s_1\left(\tfrac{1}{x}\right)\det\left(\underline{A} + \tfrac{1}{x}I\right)\right). \tag{3.21}$$

Then, to compute the contact order of $p_{\widehat{A}}$ at $(-1,1)$, just observe that for $x \in \mathbb{R}$ near 0,

$$\begin{aligned}\operatorname{Im}\left(\hat{r}_1(x)\overline{\hat{r}_2(x)}\right) &= -x^{2n}\Big(\det\left(\underline{A} + \tfrac{1}{x}I\right)\left(-\tfrac{1}{x}\det\left(\underline{\overline{A}} + \tfrac{1}{x}I\right) + s_1\left(\tfrac{1}{x}\right)\right)\\ &\qquad - \det\left(\underline{\overline{A}} + \tfrac{1}{x}I\right)\left(-\tfrac{1}{x}\det\left(\underline{A} + \tfrac{1}{x}I\right) + s_2\left(\tfrac{1}{x}\right)\right)\Big)\\ &= x^2x^{2n-2}\left(s_2\left(\tfrac{1}{x}\right)\det\left(\underline{\overline{A}} + \tfrac{1}{x}I\right) - s_1\left(\tfrac{1}{x}\right)\det\left(\underline{A} + \tfrac{1}{x}I\right)\right) \approx x^{2+K},\end{aligned}$$

by (3.21). ∎

For the second case in Theorem 1.5, let G_A be an n-vertex graph with v_1 and v_n path connected. Create $G_{\widetilde{A}}$ from G_A by adding a vertex v_0 that is only connected directly to v_1 and then renumbering the vertices to $v_1, \ldots, v_{n+1}$. Then, $G_{\widetilde{A}}$ has the adjacency matrix

$$\widetilde{A} = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{array}\right]. \tag{3.22}$$

This construction is illustrated in Fig. 5 below.

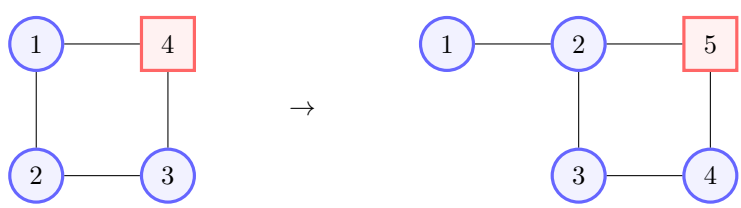

FIGURE 5. *An example of the graph modification G_A to $G_{\widetilde{A}}$ for Theorem 3.6.*

We then have the following result.

THEOREM 3.6. *Let G_A be an n-vertex graph with vertices 1 and n path connected. Assume that the contact order of p_A at $(-1,1)$ is K. Then, with $\widetilde{A}$ defined in (3.22), the contact order of $p_{\widetilde{A}}$ at $(-1,1)$ is $K+2$.*

Proof. We proceed as in the proof of Theorem 3.5. Then, from that proof, we can extract the following useful identities from (3.20) and (3.21):

$$\det(\overline{A} - \widehat{z}_Y) = -z_2\det\left(\underline{\overline{A}} + \tfrac{1}{z_1}I\right) - s_1\left(\tfrac{1}{z_1}\right) \tag{3.23}$$

$$\det(A - \widehat{z}_Y) = -z_2\det\left(\underline{A} + \tfrac{1}{z_1}I\right) - s_2\left(\tfrac{1}{z_1}\right) \tag{3.24}$$

$$x^K \approx x^{2n-2}\left(s_2\left(\tfrac{1}{x}\right)\det\left(\underline{\overline{A}} + \tfrac{1}{x}I\right) - s_1\left(\tfrac{1}{x}\right)\det\left(\underline{A} + \tfrac{1}{x}I\right)\right), \tag{3.25}$$

for $x \in \mathbb{R}$ near 0, where K is the contact order of p_A at $(-1,1)$ and $s_1, s_2, \widehat{z}_Y$ are defined in the proof of Theorem 3.5. As in the proof of Theorem 3.5, we now find polynomials $\tilde{q}_1, \tilde{q}_2$ (with no common factor of z_1 or common factor containing z_2) such that

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A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

$$g_{\widetilde{A}}(z_1, z_2) := f_{\widetilde{A}}\left(-\tfrac{1}{z_1}, z_2\right) = \frac{\tilde{q}_1}{\tilde{q}_2},$$

write $\tilde{q}_2 - i\tilde{q}_1 = \tilde{r}_1(z_1) + z_2\tilde{r}_2(z_1)$ for polynomials $\tilde{r}_1, \tilde{r}_2 \in \mathbb{C}[z]$, and compute the order of vanishing of $\operatorname{Im}(\tilde{r}_1(x)\overline{\tilde{r}_2(x)})$ for $x \in \mathbb{R}$ near 0.

To that end, observe that using Laplace expansion along the first row and column of $(\widetilde{A} - \widehat{z}_Y)$, $(\overline{\widetilde{A}} - \widehat{z}_Y)$ and then using the formulas (3.23) and (3.24) will give

$$\begin{aligned} g_{\widetilde{A}}(z_1, z_2) &= \frac{\det\left(\overline{\widetilde{A}} - \widehat{z}_Y\right)}{\det\left(\widetilde{A} - \widehat{z}_Y\right)} = \frac{\det\left(A - \widehat{z}_Y\right)}{\frac{1}{z_1}\det(A - \widehat{z}_Y) - \det\left(\overline{A} - \widehat{z}_Y\right)} \\ &= \frac{-z_2 \det\left(\underline{A} + \frac{1}{z_1}I\right) - s_2\left(\frac{1}{z_1}\right)}{\frac{1}{z_1}\left(-z_2\det\left(\underline{A} + \frac{1}{z_1}I\right) - s_2\left(\frac{1}{z_1}\right)\right) - \left(-z_2\det\left(\overline{\underline{A}} + \frac{1}{z_1}I\right) - s_1\left(\frac{1}{z_1}\right)\right)} \\ &= \frac{z_1^n\left(z_2\det\left(\underline{A} + \frac{1}{z_1}I\right) + s_2\left(\frac{1}{z_1}\right)\right)}{z_1^n\left(z_2\left(\frac{1}{z_1}\det\left(\underline{A} + \frac{1}{z_1}I\right) - \det\left(\overline{\underline{A}} + \frac{1}{z_1}I\right)\right) + \frac{1}{z_1}s_2\left(\frac{1}{z_1}\right) - s_1\left(\frac{1}{z_1}\right)\right)}. \end{aligned} \tag{3.26}$$

We can let the numerator and denominator in (3.26) be $\tilde{q}_1, \tilde{q}_2$ (there is no common z_1 factor since the denominator has a nonzero constant term and no term with z_2 can cancel since vertices v_1 and v_{n+1} are path connected) and easily find $\tilde{r}_1$ and $\tilde{r}_2$:

$$\begin{aligned} \tilde{r}_1(z_1) &= z_1^n\left(\tfrac{1}{z_1}s_2\left(\tfrac{1}{z_1}\right) - s_1\left(\tfrac{1}{z_1}\right) - is_2\left(\tfrac{1}{z_1}\right)\right) \\ \tilde{r}_2(z_1) &= z_1^n\left(\tfrac{1}{z_1}\det\left(\underline{A} + \tfrac{1}{z_1}I\right) - \det\left(\overline{\underline{A}} + \tfrac{1}{z_1}I\right) - i\det\left(\underline{A} + \tfrac{1}{z_1}I\right)\right). \end{aligned}$$

Now we can compute the contact order for $p_{\widetilde{A}}$ at $(-1, 1)$. Specifically, for $x \in \mathbb{R}$ near 0,

$$\begin{aligned} \operatorname{Im}(\tilde{r}_1(x)\overline{\tilde{r}_2(x)}) &= x^{2n}\Big(\left(\tfrac{1}{x}s_2\left(\tfrac{1}{x}\right) - s_1\left(\tfrac{1}{x}\right)\right)\det\left(\underline{A} + \tfrac{1}{x}I\right) \\ &\qquad - s_2\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\det\left(\underline{A} + \tfrac{1}{x}I\right) - \det\left(\overline{\underline{A}} + \tfrac{1}{x}I\right)\right)\Big) \\ &= x^2x^{2n-2}\left(s_2\left(\tfrac{1}{x}\right)\det\left(\overline{\underline{A}} + \tfrac{1}{x}I\right) - s_1\left(\tfrac{1}{x}\right)\det\left(\underline{A} + \tfrac{1}{x}I\right)\right) \approx x^{2+K}, \end{aligned}$$

where we used (3.25). Then, applying Remark 3.1 establishes the claim. ∎

Let G_A be an n-vertex graph such that the only connection between vertex n and the rest of the graph is a single edge between vertex $n-1$ and vertex n. Then, G_A has the adjacency matrix

$$A = \left[\begin{array}{ccc|c} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ & & & 1 \\ \hline 0 & \cdots & 0 \quad 1 & 0 \end{array}\right],$$

for some $(n-1) \times (n-1)$ matrix B.

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A publication of the International Linear Algebra Society
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Now, fix $m \geq n+1$. Relabel vertex n to vertex m and expand G_A to a new graph $G_{\breve{A}}$ by attaching an $(m-n)$-vertex graph to G_A via an edge at vertex $n-1$ (and potentially other edges to the new vertex m). This follows Modification (iii) in Theorem 1.5. Then, $G_{\breve{A}}$ has the $m \times m$ adjacency matrix

$$\breve{A} = \left[\begin{array}{cccc|cccc} & & & & 0 & \cdots & \cdots & 0 \\ & & & & \vdots & \ddots & \ddots & \vdots \\ & B & & & 0 & \cdots & \cdots & 0 \\ & & & & * & \cdots & * & 1 \\ \hline 0 & \cdots & 0 & * & & & & \\ \vdots & \ddots & \vdots & \vdots & & & & \\ \vdots & \ddots & \vdots & * & & C & & \\ 0 & \cdots & 0 & 1 & & & & \end{array}\right], \tag{3.27}$$

where the asterisks * contain at least one 1 and are 0 otherwise, and C is the adjacency matrix of an $(m-(n-1))$-vertex graph G_C. This construction for $n=4$ and $m=6$ is illustrated in Fig. 6 below. The dotted edge indicates that one could also insert an edge between v_4 and v_6, and the contact order result would still hold.

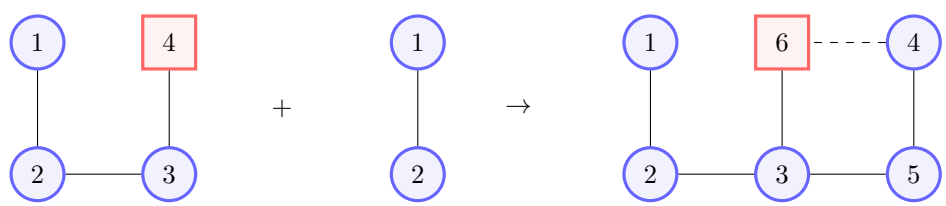

FIGURE 6. *An example of the graph modification in Theorem 3.7.*

Note that the length of the shortest path between the first and last vertex will be the same for G_A and $G_{\breve{A}}$. Thus, Conjecture 1.4 implies that this contact order at $(-1,1)$ should be the same for both related polynomials. This is proved in the following result:

THEOREM 3.7. *Let G_A be an n-vertex graph with v_1 and v_{n-1} path connected, and v_n only connected to v_{n-1} and assume that contact order of p_A at $(-1,1)$ is K. Then, with $\breve{A}$ defined in (3.27), the contact order of $p_{\breve{A}}$ at $(-1,1)$ is also K.*

Proof. To compare the contact orders, we will apply Remark 3.1 to the functions

$$g_A(z_1,z_2) := f_A\left(-\tfrac{1}{z_1}, z_2\right) = \frac{q_1}{q_2} \quad \text{and} \quad g_{\breve{A}}(z_1,z_2) := f_{\breve{A}}\left(-\tfrac{1}{z_1}, z_2\right) = \frac{\breve{q}_1}{\breve{q}_2},$$

where the pairs q_1, q_2 and $\breve{q}_1, \breve{q}_2$ have no common factor of z_1 or common factor containing z_2. Let $\breve{z_Y}$ be the diagonal matrix (of appropriate size) with entries $-\frac{1}{z_1}, -\frac{1}{z_1}, \ldots, -\frac{1}{z_1}, z_2$ along the diagonal. Then, the structure of A implies that

$$g_A(z_1,z_2) = \frac{\det(\overline{A} - \breve{z_Y})}{\det(A - \breve{z_Y})} = \frac{z_2 \det(\overline{B} + \frac{1}{z_1}I) + \det(\underline{\overline{B}} + \frac{1}{z_1}I)}{z_2 \det(B + \frac{1}{z_1}I) + \det(\underline{B} + \frac{1}{z_1}I)},$$

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A publication of the International Linear Algebra Society
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where the largest power of $\frac{1}{z_1}$ is exactly $n-1$ and so, we can take

$$q_1(z_1,z_2) = z_1^{n-1}\Big(z_2\det(\overline{B}+\tfrac{1}{z_1}I)+\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)\Big),$$
$$q_2(z_1,z_2) = z_1^{n-1}\Big(z_2\det(B+\tfrac{1}{z_1}I)+\det(\underline{B}+\tfrac{1}{z_1}I)\Big).$$

As in the proofs of the previous two theorems, these q_1, q_2 share no common factor of z_1 or common factor involving z_2. From this, we can deduce

$$r_1(z_1,z_2) = z_1^{n-1}\Big(\det(\underline{B}+\tfrac{1}{z_1}I)-i\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)\Big),$$
$$r_2(z_1,z_2) = z_1^{n-1}\Big(\det(B+\tfrac{1}{z_1}I)-i\det(\overline{B}+\tfrac{1}{z_1}I)\Big).$$

Thus, using the contact order assumption, we have

$$\operatorname{Im}(r_1(x)\overline{r_2(x)}) = x^{2n-2}(\det(\underline{B}+\tfrac{1}{x}I)\det(\overline{B}+\tfrac{1}{x}I)-\det(B+\tfrac{1}{x}I)\det(\underline{\overline{B}}+\tfrac{1}{x}I)) \approx x^K. \tag{3.28}$$

Now we connect this to the contact order of $p_{\breve{A}}$ or equivalently, to the order of vanishing of $\operatorname{Im}(\breve{r}_1(x)\overline{\breve{r}_2(x)})$ near 0. To obtain a formula for $g_{\breve{A}}$, first recall that

$$g_{\breve{A}}(z_1,z_2) = \frac{\det(\overline{\breve{A}}-z_{\breve{Y}})}{\det(\breve{A}-z_{\breve{Y}})}.$$

By applying the Laplace expansion repetitively on $\det(\breve{A}-z_{\breve{Y}})$ starting on bottom row of $\breve{A}$ and using the fact that the upper-right and the lower-left matrices in the partition of $\breve{A}$ have only a single row or column with 1's, one can show that there exist polynomials s_1, s_2, s_3, s_4 such that

$$\begin{aligned}\det(\breve{A}-z_{\breve{Y}}) =& z_2\Big(s_1(\tfrac{1}{z_1})\det(B+\tfrac{1}{z_1}I)+s_2(\tfrac{1}{z_1})\det(\underline{B}+\tfrac{1}{z_1}I)\Big)\\ &+\Big(s_3(\tfrac{1}{z_1})\det(B+\tfrac{1}{z_1}I)+s_4(\tfrac{1}{z_1})\det(\underline{B}+\tfrac{1}{z_1}I)\Big).\end{aligned}$$

The highest power of s_1 is $m-n$ with a coefficient of 1 by following the main diagonal. The highest power of s_4 is also $m-n$ with a coefficient of 1 by using the two 1s at the last row and column, then following the main diagonal. The highest power for s_2 is $m-n-1$, using the fact that there is at least a pair of 1s in the asterisks in $\breve{A}$.

Since the only nonzero values affected by moving from $\breve{A}$ to $\overline{\breve{A}}$ are in the change from B to $\overline{B}$, we immediately obtain

$$\begin{aligned}\det(\overline{\breve{A}}-z_{\breve{Y}}) =& z_2\Big(s_1(\tfrac{1}{z_1})\det(\overline{B}+\tfrac{1}{z_1}I)+s_2(\tfrac{1}{z_1})\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)\Big)\\ &+\Big(s_3(\tfrac{1}{z_1})\det(\overline{B}+\tfrac{1}{z_1}I)+s_4(\tfrac{1}{z_1})\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)\Big).\end{aligned}$$

Then, multiplying the formula for $g_{\breve{A}}$ by z_1^{m-1} in the numerator and denominator will allow us to obtain the following formulas for $\breve{q}_1$ and $\breve{q}_2$:

$$\begin{aligned}\breve{q}_1 = z_1^{n-1}\Big(&S_1(z_1)z_2\det(\overline{B}+\tfrac{1}{z_1}I)+S_2(z_1)z_2\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)+S_3(z_1)\det(\overline{B}+\tfrac{1}{z_1}I)\\ &+S_4(z_1)\det(\underline{\overline{B}}+\tfrac{1}{z_1}I)\Big),\\ \breve{q}_2 = z_1^{n-1}\Big(&S_1(z_1)z_2\det(B+\tfrac{1}{z_1}I)+S_2(z_1)z_2\det(\underline{B}+\tfrac{1}{z_1}I)+S_3(z_1)\det(B+\tfrac{1}{z_1}I)\\ &+S_4(z_1)\det(\underline{B}+\tfrac{1}{z_1}I)\Big),\end{aligned}$$

where $S_k(z_1) = z_1^{m-n}s_k(\frac{1}{z_1})$ for $k=1,\dots,4$.

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A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

By manipulating the equations $\breve{q}_1$ and $\breve{q}_2$, we obtain

$$\begin{aligned}\breve{r}_1 = z_1^{n-1}\Big(&S_3(z_1)\det(B+\tfrac{1}{z_1}I) + S_4(z_1)\det(\underline{B}+\tfrac{1}{z_1}I) - iS_3(z_1)\det(\overline{B}+\tfrac{1}{z_1}I)\\ &- iS_4(z_1)\det(\overline{\underline{B}}+\tfrac{1}{z_1}I)\Big),\\ \breve{r}_2 = z_1^{n-1}\Big(&S_1(z_1)\det(B+\tfrac{1}{z_1}I) + S_2(z_1)\det(\underline{B}+\tfrac{1}{z_1}I) - iS_1(z_1)\det(\overline{B}+\tfrac{1}{z_1}I)\\ &- iS_2(z_1)\det(\overline{\underline{B}}+\tfrac{1}{z_1}I)\Big).\end{aligned}$$

By substituting and canceling terms, we have

$$\begin{aligned}&\operatorname{Im}(\breve{r}_1(x)\overline{\breve{r}_2(x)})\\ &\quad= x^{2n-2}\Big(\det(\underline{B}+\tfrac{1}{x}I)\det(\overline{B}+\tfrac{1}{x}I) - \det(B+\tfrac{1}{x}I)\det(\overline{\underline{B}}+\tfrac{1}{x}I)\Big)\\ &\qquad\qquad\times\Big(S_1(x)S_4(x) - S_2(x)S_3(x)\Big)\\ &\quad\approx x^K\Big(S_1(x)S_4(x) - S_2(x)S_3(x)\Big),\end{aligned}$$

where we used (3.28). By the earlier statements about the highest powers of the s_k, $S_1(0), S_4(0) \neq 0$ and $S_2(0) = 0$, so the lowest order term in $S_1(x)S_4(x) - S_2(x)S_3(x)$ must be a nonzero constant term. Thus, the contact order of $p_{\breve{A}}$ at $(-1, 1)$ is also K, as needed. ∎

4. Open questions. This paper initiates the study of stable polynomials constructed via general graphs and a specific type of coloring matrix Y. The line of inquiry is still quite new, and many questions remain open. Here, are three possible directions of inquiry:

1. *What can one say about additional boundary zeros of the stable polynomials p_A^t?* A number of special cases were studied in [20], but we currently know of no general way to determine (from the graph's properties) whether the polynomial will have extra boundary zeros.

2. *What happens when one changes the coloring matrix Y or vector α?* For our particular choice of Y, the existence of at least one boundary zero was fully determined by whether the first and last vertices of the graph were path connected. It would be very interesting if there was a more general characterization of the existence of boundary zeros that held for stable polynomials constructed from general triples $(A,\ Y,\ \alpha)$. In particular, changing the matrix Y would allow for much more complicated (higher degree) polynomials.

3. *What additional information do these types of stable polynomials encode about the associated colored graphs?* This current paper was mostly concerned with using graphs to obtain polynomials with particular properties. However, one could imagine going in the other direction and using a polynomial to obtain information about an underlying graph. For example, can one use these polynomials (or related objects) to count the number of paths with certain properties within the colored graph?

Acknowledgment. Special thanks to Ryan Tully-Doyle and J.E. Pascoe for very useful insights and discussions during the writing of this paper. The authors Kelly Bickel and Yang Hong were partially supported

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

by the National Science Foundation DMS grant #2000088 during these research activities. Additional support was provided to Hong by Bucknell University. Many results in Section 2 originally appeared in Hong's Bucknell honor thesis [20].

REFERENCES

[1] L. Adlin, G. Thai, S. Tiscareno, and R. Tully-Doyle. Cauchy transforms of colored graphs in two variables. *Linear Algebra Appl.*, 731:1–21, 2026.

[2] J. Agler. On the representation of certain holomorphic functions defined on a polydisc. *Oper. Theory Adv. Appl.*, 48:47–66, 1990.

[3] J. Agler and J.E. McCarthy. *Pick Interpolation and Hilbert Function Spaces*. Graduate Studies in Mathematics, vol. 44. American Mathematical Society, Providence RI, 2002.

[4] J. Agler, J.E. McCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math. (2),* 176:1783–1826, 2012.

[5] J. Agler, R. Tully-Doyle, and N.J. Young. Nevanlinna representations in several variables. *J. Funct. Anal.*, 270(8):3000–3046, 2016.

[6] O. Arizmendi, T. Hasebe, and F. Lehner. Cyclic independence: Boolean and monotone. *Algebr. Comb.*, 6(6):1697–1734, 2023.

[7] O. Arizmendi and T. Gaxiola. Asymptotic spectral distributions of distance-k graphs of star product graphs. In: *XI Symposium on Probability and Stochastic Processes. Progr. Probab.*, vol. 69. Birkhäuser/Springer, Cham, 47–58, 2015.

[8] J. Ball, C. Sadosky, and V. Vinnikov. Scattering systems with several evolutions and multidimensional input/state/output systems. *Integr. Equ. Oper. Theory,* 52:323–393, 2005.

[9] K. Bickel and G. Knese. Inner functions on the Bidisk and associated Hilbert spaces. *J. Funct. Anal.*, 265(11):2753–2790, 2013.

[10] K. Bickel, J.E. Pascoe, and A. Sola. Derivatives of rational inner functions: Geometry of singularities and integrability at the boundary. *Proc. Lond. Math. Soc. (3)*, 116(2):281–329, 2018.

[11] K. Bickel, J.E. Pascoe, and A. Sola. Level curve portraits of rational inner functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 21:449–494, 2020.

[12] K. Bickel, J.E. Pascoe, and A. Sola. Singularities of rational inner functions in higher dimensions. *Amer. J. Math.*, 144(4): 1115–1157, 2022.

[13] K. Bickel, G. Knese, J.E. Pascoe, and A. Sola. Local theory of stable polynomials and bounded rational functions of several variables, *Ann. Polon. Math.*, 133(2):95–169, 2024.

[14] J. Borcea, P. Brändén, and T.M. Liggett. Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.*, 22(2):521–567, 2009.

[15] D. Lind, K. Schmidt, and I.E. Verbitsky. Homoclinic points, atoral polynomials, and periodic points of algebraic $\mathbb{Z}^d$ actions. *Ergodic Theory Dynam. Syst.,* 33:1060–1081, 2013.

[16] J.S. Geronimo and H.J. Woerdeman. Two-variable polynomials: Intersecting zeros and stability. *IEEE Trans. Circuits Syst. I Regul. Pap.*, 53:1130–1139, 2006.

[17] S.R. Garcia, J. Mashreghi, and W.T. Ross. *Finite Blaschke Products and Their Connections*. Springer-Verlag, Cham, 2018.

[18] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. Woerdeman. Matrix-valued Hermitian Positivstellensatz, lurking contractions, and contractive determinantal representations of stable polynomials. *Oper. Theory Adv. Appl.*, 255:123–136, 2016.

[19] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. Woerdeman. Stable and real-zero polynomials in two variables. *Multidimens. Syst. Signal Process.,* 27(1):1–26, 2016.

[20] Y. Hong. Graphs, Adjacency matrices, and corresponding functions. Bucknell Honors Thesis, 2023.

[21] J.W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5):654–674, 2002.

[22] S. R. Garcia and R. A. Horn. Block matrices in linear algebra. *PRIMUS,* 30(3):285–306, 2020.

[23] G. Knese. Polynomials with no zeros on the Bidisk. *Anal. PDE*, 3:109–149, 2010.

[24] G. Knese. Integrability and regularity of rational functions. *Proc. London. Math. Soc. (3)*, 111 (6):1261–1306, 2015.

[25] A. Kummert. 2-D stable polynomials with parameter-dependent coefficients: Generalizations and new results. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 49(6):725–731, 2002.

[26] P. Kurasov and P. Sarnak. Stable polynomials and crystalline measures. *J. Math. Phys.*, 61:083501, 13pp., 2020.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 213-238, March 2026.

[27] A. Marcus, D. Spielman, and N. Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Ann. of Math. (2)*, 182(1):307–325, 2015.

[28] A. Marcus, D. Spielman, and N. Srivastava. Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem. *Ann. of Math. (2)*, 182(1):327–350, 2015.

[29] J.E. Pascoe. A wedge-of-the-edge theorem: Analytic continuation of multivariable Pick functions in and around the boundary. *Bull. London Math. Soc.* 49:916–925, 2017.

[30] J.E. Pascoe. An inductive Julia-Carathéodory theorem for Pick functions in two variables. *Proc. Edinb. Math. Soc. (2)*, 61(3): 647–660, 2018.

[31] W. Rudin, *Function Theory in Polydiscs*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[32] A. Sola. A note on polydegree $(n, 1)$ rational inner functions, slice matrices, and singularities. *Arch. Math. (Basel)*, 120(2): 171–181, 2023.

[33] D.G. Wagner. Multivariate stable polynomials: Theory and applications. *Bull. Amer. Math. Soc.*, 48:53–84, 2011.