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THE CLASSIFICATION OF GRAPHS ON EIGHT VERTICES WITH COINCIDING ZERO FORCING NUMBER AND MAXIMUM NULLITY*

WAYNE BARRETT†, MARK HUNNELL‡, JOHN HUTCHENS§, AND JOHN SINKOVIC¶

Abstract. This article studies the minimum rank of a (simple, undirected) graph, which is the minimum rank among all matrices in a space determined by the graph. It determines the exact set of graphs on eight vertices for which the nullity of a minimum rank matrix does not coincide with a bound determined by the zero forcing number of a graph. Although the goal was to determine which eight-vertex graphs satisfy maximum nullity equal to the zero forcing number, it also establishes several additional methods to assist in the computation of minimum rank for general graphs.

Key words. Minimum rank, Maximum nullity, Zero forcing, Symmetric matrix.

AMS subject classifications. 05C50, 15A03.

1. Introduction. Determining the maximum multiplicity of an eigenvalue among all symmetric matrices in a space of matrices defined by a graph (definitions to follow) is an important relaxation of the inverse eigenvalue problem for a graph. In this article, we describe new approaches for this problem on small graphs and completely determine the minimum rank of all graphs with eight or fewer vertices. Work has continued on the minimum rank problem for a graph since [12], though some of the techniques established since then have not been implemented in code; we provide the relevant update to the algorithm described therein.

Given a simple undirected graph on n vertices, let $\mathcal{S}(G)$ be the set of all $n \times n$ symmetric matrices A such that $a_{ij} \neq 0$ if and only if $ij \in E(G)$. There is no condition on the diagonal entries of A (which allows for diagonal translation). Let

$$\mathrm{mr}(G) = \min\{ \operatorname{rank} A \mid A \in \mathcal{S}(G) \},$$

and let

$$\mathrm{M}(G) = \max\{ \operatorname{nullity}(A) \mid A \in \mathcal{S}(G) \}.$$

It is immediate that $\mathrm{mr}(G) + \mathrm{M}(G) = n$, and thus for fixed n, the problems of determining $\mathrm{mr}(G)$ and $\mathrm{M}(G)$ are equivalent. A matrix $A \in \mathcal{S}(G)$ satisfying $\operatorname{rank} A = \mathrm{mr}(G)$ is called a witness of $\mathrm{mr}(G)$. Determining the minimum rank /maximum nullity of a graph has been studied extensively since [25]. For most graphs, $\mathrm{mr}(G)$ (equivalently, $\mathrm{M}(G)$) is difficult to compute.

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†Department of Mathematics, Brigham Young University, Provo, UT, USA (wb@mathematics.byu.edu).
‡Department of Mathematics, Winston-Salem State University, Winston-Salem, NC, USA (hunnellm@wssu.edu). Supported by NSF award number 2447261.
§University of San Francisco, 2130 Fulton St, San Francisco, CA 94117, USA (johnd.hutchens@gmail.com).
¶Department of Mathematics, Brigham Young University Idaho, Rexburg, ID, USA (sinkovicj@byui.edu).

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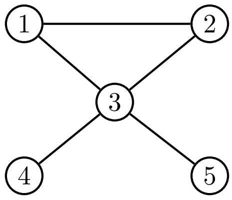

FIGURE 1. *The campstool graph W.*

In [16], a close connection was found between $\mathrm{M}(G)$ and a graph parameter called the zero forcing number. This is defined as follows. Each vertex of a graph is colored blue or white. We then define the standard color change rule:

> **Standard color change rule**: If a blue vertex b has a unique white neighbor w, then change the color of w to blue.

If w is the unique white neighbor of a blue vertex b, we say b forces w and denote this $b \to w$. A subset $Z \subset V(G)$ is called a zero forcing set if when the vertices of Z are colored blue, all vertices in G can be made blue by successively applying the standard color change rule. The zero forcing number $\mathrm{Z}(G)$ of a graph is the minimum size of a zero forcing set. For example, in the campstool graph in Fig. 1, no single vertex is a zero forcing set, but $\{1,4\}$ is a zero forcing set. Therefore, $\mathrm{Z}(\text{campstool}) = 2$. For most graphs G, calculating $\mathrm{Z}(G)$ is NP-hard, although several known algorithms exist. In general, at various points in the application of the color change rule, it may be possible for more than one blue vertex to force a white vertex. Given a zero forcing set B, we can choose a chronological set of forces F such that each force in the set is valid given the preceding forces. A chronological set of forces F induces a chain C_v for each $v \in B$, where $C_v = \{v = v_0, v_1, \ldots, v_k\}$ and $v_{i-1} \to v_i \in F$ for each $i \in \{1, 2, \ldots k\}$. We allow chains to consist of one vertex when $v \in B$ does not perform a force in F. The collection of chains $\{C_v\}_{v \in B}$ is called a set of zero forcing chains of B.

An exact algorithm for calculating the minimum rank of a graph is known and described in [9], but the number of variables makes the computation cost prohibitive. As described therein, for some graphs on seven vertices, the minimum rank can be found in a few minutes, but others take hours. The runtime required increases significantly as the number of vertices or edges increases and also as the minimum rank increases. So, it cannot be used to find the minimum rank of all graphs on eight vertices. In related work, [17], a bound for minimum rank that takes $O(2^{2^n+5n}n^3)$ time to compute is given. Additionally, as described in [22], there is an algorithm utilizing a scheme of alternating projections that can approximate the minimum rank of a graph that is efficient for graphs with up to 100 vertices. However, in addition to the numerical vulnerabilities and the possibility of converging to a matrix in the space of matrices for a subgraph of the intended graph described in [22], the iterative process yields a certificate matrix whose entries are difficult to parse for greater insight. The approach in this article is to employ various bounds on the maximum nullity of a graph determined by graph parameters whose computation is efficient for small graphs. See Section 7 for a description of these parameters. Although the minimum rank of a rational-valued (equivalently, integer-valued) matrix in the space of a graph can exceed the real-valued minimum rank [6], for small graphs, there is no difference. Thus, our techniques prefer integer matrices with reasonably sized entries.

In [16], a purely combinatorial bound on the maximum nullity was proven:

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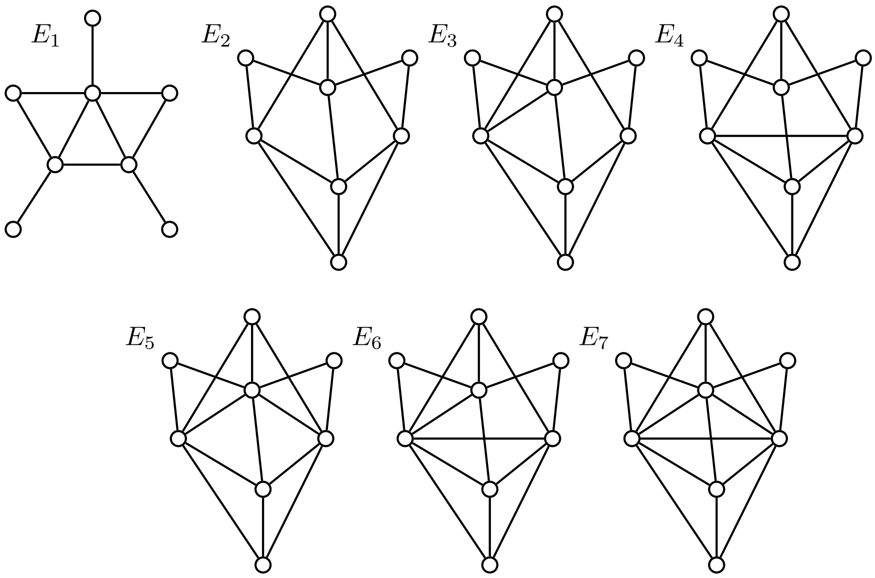

FIGURE 2. *The set of graphs $\mathcal{E}$ in Theorem 1.3.*

THEOREM 1.1. *For any graph G,* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

The computation of $\mathrm{Z}(G)$ is known to be NP-complete [14, 31], but the implementation has been improved by algorithms such as the integer programming formulation in [8] and the wavefront algorithm [10], which also appears under a modification for a related variant of zero forcing in [11].

A natural question is when equality between $\mathrm{M}(G)$ and $\mathrm{Z}(G)$ holds, and a satisfactory answer has remained elusive. Early on in the study of zero forcing, it was determined in [16] that:

THEOREM 1.2. *If G is a graph on 6 or fewer vertices, then* $\mathrm{M}(G) = \mathrm{Z}(G)$.

These results generated a lot of interest in $\mathrm{Z}(G)$. They were extended in [12], where the minimum rank of all 1044 graphs on seven vertices was calculated. By examination of their table, it can be seen that $\mathrm{M}(G) = \mathrm{Z}(G)$ for all graphs of order $n \leq 7$. In addition, there are several known graph families for which $\mathrm{M}(G) = \mathrm{Z}(G)$, such as the complete graphs K_n, complete bipartite graphs, cycles, and trees [16].

The equality $\mathrm{M}(G) = \mathrm{Z}(G)$ no longer holds for eight-vertex graphs, and one principal goal of this paper is to identify the connected eight-vertex graphs for which $\mathrm{M}(G) < \mathrm{Z}(G)$.

We are now ready to state our main result.

THEOREM 1.3. *Let $\mathcal{E} = \{E_1, E_2, \ldots, E_7\}$ be the set of graphs given in Fig. 2. A connected graph G on eight vertices satisfies* $\mathrm{M}(G) < \mathrm{Z}(G)$ *if and only if $G \in \mathcal{E}$.*

We now proceed to describe the organization of this article. In Section 2, we describe some of the basic results for studying the minimum rank of a graph. Section 3 establishes a general result that simplifies the computation of the minimum rank of some graphs, particularly a subset of graphs on eight vertices for which previously implemented bounds were inconclusive. Section 4 simplifies the proof that all graphs of order $n \leq 7$ satisfy $\mathrm{M}(G) = \mathrm{Z}(G)$. In Section 5, we discuss techniques for computing $\mathrm{M}(G)$ for graphs with a

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cut-vertex or a 2-separation. Section 6 describes an implementation of using a known witness for $\mathrm{mr}(G-v)$ to determine a witness of $\mathrm{mr}(G)$ under suitable conditions. A detailed description of the algorithm used to compute $\mathrm{M}(G)$ for all but 31 of the connected graphs on eight vertices is described in Section 7. Finally, Subsection 7.9 describes the technique used to compute a witness for $\mathrm{mr}(G)$ for graphs G that were not amenable to other methods.

2. Preliminaries. Unless otherwise stated, all graphs are simple and undirected. We denote the set of vertices of a graph G by $V(G)$ and its edges by $E(G)$. We denote the complete graph on n vertices by K_n.

A vertex $v \in V(G)$ is a cut vertex if $G - v$ has more connected components than G. If G has a cut vertex v, it is natural to consider G as composed of two graphs G_1, G_2, where one vertex of each is identified and denoted by v. We denote this decomposition by $G = G_1 \oplus_v G_2$.

We recall some additional parameters that we will need. Given a connected graph $G \neq K_n$ on n vertices, its vertex connectivity $\kappa(G)$ is the minimum number of vertices in $V(G)$ whose removal disconnects G, while $\kappa(K_n) = n-1$. A real symmetric $n \times n$ matrix A satisfies the Strong Arnold Hypothesis provided there is no nonzero symmetric $n \times n$ matrix X such that $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the entry-wise product. The parameter $\xi(G)$ is defined to be the maximum nullity among all real symmetric matrices such that $A \in \mathcal{S}(G)$ and A satisfies the Strong Arnold Hypothesis. The Colin de Verdiere parameter $\mu(G)$ is the largest nullity of any real symmetric matrix $M \in \mathcal{S}(G)$ such that:

1. M satisfies the Strong Arnold Hypothesis,
2. M has exactly one negative eigenvalue, of multiplicity 1,
3. for all i, j with $i \neq j, m_{ij} < 0$ if i and j are adjacent and $m_{ij} = 0$ if i and j are nonadjacent.

We will make use of the following results.

THEOREM 2.1. *[1] For any connected graph G,*

$$\kappa(G) \leq \xi(G) \leq \mathrm{M}(G) \leq \mathrm{Z}(G). \tag{2.1}$$

We will elaborate on the properties of $\mu(G)$ in Subsection 7.8.

If a graph G has a cut vertex v, then its minimum rank is expressible in terms of its induced subgraphs. Since we are interested in graphs with eight vertices, the minimum rank of all induced subgraphs is known.

THEOREM 2.2. *[20][2] Let G be a graph with a cut vertex v and suppose that $G = G_1 \oplus_v G_2$. Then,*

$$\mathrm{mr}(G) = \min\{\mathrm{mr}(G_1) + \mathrm{mr}(G_2), \mathrm{mr}(G_1 - v) + \mathrm{mr}(G_2 - v) + 2\}. \tag{2.2}$$

Rewriting this identity in terms of M gives the identity

$$\mathrm{M}(G) = \max\{\mathrm{M}(G_1) + \mathrm{M}(G_2), \mathrm{M}(G_1 - v) + \mathrm{M}(G_2 - v)\} - 1. \tag{2.3}$$

Thus, for any graph with a cut vertex, $\mathrm{M}(G)$ can be determined from $\mathrm{M}(H)$ for four proper induced subgraphs of G. A generalization of this formula exists, but requires the following definition.

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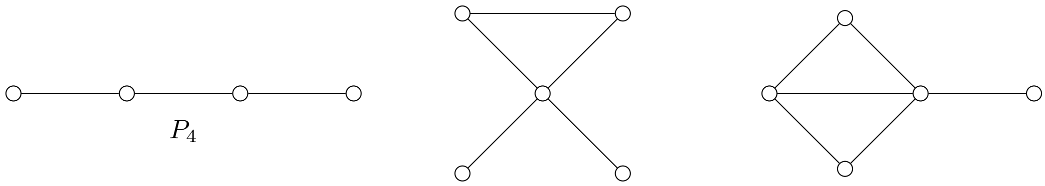

FIGURE 3. *Forbidden induced subgraph obstructions for graphs of minimum rank at most two among graphs of order at most eight.*

DEFINITION 2.3. *A 2-separation of a graph $G = (V, E)$ is a pair of subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cup V_2 = V$, $E_1 \cup E_2 = E$, $|V_1 \cap V_2| = 2$, and $E_1 \cap E_2 = \emptyset$.*

If a graph G has a 2-separation, then its maximum nullity is determined exactly by a formula.

THEOREM 2.4. *[29] Let G be a graph with a 2-separation (G_1, G_2), with $R = \{r_1, r_2\} = V(G_1) \cap V(G_2)$. Let H_1 and H_2 be obtained from G_1 and G_2, respectively, by inserting an edge between r_1 and r_2, and let $\overline{G_i}$, i=1,2, be the graph obtained from G_i by identifying r_1 and r_2. Then,*

$$\begin{aligned}\mathrm{M}(G) = \max\{\, &\mathrm{M}(G_1) + \mathrm{M}(G_2) - 2, \mathrm{M}(H_1) + \mathrm{M}(H_2) - 2,\\ &\mathrm{M}(\overline{G_1}) + \mathrm{M}(\overline{G_2}) - 2, \mathrm{M}(G_1 - r_1) + \mathrm{M}(G_2 - r_1) - 2,\\ &\mathrm{M}(G_1 - r_2) + \mathrm{M}(G_2 - r_2) - 2, \mathrm{M}(G_1 - R) + \mathrm{M}(G_2 - R) - 2\}.\end{aligned} \tag{2.4}$$

Similar to a graph with a cut vertex, $\mathrm{M}(G)$ for a graph G with a 2-separation can be determined from $\mathrm{M}(H)$ for several related graphs H associated with G.

Another tool we employ involves identifying dominating vertices whose deletion preserves equality between the maximum nullity and zero forcing number.

PROPOSITION 2.5. *Suppose that a graph G has a dominating vertex v. Then, $\mathrm{M}(G - v) = \mathrm{Z}(G - v)$ implies $\mathrm{M}(G) = \mathrm{Z}(G)$.*

Proof. By Corollary 7.2 in [3], $\mathrm{mr}(G) = \mathrm{mr}(G - v)$, and it is easy to see that $\mathrm{Z}(G) \leq \mathrm{Z}(G - v) + 1$. Then,

$$\begin{aligned}\mathrm{M}(G) &= n - \mathrm{mr}(G)\\ &= 1 + (n-1) - \mathrm{mr}(G - v)\\ &= 1 + \mathrm{M}(G - v)\\ &= 1 + \mathrm{Z}(G - v) \geq \mathrm{Z}(G).\end{aligned}$$ ◻

Proposition 2.5 suggests a more general technique we term lifting, which we discuss in Section 6.

Finally, there is a classification for graphs G with $\mathrm{mr}(G) = 2$, which is especially useful when determining the minimum rank of small graphs.

THEOREM 2.6. *[5] Let G be a connected graph on fewer than nine vertices. Then, $\mathrm{mr}(G) \leq 2$ if and only if G does not contain as an induced subgraph any of the graphs in Fig. 3.*

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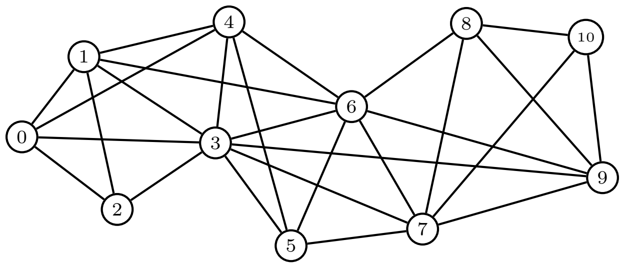

FIGURE 4. *The 3-path used in Example 4.*

3. Partial k-paths. The class of graphs known as k-trees has been studied previously in the context of the minimum rank of a graph. See for example [28, 30].

DEFINITION 3.1. *[7, 24] The complete graph on $k+1$ vertices, K_{k+1}, is a k-tree. A k-tree with n vertices, $n > k+1$, can be constructed from a k-tree with $n-1$ vertices by adding a vertex adjacent to all vertices of a k-clique of the existing k-tree and only to these vertices.*

DEFINITION 3.2. *[7, 24] A k-path graph G is an alternating sequence of distinct k- and $k+1$-cliques $e_0, t_1, e_1, t_2, \ldots, t_p, e_p$, $p \geq 1$, starting and ending with a k-clique and such that t_i contains exactly two k-cliques e_{i-1} and e_i for $1 \leq i \leq p$.*

LEMMA 3.3. *[7, 24] A k-path is a k-tree with exactly two vertices of degree k.*

EXAMPLE 4. *In the graph in Fig. 4, the sequence of distinct 3- and 4-cliques is as follows:*

$$\begin{array}{ll}
e_0 = \{0,1,2\} & t_1 = \{0,1,2,3\} \\
e_1 = \{0,1,3\} & t_2 = \{0,1,3,4\} \\
e_2 = \{1,3,4\} & t_3 = \{1,3,4,6\} \\
e_3 = \{3,4,6\} & t_4 = \{3,4,5,6\} \\
e_4 = \{3,5,6\} & t_5 = \{3,5,6,7\} \\
e_5 = \{3,6,7\} & t_6 = \{3,6,7,9\} \\
e_6 = \{6,7,9\} & t_7 = \{6,7,8,9\} \\
e_7 = \{7,8,9\} & t_8 = \{7,8,9,10\} \\
e_8 = \{8,9,10\} &
\end{array}$$

Starting with an initial colored set of e_0, the sequence of distinct 3- and 4-cliques induces a set of zero forcing paths. In general, $t_i \backslash e_i$ forces $t_i \backslash e_{i-1}$ for $1 \leq i \leq 8$.

$$\begin{array}{ll}
t_1\backslash e_1 \to t_1\backslash e_0 & 2 \to 3 \\
t_2\backslash e_2 \to t_2\backslash e_1 & 0 \to 4 \\
t_3\backslash e_3 \to t_3\backslash e_2 & 1 \to 6 \\
t_4\backslash e_4 \to t_4\backslash e_3 & 4 \to 5 \\
t_5\backslash e_5 \to t_5\backslash e_4 & 5 \to 7 \\
t_6\backslash e_6 \to t_6\backslash e_5 & 3 \to 9 \\
t_7\backslash e_7 \to t_7\backslash e_6 & 6 \to 8 \\
t_8\backslash e_8 \to t_8\backslash e_7 & 7 \to 10
\end{array}$$

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Thus, the three zero forcing chains are

$$0 \to 4 \to 5 \to 7 \to 10, 2 \to 3 \to 9 \text{ and } 1 \to 6 \to 8.$$

Lemma 3.5. *[7] Let G be a k-tree. Then G is k-connected.*

Theorem 3.6. *Let G be a k-path with sequence $e_0, t_1, e_1, t_2, \dots, t_p, e_p$ of distinct k- and $k+1$-cliques. Then, e_0 is a zero forcing set for G, and the paths generated by the recursive relation $t_i \backslash e_i$ forces $t_i \backslash e_{i-1}$ for $1 \le i \le p$ are a set of zero forcing chains for G. Consequently, $\mathrm{Z}(G) = \mathrm{M}(G) = k$.*

Proof. Proceed by induction on the order of G. The smallest k-path is K_{k+1} and $\mathrm{Z}(K_{k+1}) = \mathrm{M}(K_{k+1}) = k$. Assume that the statement of the theorem is true for all k-paths of order at least $n \ge k+1$. Let G be a k-path of order $n+1$ with sequence $e_0, t_1, e_1, t_2, \dots, t_p, e_p$ of distinct k- and $k+1$-cliques. Let $v = t_p \backslash e_{p-1}$. The vertex v is one of two vertices of degree k as v is adjacent to only the k vertices in e_{p-1}. The graph $G - v$ is a k-path on n vertices with sequence of distinct k- and $k+1$-cliques $e_0, t_1, e_1, t_2, \dots t_{p-1}, e_{p-1}$. By the inductive hypothesis, e_0 is a zero forcing set for $G - v$. The paths created by $t_i \backslash e_i$ forces $t_i \backslash e_{i-1}$, for $1 \le i \le p-1$, are zero forcing chains for $G - v$. None of the vertices of e_{p-1} are used to force other vertices of $G - v$. Since these are the only vertices adjacent to v in G, the zero forcing chains for $G - v$ are also valid in G. Thus, it only remains to force the vertex v in G. The vertex of e_{p-1} that forces v is $t_p \backslash e_p$. Thus, e_0 is a zero forcing set for G and $\mathrm{Z}(G) \le |e_o| = k$. Since G is k-connected, $\mathrm{Z}(G) \ge \mathrm{M}(G) \ge \kappa(G) = k$ by Theorem 2.1. Therefore, $\mathrm{Z}(G) = \mathrm{M}(G) = k$. ∎

Lemma 3.7. *Let G be a k-path with sequence $e_0, t_1, e_1, t_2, \dots, t_p, e_p$ of k- and $k+1$-cliques. Let P be the set of zero forcing chains induced by the sequence of k- and $k+1$-cliques. Let e be an edge of one of the chains in P. Then, $G - e$ is not k-connected.*

Proof. Let G be as described in the theorem. Let e be an edge of one of the zero forcing chains. Since the set of zero forcing chains was induced by the sequence of k- and $k+1$-cliques, there exists an r, $1 \le r \le p$, such that $e = (t_r \backslash e_r, t_r \backslash e_{r-1})$. Let $u = t_r \backslash e_r$ and $w = t_r \backslash e_{r-1}$. By construction, there exists a set S of $k-1$ vertices such that $e_r = w \bigcup S$ and $e_{r-1} = u \bigcup S$. Since u is in e_{r-1} but not in e_r, u is not in e_j for $r \le j \le p$. Thus, u is only adjacent to vertices in e_j for $1 \le j \le r$. Similarly, w is only adjacent to vertices in e_j for $r \le j \le p$. Consider the graph H obtained from $G - e$ by deleting the vertices in S. Since all the vertices in e_r have been deleted with the exception of w, and the edge uw has been deleted, u and w are in distinct components. Thus, the vertex connectivity of $G - e$ is less than or equal $k-1$. Therefore, $G - e$ is not k-connected. ∎

Definition 3.8. *If G is a k-tree (k-path), then any subgraph of G is a partial k-tree (partial k-path).*

Lemma 3.9. *[24, 30] If G is a partial k-path, then G can be completed to a k-path on the same number of vertices.*

Theorem 3.10. *If G is a k-connected partial k-path, then $\mathrm{Z}(G) = \mathrm{M}(G) = k$.*

Proof. Let G be as stated in the theorem. By Lemma 3.9, complete G to a k-path H. By Theorem 3.6, $\mathrm{Z}(H) = k$ and the sequence of distinct k- and $k+1$-cliques induce a set of k zero forcing chains. Since G is k-connected, all of the edges used in the zero forcing chains for H are present in G. Thus, the same zero forcing chains for H can be used to color the vertices of G. Thus, $\mathrm{Z}(G) \le k$. As G is k-connected, $\mathrm{Z}(G) \ge \mathrm{M}(G) \ge k$ by Theorem 2.1. Therefore, $\mathrm{Z}(G) = \mathrm{M}(G) = k$. ∎

Theorem 3.11. *[30] For a 3-connected graph $G = (V, E)$, the following are equivalent:*

(i) *G is a partial 3-path;*
(ii) *$M(G) = \xi(G) = 3$;*
(iii) *G has no $K_5, K_{2,2,2}, K_{3,3}, Q_3,$ and no $Q_3 Y\Delta$ minor.*

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Theorem 3.12. *Let G be a 3-connected graph such that* $\mathrm{Z}(G) = 4$. *Then,* $\mathrm{M}(G) = 4$.

Proof. By Theorem 2.1, $4 = \mathrm{Z}(G) \geq \mathrm{M}(G) \geq \xi(G) \geq \kappa(G) \geq 3$. If $\mathrm{M}(G) = 3$, then by Theorem 3.11, (ii) $\Rightarrow$ (i), G is a partial 3-path. Then, by Theorem 3.10, $\mathrm{Z}(G) = 3$, contradicting the hypothesis. Therefore, $\mathrm{M}(G) = 4$. ∎

4. Maximum nullity and zero forcing number coincide for seven-vertex graphs. Before proceeding to the proof of our main theorem, we first show how the proof of the known result $\mathrm{M}(G) = \mathrm{Z}(G)$ for all seven-vertex graphs can be simplified considerably using the results above. It suffices to show that equality holds for the 853 connected seven-vertex graphs. We establish the equality by examining various classes of graphs.

1. Graphs with a cut-vertex:
   Calculating $\mathrm{M}(G)$ for all seven-vertex graphs with a cut-vertex by (2.3) and calculating each $\mathrm{Z}(G)$ by an exhaustive search, we find that $\mathrm{M}(G) = \mathrm{Z}(G)$ for each.
2. Graphs with a $2-$separation:
   As in the previous case, one must calculate $\mathrm{M}(G)$ for all seven-vertex connected graphs G with a $2-$separation using Equation (2.4) and also calculate $\mathrm{Z}(G)$ for each of these to determine that each pair of these are equal.
3. $3-$connected graphs:
   It then suffices to determine whether or not $\mathrm{M}(G) = \mathrm{Z}(G)$ for the 136 $3-$connected graphs on seven vertices.

   If G has a dominating vertex, then $\mathrm{M}(G - v) = \mathrm{Z}(G - v)$ by Theorem 1.2, so $\mathrm{M}(G) = \mathrm{Z}(G)$ by Proposition 2.5.

   If $\kappa(G) = \mathrm{Z}(G)$, then $\mathrm{M}(G) = \mathrm{Z}(G)$ by Theorem 2.1.

   Suppose then that $3 \leq \kappa(G) < \mathrm{Z}(G) \leq 6$.

   If $\mathrm{Z}(G) = 6$, then $G = K_7$ and $\mathrm{M}(K_7) = 6 = \mathrm{Z}(K_7)$.

   So we can assume that $\kappa(G) < \mathrm{Z}(G) \leq 5$.

   If $\mathrm{Z}(G) = 5$, then $\mathrm{M}(G) \leq 5$ and $\mathrm{mr}(G) \geq 2$. We now apply Theorem 2.6. Checking all the remaining graphs with $\mathrm{Z}(G) = 5$, we find that none of these contain P_4, campstool, or dart. Then for each, $2 \leq \mathrm{mr}(G) \leq 2$, which implies that $\mathrm{M}(G) = 5 = \mathrm{Z}(G)$.

   The last case is $\mathrm{Z}(G) = 4$. By Theorem 3.12, $\mathrm{M}(G) = 4$.

   So in all cases, $\mathrm{M}(G) = \mathrm{Z}(G)$.

5. Graphs on eight vertices with a cut vertex or a $2-$separation. In light of Equation (2.3) and Equation (2.4), it is natural to begin a search for graphs G with the property $\mathrm{M}(G) < \mathrm{Z}(G)$ among graphs that possess a cut-vertex or 2-separation.

1. Eight-vertex graphs with a cut-vertex.

   We run through the graphs on eight vertices with a cut-vertex calculating $\mathrm{M}(G)$ by Equation (2.3) and comparing each with $\mathrm{Z}(G)$. We find that $\mathrm{M}(G) = \mathrm{Z}(G)$ for all but one of these graphs, the

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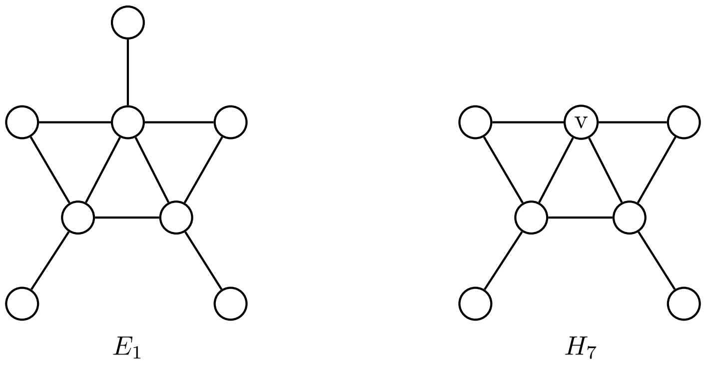

FIGURE 5. *The graphs E_1 and H_7.*

graph E_1 in Fig. 5. For E_1, we have $\mathrm{M}(E_1) = 2$ and $\mathrm{Z}(E_1) = 3$. This graph is well known in multiple contexts; for example, it is the simplest of the exceptional graphs given in Appendix B of [21]. As this is the first exceptional graph so far, we give an argument that $\mathrm{Z}(E_1) = 3$ and $\mathrm{M}(E_1) = 2$. It is obvious that the set of three pendent vertices is a zero forcing set, so $\mathrm{Z}(E_1) \leq 3$. Examination of all two element subsets of $V(E_1)$ shows that none is a zero forcing set, so $\mathrm{Z}(E_1) = 3$.

Consider the graph H_7 in Fig. 5, which is graph 482 in [26]. According to the table in [12], $\mathrm{mr}(H_7) = 5$. The graph $H_7 - v$ is a tree H on six vertices and has zero forcing number equal to 2, so $\mathrm{mr}(H) = 4$. Then by the cut-vertex theorem,

$$\mathrm{mr}(E_1) = \min\{\mathrm{mr}(H_7) + \mathrm{mr}(K_2), \mathrm{mr}(H) + \mathrm{mr}(K_2 - v) + 2\} = \min\{6, 6\} = 6.$$

Thus, $\mathrm{M}(E_1) = 8 - \mathrm{mr}(E_1) = 8 - 6 = 2 < 3 = \mathrm{Z}(E_1)$, so $\mathrm{M}(E_1) \neq \mathrm{Z}(E_1)$.

2. Eight-vertex graphs with a $2-$separation

Comparing $\mathrm{M}(G)$ and $\mathrm{Z}(G)$ for these graphs, we find six graphs with $\mathrm{M}(G) = 3 < 4 = \mathrm{Z}(G)$; see Fig. 6. The dashed lines denote edges not present in the first graph, E_2.

Once the 6 graphs in Fig. 6 have been identified, there is a simpler way to see that $\mathrm{M}(E_i) = 3$ and $\mathrm{Z}(E_i) = 4$ for $i = 2, 3, \ldots, 7$. It depends on one more graph theory parameter, defined by a variation of the standard color change rule. We include this because it gives a much clearer picture of why $\mathrm{M}(G)$ and $\mathrm{Z}(G)$ differ for these graphs.

Given a simple undirected graph G on n vertices, we consider a graph $\widehat{G}$ with the same vertices and edges, but with loops on some of the vertices. There are 2^n such graphs for each G. Then,

$$S(\widehat{G}) = \{A \in \mathcal{S}(G) \,|\, a_{ii} \neq 0 \text{ if and only if there is a loop at } i\}.$$

The color change rule for graphs with loops is a relaxation of the standard zero forcing color change rule:

Looped graph color change rule: If *any* vertex has exactly one white neighbor (possibly itself), then that neighbor can be colored blue.

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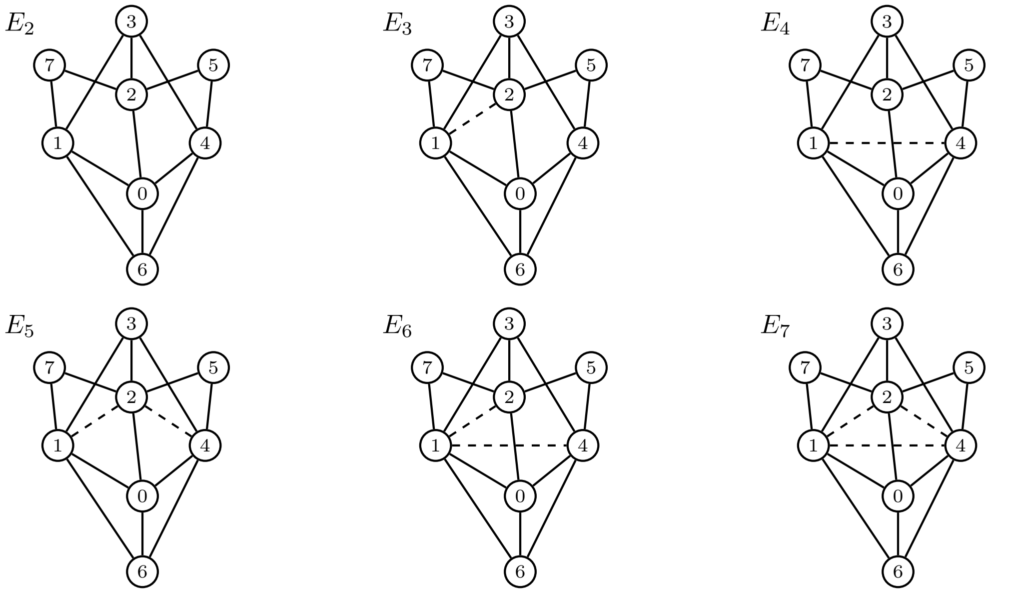

Figure 6. *The graphs $E_2, E_3, \dots, E_7$.*

The zero forcing number of a loop graph is the minimum number of vertices needed to force every vertex to be blue applying this amended rule. $\widehat{\mathrm{Z}}(G)$ is the maximum of $\mathrm{Z}(\widehat{G})$ over all 2^n loop graphs. By [1], $\mathrm{M}(G) \le \widehat{\mathrm{Z}}(G)$.

We now use $\widehat{\mathrm{Z}}(G)$ to simultaneously show that $\mathrm{M}(E_i) \le 3, i = 2 \dots 7$. We only need to consider two cases.

a. There is no loop on vertex 7
b. There is a loop on vertex 7

Regardless of which of these E_i we choose, the same forcing sets and forcing sequences work for each.

a. Color vertices 0, 1, 3 blue. Then, the sequence of forces

$$7 \to 2,\ 3 \to 4,\ 0 \to 6,\ 4 \to 5,\ 2 \to 7,$$

colors all vertices blue.

b. Color vertices 3, 4, 5 blue. Then, the sequence

$$5 \to 2,\ 3 \to 1,\ 7 \to 7,\ 2 \to 0,\ 0 \to 6,$$

colors all vertices blue.

It follows that $\mathrm{M}(E_j) \le \widehat{\mathrm{Z}}(E_j) = 3$. A zero forcing program shows that $\mathrm{Z}(E_i) = 4, i = 2 \dots 7$.

We now recall our main theorem.

Theorem 1.3. *Let $\mathcal{E} = \{E_1, E_2, \dots, E_7\}$ be the set of graphs given in Fig. 2. A connected graph G on eight vertices satisfies $\mathrm{M}(G) < \mathrm{Z}(G)$ if and only if $G \in \mathcal{E}$.*

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We have already verified the reverse implication; we now approach the forward implication.

We can construct minimum rank matrices for each graph $E_2, \dots, E_7$ using the same clique cover. Notice that E_7 can be covered by the following five cliques $V_1 = \{1,2,3,4\}, V_2 = \{1,2,7\}, V_3 = \{2,4,5\}, V_4 = \{0,1,4,6\}$, and $V_5 = \{0,2\}$. We can construct a matrix with rank at most 5 of the form

$$\begin{bmatrix} a_4+a_5 & a_4 & a_5 & 0 & a_4 & 0 & a_4 & 0 \\ a_4 & a_1+a_2+a_4 & a_1+a_2 & a_1 & a_1+a_4 & 0 & a_4 & a_2 \\ a_5 & a_1+a_2 & a_1+a_2+a_3+a_5 & a_1 & a_1+a_3 & a_3 & 0 & a_2 \\ 0 & a_1 & a_1 & a_1 & a_1 & 0 & 0 & 0 \\ a_4 & a_1+a_4 & a_1+a_3 & a_1 & a_1+a_3+a_4 & a_3 & a_4 & 0 \\ 0 & 0 & a_3 & 0 & a_3 & a_3 & 0 & 0 \\ a_4 & a_4 & 0 & 0 & a_4 & 0 & a_4 & 0 \\ 0 & a_2 & a_2 & 0 & 0 & 0 & 0 & a_2 \end{bmatrix},$$

and choose different nonzero values for a_1, a_2, a_3, a_4, a_5 to achieve a rank 5 matrix for each E_i, $i \in \{2, 3, \dots 7\}$, with the associated zero/nonzero pattern.

6. Lifting minimum rank matrices. Since the minimum rank of all graphs on seven vertices is known, Proposition 2.5 suggests a technique to determine witnesses for the minimum rank of graphs G on eight vertices if G has the same minimum rank as one its induced seven-vertex graphs.

Observation 1. *Let G be a graph, $v \in V(G)$, and $A \in S(G-v) \subseteq \mathbb{R}^{n-1\times n-1}$ be a witness for $\operatorname{mr}(G-v)$. Given $w \in \mathbb{R}^{n-1}$ construct the matrix $B = \begin{bmatrix} A & Aw \\ w^TA & w^TAw \end{bmatrix}$. Then, $\operatorname{rank} B = \operatorname{rank} A$. Furthermore, if $B \in \mathcal{S}(G)$, then B is a witness for $\operatorname{mr}(G)$.*

Proof. $\operatorname{mr}(G) \le \operatorname{rank} B = \operatorname{rank} A = \operatorname{mr}(G-v) \le \operatorname{mr}(G)$. ☐

We term the matrix B constructed in Observation 1 a lifting of A. To ensure a lifting is in $\mathcal{S}(G)$, we make an observation in Lemma 6.2. Let $\mathcal{N}(C)$ denote the (right) nullspace of a matrix C. A submatrix $A[\alpha,\beta]$ of a matrix A is the matrix lying in the rows α and the columns β. Let $L = \{1, 2, \dots, n\}$ and $L_v = L \setminus \{v\}$.

Lemma 6.2. *Let A be a witness for $\operatorname{mr}(G-v)$ and B a lifting of A. Then, B is a witness of $\operatorname{mr}(G)$ if and only if:*

1. $w \in \mathcal{N}(A[L_v \setminus N_G(v), L_v])$
2. $[Aw]_i \neq 0$ *for all* $i \in N_G(v)$.

Proof. The first condition ensures that B has zeroes in all entries corresponding to non-neighbors of v in G, while the second ensures that B has nonzero entries corresponding to neighbors of v. ☐

It is easy to ensure that the first condition in Lemma 6.2 is satisfied, but it is not always possible to satisfy the second condition. For instance, the second condition will never be satisfied if a row of $A[N_G(v), L_v]$ is equal to a row of $A[L_v \setminus N_G(v), L_v]$. For implementation, we adopt a naive approach for choosing $w \in \mathcal{N}(A[L_v \setminus N_G(v), L_v])$. Our code computes a basis for $\mathcal{N}(A[L_v \setminus N_G(v), L_v])$, say $\{x_1, x_2, \dots, x_r\}$ and uses $w = \sum_{i=1}^r x_i$. It then determines if $B \in \mathcal{S}(G)$, in which case we conclude $r_v(G) = 0$ and $\mathrm{Z}(G) = \mathrm{M}(G)$.

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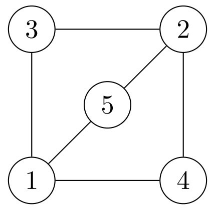

FIGURE 7. *The graph G in Example 3.*

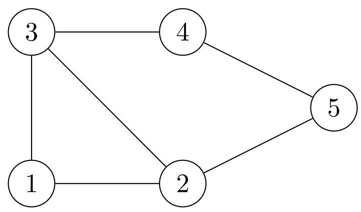

FIGURE 8. *The graph G in Example 4.*

In practice, the naive guess is quite good, but in some cases, we needed to be careful with the null vector chosen. Using this technique, we were able to show that 181 of the remaining 403 graphs at this stage satisfy $\mathrm{M}(G) = \mathrm{Z}(G)$. See Subsection 7.7 for more details. We now give examples of implementation of the technique when it is successful and when it is unsuccessful.

EXAMPLE 3. *We illustrate the technique with examples. Consider the graph $G = K_{2,3}$ as labeled in Fig. 7. Consider the subgraph H of G obtained by deleting vertex 5. We adopt the notation of Observation 1. Then, we have a known minimum rank witnessing matrix for H, namely,*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

We have $L_v = \{1, 2, 3, 4\}$, $N_G(5) = \{1, 2\}$, and $L_v \setminus N_G(5) = \{3, 4\}$, and thus

$$\mathcal{N}(L_5 \setminus N_G(5), L_5) = \left\langle (1, -1, 0, 0)^T, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T \right\rangle.$$

Consider $w = (1, -1, 1, 1)^T \in \mathcal{N}(A[\{3, 4\}, \{1, 2, 3, 4\}])$. Then by computing $Aw = (2, 2, 0, 0)^T$, we see that w satisfies the conditions in Lemma 6.2. We compute $w^T Aw = x = 0$, and our minimum rank matrix for G becomes

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 4. *For the graph in Fig. 8, a matrix witnessing the minimum rank of $G - \{5\}$ is given by*

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

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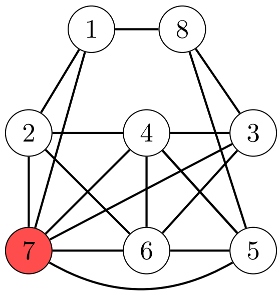

FIGURE 9. *The graph in Example 5.*

Since $1 \in L_5 \setminus N_G(5)$, $2 \in N_G(5)$, *and* $A[\{1\}, L_5] = A[\{2\}, L_5]$, *no linear combination of the columns of* A *can have a zero entry in the first component but a nonzero entry in the second.*

In practice, the new row and column can be inserted anywhere in A to obtain its lift B, which allows us to avoid relabeling the graph.

EXAMPLE 5. *Consider the graph* G *in Fig. 9. We delete the red-colored vertex to obtain* $G - v$, *for which a witness of* $\mathrm{mr}(G - v)$ *is known, namely,*

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{bmatrix}.$$

Applying the lifting technique, we obtain a witness for $\mathrm{mr}(G)$, *with the added row and column indicated in red:*

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -6 & 2 \\ -1 & 1 & 0 & 2 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & -2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 10 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & -2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 10 & 0 \\ -6 & 4 & -2 & 10 & -2 & 10 & 14 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

7. Algorithmic details. We employ code implemented in Sage to assist with our calculations. All code is available at https://github.com/hunnellm/maximum-nullity.

7.1. Previously implemented bounds. Our goal is to determine the minimum rank of all graphs with eight vertices, and it is enough to consider the 11,117 connected eight-vertex graphs. Using the program in [13], we are left with 1803 connected graphs whose minimum rank is undetermined by previously implemented techniques. The program does, however, compute lower and upper bounds for the minimum rank of all graphs. We note that the program establishes again that $\mathrm{M}(E_1) = 2 < 3 = \mathrm{Z}(E_1)$.

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For a graph G, there is previously implemented code to compute the enhanced zero forcing number $\widehat{\mathrm{Z}}(G)$. This code identifies $\widehat{\mathrm{Z}}(E_i) = 3 < 4 = \mathrm{Z}(E_i)$ for $i \in \{2, 3, \ldots, 7\}$ as before. This establishes that $\mathrm{M}(E_i) < \mathrm{Z}(E_i)$ for $i \in \{1, 2, \ldots, 7\}$.

Our remaining work is to show that $\mathrm{M}(G) = \mathrm{Z}(G)$ for the remaining 1797 connected graphs with eight vertices.

7.2. Dominating vertices. Determining the degree sequence of a graph is computationally efficient, which suggests that any simplification of the minimum rank problem from the degree sequence should be implemented early in the algorithm.

Recall that all graphs G with order at most seven satisfy $\mathrm{M}(G)$= $\mathrm{Z}(G)$. Proposition 2.5 leads to an observation for graphs on eight vertices.

Corollary 7.1. *Let G be graph on eight vertices. If G has a dominating vertex, then* $\mathrm{M}(G) = \mathrm{Z}(G)$.

Implementing this corollary establishes that $\mathrm{M}(G) = \mathrm{Z}(G)$ for an additional 254 remaining graphs. 1543 graphs remain at this stage.

7.3. Graphs with a 2-separation. A formula for the maximum nullity of graphs with $\kappa(G) = 2$ was given in Equation (2.4). Our program implements this formula and establishes $\mathrm{M}(G) = \mathrm{Z}(G)$ for a further 948 graphs, which leaves 595 graphs on eight vertices whose minimum rank is undetermined.

7.4. Vertex connectivity. From Theorem 2.1, it follows that for any graph G, $\kappa(G) = \mathrm{Z}(G)$ implies $\mathrm{M}(G) = \mathrm{Z}(G)$. Of our remaining graphs, 14 satisfy $\kappa(G) = \mathrm{Z}(G)$ and thus there 581 graphs on eight vertices whose minimum rank is undetermined.

7.5. Twin vertex reduction. In a graph G, if $v, w \in V(G)$ have the same open neighborhoods they are twins. Let v, w be twins in G. If $vw \in E(G)$, then v, w are adjacent twins and are independent twins otherwise.

Suppose G is a graph and $A \in \mathcal{S}(G)$ is a minimum rank witness of G, i.e., $\mathrm{mr}(G) = \mathrm{rank}(A)$. For $v \in V(G)$, denote by a_{vv} the diagonal entry of A corresponding to v.

Proposition 7.2 ([13]). *Let G be a graph with twin vertices v, w*

1. *If v, w are independent twins and $G - w$ has a minimum rank witness such that $a_{vv} = 0$, then* $\mathrm{mr}(G) = \mathrm{mr}(G - w)$.
2. *If v, w are adjacent twins and $G - w$ has a minimum rank witness such that $a_{vv} \neq 0$, then* $\mathrm{mr}(G) = \mathrm{mr}(G - w)$.

Proposition 7.2 suggests that for graphs with twins v, w, the determination of minimum rank can be reduced to smaller graphs in cases where a_{vv} can be determined to be zero or nonzero. Sufficient conditions for this determination were given in [4], where they appear as corollaries.

Proposition 7.3. *Let G be a graph, v be a vertex of G, and G^ℓ the loop configuration with a loop at v and no other loops or nonloops are specified. If there exists a set of fewer than $\mathrm{M}(G)$ vertices such that starting with these vertices blue, every vertex is eventually colored blue under the standard color change rule with the additional condition that v can color itself if it has no white neighbors, then $a_{vv} = 0$ for witnesses of* $\mathrm{mr}(G)$.

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Proposition 7.4. *Let G be a graph, v be a vertex of G, and G^ℓ the loop configuration with no loop at v and no other loops or nonloops are specified. If there exists a set of fewer than* $\mathrm{M}(G)$ *vertices such that starting with these vertices blue, every vertex is eventually colored blue under the standard color change rule with the additional condition that v can color a white neighbor u if it has no other white neighbors, then $a_{vv} \neq 0$ for witnesses of* $\mathrm{mr}(G)$.

Note that these are not mutually exclusive, and that in each proposition, the color change rule is modified by a special case of the looped color change rule. In some cases, no information about the diagonal entry is gained. Our code implements these results and thereby determines that $\mathrm{M}(G) = \mathrm{Z}(G)$ for 84 of the remaining graphs. Thus, there are 497 left to consider.

7.6. Partial 3-path reduction. We determine that 94 of the remaining graphs satisfy the hypotheses of Theorem 3.12, and thus we are left with 403 graphs to consider.

7.7. Lifting techniques. We now implement the techniques discussed in Section 6. This allows us to find witnesses establishing $\mathrm{M}(G) = \mathrm{Z}(G)$ for 181 of the remaining graphs.

We started by compiling a list (not necessarily complete) of known witnesses for the minimum rank of graphs on seven vertices. We then form the set of graphs, denoted by $\mathcal{K}$, corresponding to these matrices using the zero-nonzero pattern of the matrix.

For each of our remaining graphs G, we compute the set of induced subgraphs $\mathcal{H} = \{G - v \mid v \in V(G)\}$. For each $H \in \mathcal{K} \cap \mathcal{H}$, we use the techniques in Section 6 to determine a candidate matrix A. We then tested whether $A \in \mathcal{S}(G)$ and if $\mathrm{nullity}(A) = \mathrm{Z}(G)$ for each of the candidates produced. When both tests are affirmative, A it a witness for the minimum rank of G. The catalog of the 181 graphs and their witnesses determined in this way are also available at https://github.com/hunnellm/maximum-nullity.

There are 222 graphs left to consider.

7.8. Colin de Verdiere parameter. The Colin de Verdiere parameter $\mu(G)$ is a lower bound on $\mathrm{M}(G)$:

Theorem 7.5 ([1]). *For any graph G, $\mu(G) \leq \mathrm{M}(G)$.*

Definition 7.6. *A* contraction *of an edge in a simple graph G is obtained by identifying the two adjacent vertices of the edge and suppressing any loops or multiple edges formed in the process. A* minor *of a graph G is any graph formed by a series of edge deletions, deletions of isolated vertices, and edge contractions.*

Definition 7.7. *The Petersen family, $\mathcal{P}$, consists of the graphs in Fig. 10.*

Theorem 7.8 ([27], p. 165). *The parameter μ is minor-monotone; i.e., if H is a minor of G, then $\mu(H) \leq \mu(G)$.*

Theorem 7.9 ([27], p. 188). *For a graph G, we have $\mu(G) \leq 4$ if and only if G does not have a minor in the Petersen family. Equivalently, $\mu(G) \geq 5$ if and only if G has a minor in the Petersen family.*

Of course, in order for H to be a minor of an 8-vertex graph and be a member of the Petersen family, it must be one of the first 5 graphs in the figure above. Call this group of 5 graphs $\mathcal{P}_5$.

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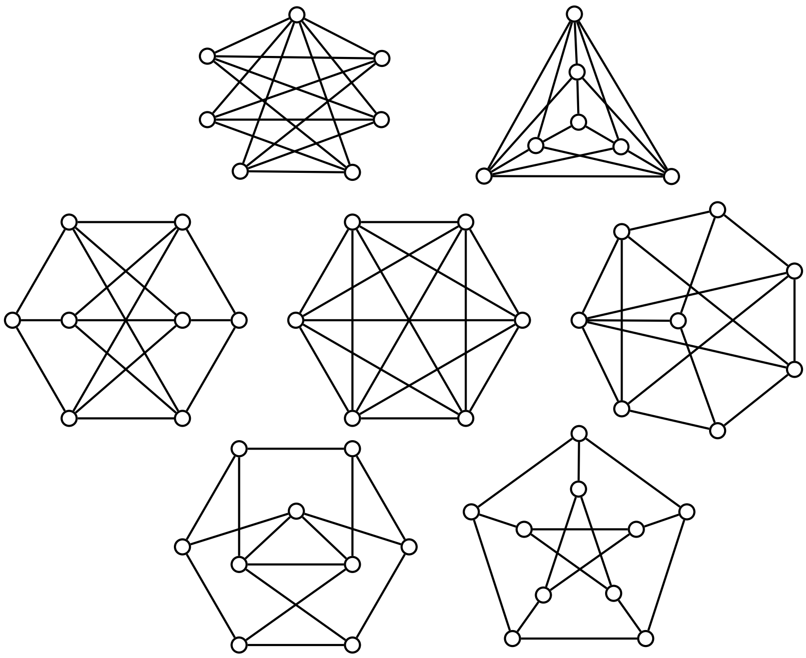

FIGURE 10. *The Petersen family of graphs $\mathcal{P}_5$.*

THEOREM 7.10. *If G is an 8-vertex graph with $\mathrm{Z}(G) = 5$ and G has a minor in $\mathcal{P}_5$, then $\mathrm{M}(G) = \mathrm{Z}(G)$.*

Proof. A minor in $\mathcal{P}_5$ implies $\mu(G)$ is at least five, so $5 \le \mu(G) \le \mathrm{M}(G) \le \mathrm{Z}(G) = 5$. Thus, $\mathrm{M}(G) = \mathrm{Z}(G) = 5$. ☐

Implementing this result in our code, we find 191 additional graphs for which $\mathrm{M}(G) = \mathrm{Z}(G)$.

7.9. The final 31 graphs. There are 31 graphs that are not covered by any of the arguments above. Here, we provide a general construction method for minimum rank matrices for these 31 graphs. We label each graph with a graph6 string and provide a minimum rank matrix that corresponds to the vertex labeling of the provided graph6 string.

For all graphs G in the last 31, we have $\mathrm{Z}(G) = 5$, so $\mathrm{mr}(G) \ge 3$. To verify we have constructed a matrix for each G that has at most rank 3, we construct $A \in \mathcal{S}(G)$ by choosing a vector representation for the vertices of G. In each case, we were able to construct $A = M^T DM$, where M is a 3×8 matrix containing the column vectors corresponding to the vertices in G and $D = \mathrm{diag}(1, 1, -1)$. We were able to find a minimum rank matrix in $\mathcal{S}(G)$ with two positive eigenvalues and one negative eigenvalue for the remaining graphs. Note that the matrix D induces an (indefinite) bilinear form on $\mathbb{R}^3$.

The construction of the matrix M is done with the aid of several heuristics derived from the graph and the bilinear form. For instance, a set of three independent vertices in a graph under consideration can be identified with the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ since $(e_i, e_j) = 0$ when $i \ne j$. This also guarantees that M will have rank 3. Additionally, independent twins v and w in the graph can be identified with the same isotropic vector since the bilinear form will generate the same neighborhoods while ensuring no edge between v and w. Since we are working in three dimensions, it is convenient to choose a Pythagorean triple for the

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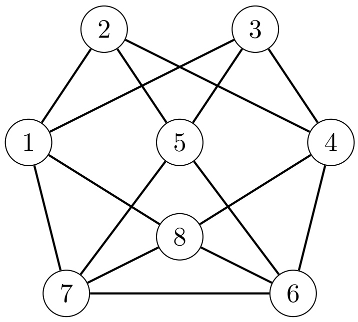

FIGURE 11. *The graph in Example 11.*

entries of the isotropic vector. Similarly, the same nonisotropic vector may be chosen for a pair of adjacent twins. The remaining vectors can then be determined by considering the restrictions imposed by the edges of the graph and any vectors determined by heuristics. See [18] for a more complete treatment of this topic.

EXAMPLE 11. *To illustrate the process, we compute* $A = M^T DM$ *for the graph* G *with graph6 string* `GrW[[k`*, which is shown in Fig. 11.*

Observe that $\{1, 4, 5\}$ *is a set of independent vertices and that* $\{2, 3\}$ *is a pair of independent twins. Following the heuristics above, we assign the standard basis vectors in* $\mathbb{R}^3$ *to* 1*,* 4*, and* 5*, respectively, and assign* $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}^T$ *to vertices* 2 *and* 3*. The remaining vertices,* 6*,* 7*, and* 8*, are non-neighbors of* 2 *and* 3*, and therefore, a vector* $\begin{bmatrix} a & b & c \end{bmatrix}^T$ *assigned to them must satisfy* $3a + 4b = 5c$*. Furthermore, each vertex in* $\{6, 7, 8\}$ *has exactly two neighbors in* $\{1, 4, 5\}$*, which requires exactly one of* a*,* b*, or* c *to be zero. One possible choice is*

$$M = \begin{bmatrix} 1 & 3 & 3 & 0 & 0 & 0 & 5 & 4 \\ 0 & 4 & 4 & 1 & 0 & 5 & 0 & -3 \\ 0 & 5 & 5 & 0 & 1 & 4 & 3 & 0 \end{bmatrix}.$$

Therefore,

$$A = M^T DM = \begin{bmatrix} 1 & 3 & 3 & 0 & 0 & 0 & 5 & 4 \\ 3 & 0 & 0 & 4 & -5 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 & -5 & 0 & 0 & 0 \\ 0 & 4 & 4 & 1 & 0 & 5 & 0 & -3 \\ 0 & -5 & -5 & 0 & -1 & -4 & -3 & 0 \\ 0 & 0 & 0 & 5 & -4 & 9 & -12 & -15 \\ 5 & 0 & 0 & 0 & -3 & -12 & 16 & 20 \\ 4 & 0 & 0 & -3 & 0 & -15 & 20 & 25 \end{bmatrix},$$

and observe that $A \in \mathcal{S}(G)$ *and* $\operatorname{rank} A = 3$*. Thus,* A *is a witness for the minimum rank of* G*.*

This process is repeated for the 30 remaining graphs. The graph6 string, a witness, and the adjacency matrix for each of the 31 graphs whose maximum nullity was undetermined by previous methods is given in the appendix.

8. Conclusions and future work. We have determined the minimum rank (equivalently, maximum nullity) for all graphs on eight vertices and established that for exactly seven graphs the maximum nullity

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and zero forcing number differ. Moreover, in each case, the difference is exactly one. It remains an open question to determine the graph(s) G with the fewest number of vertices such that $\mathrm{Z}(G) - \mathrm{M}(G) = 2$ (and the natural generalizations). The best bound of which we are aware is $|V(G)| = 15$. Such a graph can be obtained by identifying pendent vertices in two copies of E_1. We propose the following question:

QUESTION 1. *What is the smallest order of a graph that satisfies* $\mathrm{Z}(G) - \mathrm{M}(G) = k$ *for* $k \geq 2$*?*

The strategy put forth in the preceding sections is necessarily piece-meal, since the only known algorithm for the exact computation of the minimum rank of a graph is inefficient. This inefficiency becomes worse as the order of the graph grows, thus it is natural to consider how far the approach in this article can be extended.

The value of $\mathrm{M}(G)$ can be reduced to the maximum nullity of smaller graphs using Equations (2.3) or (2.4) when G has low connectivity. Alternatively, one can use algebraic geometry (see [9]) if the graph is relatively sparse. However, most graphs are not amenable to either approach.

Preliminarily, our methods are able to determine the maximum nullity for at least 87% of the graphs on nine vertices. There are two primary obstacles to completing the classification, and these remain the same for larger values of n:

1. Few known parameters bounded between $\mathrm{M}(G)$ and $\mathrm{Z}(G)$
2. Limited characterizations for bounding parameters at non-extreme values

Consider first Item 1. We are aware of only two parameters bounded below by $\mathrm{M}(G)$ and above by $\mathrm{Z}(G)$. As discussed earlier, the enhanced zero forcing number of graph G, denoted $\widehat{\mathrm{Z}}(G)$, satisfies $\mathrm{M}(G) \leq \widehat{\mathrm{Z}}(G) \leq \mathrm{Z}(G)$. Furthermore, we have established that $\widehat{\mathrm{Z}}(G)$ detects all seven of the graphs on eight vertices with $\mathrm{M}(G) < \mathrm{Z}(G)$ (as does the combination of the cut-vertex reduction formula and the two-separation formula).

In [23], Lin introduced the odd-cycle enhanced zero forcing number of a graph G, denoted $\widehat{\mathrm{Z}}_{oc}(G)$, which satisfies $\mathrm{M}(G) \leq \widehat{\mathrm{Z}}_{oc}(G) \leq \widehat{\mathrm{Z}}(G)$. This parameter is defined by a refinement of the looped graph color change rule. The refinement is an additional condition under which vertices are forced which exploits a phenomenon first described in [15]. See [23] for full details.

Theoretically, similarly obtained refinements should exist and assist in the determination of maximum nullity of small graphs.

QUESTION 2. *Given a graph G, which (other) graph parameters $\zeta(G)$ satisfy* $\mathrm{M}(G) \leq \zeta(G) \leq \mathrm{Z}(G)$*?*

We now consider Item 2. The issue is that most graph parameters relevant to determining the maximum nullity are completely characterized only at their extreme (high and/or low) values. For example, given a graph G, the Colin de Verdiere parameter $\mu(G) = k$ is characterized for each value of $k \leq 4$. Since $\mu(G) \leq \mathrm{M}(G)$, this parameter is potentially useful to establish $\mathrm{M}(G) = k$ when $k \leq 5$.

Let G be a graph of order n. There are characterizations when G satisfies $\mathrm{M}(G) = k$ or $\mathrm{Z}(G) = k$ for $k \in \{0, 1, 2, n-2, n-1\}$ (see [19]), so the techniques described in this article are necessary for the intermediate values. Of course, the number of intermediate values grows with the order of the graph. Thus, the proportion of graphs for which (e.g.) the characterization of $\mu(G)$ might be useful in determining $\mathrm{M}(G)$ quickly decreases as n increases.

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However, Section 3 suggests another path forward. Theorem 3.12 gives a simple criterion to guarantee that $\mathrm{M}(G) = \mathrm{Z}(G)$, and we had some noteworthy success in employing it in the analysis of graphs on eight vertices (and also seven-vertex graphs in Section 4.) There may be analogous results that can be applied to graphs on $n \geq 9$ vertices. For example, the following statement has been verified for graphs on 8 or fewer vertices.

Conjecture 3. *Let G be a 4-connected graph such that $Z(G) = 5$. Then, $M(G) = 5$*

Potential results along these lines are worth investigating because they may help pass the barrier for determining when $M(G) = Z(G)$ for graphs on 9 or more vertices.

Acknowledgments. We would like to thank H. Tracy Hall for suggesting the construction in Subsection 7.9, and the anonymous referees whose careful feedback improved the exposition of this article. The research of M. Hunnell was partially supported by NSF grant 2447261.

Appendix – Minimum rank witnesses. For the 31 graphs, whose minimum rank (maximum nullity) is undetermined by the the preceding methods, the following table compiles witnesses for the minimum rank of the graph. For a graph G, we include two common ways of identifying G, namely, the graph6-string codes and adjacency matrices, along with a matrix $A \in \mathcal{S}(G)$, which establishes that $\mathrm{M}(G) = \mathrm{Z}(G)$.

<table>
<tr><th></th><th>Graph6</th><th>Adjacency matrix</th><th>$M^TDM = A$</th></tr>
<tr><td>1</td><td><code>Gvd|TO</code></td><td>$\begin{bmatrix} 0&1&1&1&1&0&1&1\\ 1&0&0&1&0&1&0&0\\ 1&0&0&1&0&1&1&1\\ 1&1&1&0&1&1&0&0\\ 1&0&0&1&0&1&1&1\\ 0&1&1&1&1&0&0&0\\ 1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&0&0 \end{bmatrix}$</td><td>$\begin{bmatrix} -15&-1&-4&-3&-4&0&-2&-2\\ -1&1&0&1&0&2&0&0\\ -4&0&0&-1&0&1&-1&-1\\ -3&1&-1&1&-1&2&0&0\\ -4&0&0&-1&0&1&-1&-1\\ 0&2&1&2&1&4&0&0\\ -2&0&-1&0&-1&0&0&0\\ -2&0&-1&0&-1&0&0&0 \end{bmatrix}$</td></tr>
<tr><td>2</td><td><code>GzD~VS</code></td><td>$\begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&1&1&0&1&1&1\\ 1&1&0&1&0&1&1&1\\ 0&1&1&0&1&1&0&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&1&1&0&0&0\\ 1&1&1&0&1&0&0&1\\ 1&1&1&0&1&0&1&0 \end{bmatrix}$</td><td>$\begin{bmatrix} 29&5&5&0&0&0&11&11\\ 5&1&1&1&0&1&2&2\\ 5&1&1&1&0&1&2&2\\ 0&1&1&5&3&5&0&0\\ 0&0&0&3&-4&3&1&1\\ 0&1&1&5&3&5&0&0\\ 11&2&2&0&1&0&4&4\\ 11&2&2&0&1&0&4&4 \end{bmatrix}$</td></tr>
<tr><td>3</td><td><code>Gr\~VS</code></td><td>$\begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&0&1&1&1&1&1\\ 1&0&0&1&1&1&1&1\\ 0&1&1&0&1&1&0&0\\ 0&1&1&1&0&1&1&1\\ 0&1&1&1&1&0&0&0\\ 1&1&1&0&1&0&0&1\\ 1&1&1&0&1&0&1&0 \end{bmatrix}$</td><td>$\begin{bmatrix} 1&1&1&0&0&0&2&2\\ 1&16&0&2&1&2&9&9\\ 1&0&-23&-9&-14&-9&-1&-1\\ 0&2&-9&-3&-5&-3&0&0\\ 0&1&-14&-5&-8&-5&-1&-1\\ 0&2&-9&-3&-5&-3&0&0\\ 2&9&-1&0&-1&0&7&7\\ 2&9&-1&0&-1&0&7&7 \end{bmatrix}$</td></tr>
</table>

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<table>
<tr><td>4</td><td>GfF|tW</td><td>$$\begin{bmatrix} 0&1&0&1&0&1&1&1\\ 1&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&1&1\\ 1&1&1&0&1&1&1&0\\ 0&0&0&1&0&1&1&1\\ 1&1&1&1&1&0&0&1\\ 1&0&1&1&1&0&0&0\\ 1&0&1&0&1&1&0&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&0&2&0&1&1&-1\\ 1&1&0&1&0&2&0&0\\ 0&0&0&1&0&-1&1&-1\\ 2&1&1&2&1&2&1&0\\ 0&0&0&1&0&-1&1&-1\\ 1&2&-1&2&-1&3&0&-1\\ 1&0&1&1&1&0&1&0\\ -1&0&-1&0&-1&-1&0&-1 \end{bmatrix}$$</td></tr>
<tr><td>5</td><td>GfD|tW</td><td>$$\begin{bmatrix} 0&1&0&1&0&0&1&1\\ 1&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&1&1\\ 1&1&1&0&1&1&1&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&1&1&0&0&1\\ 1&0&1&1&1&0&0&0\\ 1&0&1&0&1&1&0&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&0&1&0&0&2&2\\ 1&1&0&3&0&2&0&0\\ 0&0&0&2&0&2&-2&-2\\ 1&3&2&5&2&6&4&0\\ 0&0&0&2&0&2&-2&-2\\ 0&2&2&6&2&8&0&-4\\ 2&0&-2&4&-2&0&-4&0\\ 2&0&-2&0&-2&-4&0&4 \end{bmatrix}$$</td></tr>
<tr><td>6</td><td>GbD|tW</td><td>$$\begin{bmatrix} 0&1&0&0&0&0&1&1\\ 1&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&0&1&1&1&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&1&1&0&0&1\\ 1&0&1&1&1&0&0&0\\ 1&0&1&0&1&1&0&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&0&0&0&0&1&-1\\ 1&1&0&1&0&1&0&0\\ 0&0&0&-1&0&-1&1&-1\\ 0&1&-1&2&-1&1&-1&0\\ 0&0&0&-1&0&-1&1&-1\\ 0&1&-1&1&-1&0&0&-1\\ 1&0&1&-1&1&0&1&0\\ -1&0&-1&0&-1&-1&0&-1 \end{bmatrix}$$</td></tr>
<tr><td>7</td><td>GbD|t[</td><td>$$\begin{bmatrix} 0&1&0&0&0&0&1&1\\ 1&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&0&1&1&1&0\\ 0&0&0&1&0&1&1&1\\ 0&1&1&1&1&0&0&1\\ 1&0&1&1&1&0&0&1\\ 1&0&1&0&1&1&1&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&0&0&0&0&2&1\\ 1&1&0&-1&0&-2&0&0\\ 0&0&0&-1&0&-2&-2&-1\\ 0&-1&-1&-3&-1&-5&-1&0\\ 0&0&0&-1&0&-2&-2&-1\\ 0&-2&-2&-5&-2&-8&0&1\\ 2&0&-2&-1&-2&0&12&7\\ 1&0&-1&0&-1&1&7&4 \end{bmatrix}$$</td></tr>
<tr><td>8</td><td>Gv]mtW</td><td>$$\begin{bmatrix} 0&1&1&1&0&1&1&1\\ 1&0&0&1&1&0&1&0\\ 1&0&0&1&1&1&0&1\\ 1&1&1&0&1&0&1&0\\ 0&1&1&1&0&1&1&1\\ 1&0&1&0&1&0&0&1\\ 1&1&0&1&1&0&0&0\\ 1&0&1&0&1&1&0&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 0&-5&11&-6&0&7&-5&7\\ -5&-1&0&-1&-5&0&-1&0\\ 11&0&5&-1&11&3&0&3\\ -6&-1&-1&1&-6&0&-1&0\\ 0&-5&11&-6&0&7&-5&7\\ 7&0&3&0&7&2&0&2\\ -5&-1&0&-1&-5&0&-1&0\\ 7&0&3&0&7&2&0&2 \end{bmatrix}$$</td></tr>
</table>

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9	`G~YmtW`	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & -3 & -3 & -1 & 0 & 1 & 1 & 1 \\ -3 & -3 & -4 & -1 & -1 & 0 & 1 & 0 \\ -3 & -4 & -3 & -2 & -1 & 1 & 0 & 1 \\ -1 & -1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$
10	`GfymtW`	$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 & 3 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -3 & -2 & 0 & -2 \\ 3 & 1 & -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & -7 & -5 & 1 & -5 \\ 1 & 0 & -2 & 0 & -5 & -3 & 0 & -3 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 & -3 & 0 & -3 \end{bmatrix}$
11	`GvY}tG`	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 4 & 3 & -1 & -17 & 0 & -1 & 3 & -3 \\ 3 & 2 & 0 & -6 & 4 & 0 & 2 & 0 \\ -1 & 0 & 2 & 8 & 6 & 2 & 0 & 2 \\ -17 & -6 & 8 & 34 & 0 & 8 & -6 & 0 \\ 0 & 4 & 6 & 0 & 17 & 6 & 4 & 0 \\ -1 & 0 & 2 & 8 & 6 & 2 & 0 & 2 \\ 3 & 2 & 0 & -6 & 4 & 0 & 2 & 0 \\ -3 & 0 & 2 & 0 & 0 & 2 & 0 & -2 \end{bmatrix}$
12	`Gvw}tG`	$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 4 & 3 & -2 & 1 & 2 & 0 & 3 & -1 \\ 3 & 2 & 0 & 1 & 1 & 0 & 2 & 0 \\ -2 & 0 & 1 & 1 & -1 & 3 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 & -1 & 1 & 0 & 1 \\ 3 & 2 & 0 & 1 & 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$
13	`Gf{}tG`	$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 & 2 & 2 & 0 & 1 & -2 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 2 & 2 & 3 & 0 & -1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 2 & 3 & 0 & -1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$

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14	`G~UlTW`	$\begin{bmatrix} 0&1&1&1&0&1&1&1\\ 1&0&1&1&1&0&0&0\\ 1&1&0&1&0&1&1&1\\ 1&1&1&0&1&0&0&0\\ 0&1&0&1&0&1&1&1\\ 1&0&1&0&1&0&0&1\\ 1&0&1&0&1&0&0&0\\ 1&0&1&0&1&1&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&2&1&0&1&-1&1\\ 1&1&3&1&1&0&0&0\\ 2&3&6&3&0&1&-2&1\\ 1&1&3&1&1&0&0&0\\ 0&1&0&1&-2&1&-2&1\\ 1&0&1&0&1&1&0&1\\ -1&0&-2&0&-2&0&-1&0\\ 1&0&1&0&1&1&0&1 \end{bmatrix}$
15	`Gz[m~[`	$\begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&1&1&1&0&1&1\\ 1&1&0&1&1&1&0&1\\ 0&1&1&0&1&0&1&0\\ 0&1&1&1&0&1&1&1\\ 0&0&1&0&1&0&1&1\\ 1&1&0&1&1&1&0&1\\ 1&1&1&0&1&1&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&1&0&0&0&1&1\\ 1&2&2&1&1&0&2&1\\ 1&2&1&1&-1&-1&0&-1\\ 0&1&1&1&1&0&1&0\\ 0&1&-1&1&-3&-2&-3&-4\\ 0&0&-1&0&-2&-1&-2&-2\\ 1&2&0&1&-3&-2&-2&-3\\ 1&1&-1&0&-4&-2&-3&-3 \end{bmatrix}$
16	`Gv}lZ_`	$\begin{bmatrix} 0&1&1&1&1&1&1&0\\ 1&0&0&1&1&0&0&1\\ 1&0&0&1&1&1&1&1\\ 1&1&1&0&1&0&0&1\\ 1&1&1&1&0&1&1&0\\ 1&0&1&0&1&0&1&0\\ 1&0&1&0&1&1&0&0\\ 0&1&1&1&0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&1&1&1&1&1&0\\ 1&1&0&-1&-1&0&0&-1\\ 1&0&4&2&5&3&3&1\\ 1&-1&2&-7&-1&0&0&-3\\ 1&-1&5&-1&5&3&3&0\\ 1&0&3&0&3&2&2&0\\ 1&0&3&0&3&2&2&0\\ 0&-1&1&-3&0&0&0&-1 \end{bmatrix}$
17	`GnulZ_`	$\begin{bmatrix} 0&1&0&1&1&1&1&0\\ 1&0&1&1&1&0&0&1\\ 0&1&0&1&0&1&1&1\\ 1&1&1&0&1&0&0&1\\ 1&1&0&1&0&1&1&0\\ 1&0&1&0&1&0&1&0\\ 1&0&1&0&1&1&0&0\\ 0&1&1&1&0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&0&1&2&1&1&0\\ 1&1&-3&1&2&0&0&-1\\ 0&-3&-3&-3&0&1&1&-2\\ 1&1&-3&1&2&0&0&-1\\ 2&2&0&2&4&2&2&0\\ 1&0&1&0&2&2&2&0\\ 1&0&1&0&2&2&2&0\\ 0&-1&-2&-1&0&0&0&-1 \end{bmatrix}$
18	`Gvx|Ro`	$\begin{bmatrix} 0&1&1&1&1&0&1&0\\ 1&0&0&1&1&1&0&1\\ 1&0&0&1&1&1&1&1\\ 1&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&1\\ 0&1&1&1&1&0&0&0\\ 1&0&1&0&1&0&0&0\\ 0&1&1&1&1&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&2&-1&4&-3&0&1&0\\ 2&3&0&5&-1&1&0&1\\ -1&0&-2&1&-4&-1&1&-1\\ 4&5&1&8&0&2&0&2\\ -3&-1&-4&0&-7&-2&1&-2\\ 0&1&-1&2&-2&0&0&0\\ 1&0&1&0&1&0&1&0\\ 0&1&-1&2&-2&0&0&0 \end{bmatrix}$

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<table>
<tr><td>19</td><td>Gfx|Ro</td><td>$$\begin{bmatrix} 0&1&0&1&1&0&1&0\\ 1&0&0&1&1&1&0&1\\ 0&0&0&1&1&1&1&1\\ 1&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&1\\ 0&1&1&1&1&0&0&0\\ 1&0&1&0&1&0&0&0\\ 0&1&1&1&1&0&0&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&0&1&1&0&1&0\\ 1&-4&0&-3&1&-1&0&-1\\ 0&0&5&-1&5&1&1&1\\ 1&-3&-1&-2&0&-1&0&-1\\ 1&1&5&0&6&1&2&1\\ 0&-1&1&-1&1&0&0&0\\ 1&0&1&0&2&0&1&0\\ 0&-1&1&-1&1&0&0&0 \end{bmatrix}$$</td></tr>
<tr><td>20</td><td>G~^[\s</td><td>$$\begin{bmatrix} 0&1&1&1&0&1&1&1\\ 1&0&1&1&1&1&0&0\\ 1&1&0&1&1&0&0&1\\ 1&1&1&0&1&1&0&1\\ 0&1&1&1&0&1&1&1\\ 1&1&0&1&1&0&1&0\\ 1&0&0&0&1&1&0&1\\ 1&0&1&1&1&0&1&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&4&3&0&-1&-1&-1\\ 1&1&2&1&1&-2&0&0\\ 4&2&8&6&4&0&0&2\\ 3&1&6&5&3&2&0&2\\ 0&1&4&3&-2&-2&-2&-3\\ -1&-2&0&2&-2&7&-1&0\\ -1&0&0&0&-2&-1&-1&-2\\ -1&0&2&2&-3&0&-2&-3 \end{bmatrix}$$</td></tr>
<tr><td>21</td><td>G~X]\s</td><td>$$\begin{bmatrix} 0&1&1&1&0&0&1&1\\ 1&0&1&1&1&1&1&0\\ 1&1&0&1&1&0&0&1\\ 1&1&1&0&0&1&0&1\\ 0&1&1&0&0&1&1&1\\ 0&1&0&1&1&0&1&0\\ 1&1&0&0&1&1&0&1\\ 1&0&1&1&1&0&1&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 1&1&2&1&0&0&1&5\\ 1&17&-3&3&2&-3&2&0\\ 2&-3&5&1&-1&0&0&11\\ 1&3&1&1&0&-1&0&4\\ 0&2&-1&0&0&-1&-1&-1\\ 0&-3&0&-1&-1&-1&-3&0\\ 1&2&0&0&-1&-3&-4&3\\ 5&0&11&4&-1&0&3&26 \end{bmatrix}$$</td></tr>
<tr><td>22</td><td>Grz[\{</td><td>$$\begin{bmatrix} 0&1&1&0&1&1&1&1\\ 1&0&0&1&1&1&0&0\\ 1&0&0&1&1&0&0&1\\ 0&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&1\\ 1&1&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&0&1&1&1&1&1&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} 4&1&2&0&2&-4&-1&-1\\ 1&1&0&2&1&1&0&0\\ 2&0&1&-1&2&0&0&1\\ 0&2&-1&5&0&2&0&-1\\ 2&1&2&0&-4&-14&-3&-7\\ -4&1&0&2&-14&-24&-5&-15\\ -1&0&0&0&-3&-5&-1&-3\\ -1&0&1&-1&-7&-15&-3&-8 \end{bmatrix}$$</td></tr>
<tr><td>23</td><td>G~xX{s</td><td>$$\begin{bmatrix} 0&1&1&1&1&0&0&1\\ 1&0&1&1&1&1&0&0\\ 1&1&0&1&1&0&1&0\\ 1&1&1&0&0&1&1&1\\ 1&1&1&0&0&1&1&1\\ 0&1&0&1&1&0&1&0\\ 0&0&1&1&1&1&0&1\\ 1&0&0&1&1&0&1&0 \end{bmatrix}$$</td><td>$$\begin{bmatrix} -3&2&1&-1&-4&0&0&-2\\ 2&5&2&3&5&1&0&0\\ 1&2&1&1&2&0&2&0\\ -1&3&1&1&0&1&-3&-1\\ -4&5&2&0&-4&1&-3&-3\\ 0&1&0&1&1&1&-4&0\\ 0&0&2&-3&-3&-4&19&-1\\ -2&0&0&-1&-3&0&-1&-1 \end{bmatrix}$$</td></tr>
</table>

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24	`GvxX|s`	$\begin{bmatrix} 0&1&1&1&1&0&0&1\\ 1&0&0&1&1&1&0&0\\ 1&0&0&1&1&0&1&1\\ 1&1&1&0&0&1&1&1\\ 1&1&1&0&0&1&1&1\\ 0&1&0&1&1&0&1&0\\ 0&0&1&1&1&1&0&1\\ 1&0&1&1&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&1&3&3&0&0&1\\ 1&0&0&-2&-2&-1&0&0\\ 1&0&1&2&2&0&1&1\\ 3&-2&2&0&0&-1&4&2\\ 3&-2&2&0&0&-1&4&2\\ 0&-1&0&-1&-1&0&1&0\\ 0&0&1&4&4&1&1&1\\ 1&0&1&2&2&0&1&1 \end{bmatrix}$
25	`GrxX[s`	$\begin{bmatrix} 0&1&1&0&1&0&0&1\\ 1&0&0&1&1&1&0&0\\ 1&0&0&1&1&0&1&0\\ 0&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&1\\ 0&1&0&1&1&0&1&0\\ 0&0&1&0&1&1&0&1\\ 1&0&0&1&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&-4&3&0&1&0&0&3\\ -4&25&0&3&-4&15&0&0\\ 3&0&0&4&-7&0&-5&0\\ 0&3&4&1&0&5&0&4\\ 1&-4&-7&0&-3&-8&-2&-7\\ 0&15&0&5&-8&9&-4&0\\ 0&0&-5&0&-2&-4&-1&-5\\ 3&0&0&4&-7&0&-5&0 \end{bmatrix}$
26	`Grx^]c`	$\begin{bmatrix} 0&1&1&0&1&0&1&1\\ 1&0&0&1&1&1&1&1\\ 1&0&0&1&1&0&1&0\\ 0&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&0\\ 0&1&0&1&1&0&1&0\\ 1&1&1&0&1&1&0&1\\ 1&1&0&1&0&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 0&-1&1&0&2&0&-2&-1\\ -1&0&0&1&-1&1&-3&-1\\ 1&0&1&1&2&0&2&0\\ 0&1&1&4&0&2&0&-1\\ 2&-1&2&0&5&-1&3&0\\ 0&1&0&2&-1&1&1&0\\ -2&-3&2&0&3&1&-11&-4\\ -1&-1&0&-1&0&0&-4&-1 \end{bmatrix}$
27	`GryW[k`	$\begin{bmatrix} 0&1&1&0&1&1&0&1\\ 1&0&0&1&1&0&0&0\\ 1&0&0&1&1&0&0&0\\ 0&1&1&0&0&1&0&1\\ 1&1&1&0&0&1&1&0\\ 1&0&0&1&1&0&1&1\\ 0&0&0&0&1&1&0&1\\ 1&0&0&1&0&1&1&0 \end{bmatrix}$	$\begin{bmatrix} -1&-1&-1&0&-1&-1&0&-1\\ -1&0&0&1&-1&0&0&0\\ -1&0&0&1&-1&0&0&0\\ 0&1&1&1&0&1&0&1\\ -1&-1&-1&0&0&1&1&0\\ -1&0&0&1&1&4&2&2\\ 0&0&0&0&1&2&1&1\\ -1&0&0&1&0&2&1&1 \end{bmatrix}$
28	`GrW[[k`	$\begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&0&1&1&0&0&0\\ 1&0&0&1&1&0&0&0\\ 0&1&1&0&0&1&0&1\\ 0&1&1&0&0&1&1&0\\ 0&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&0&0&1&0&1&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&3&3&0&0&0&5&4\\ 3&0&0&4&-5&0&0&0\\ 3&0&0&4&-5&0&0&0\\ 0&4&4&1&0&5&0&-3\\ 0&-5&-5&0&-1&-4&-3&0\\ 0&0&0&5&-4&9&-12&-15\\ 5&0&0&0&-3&-12&16&20\\ 4&0&0&-3&0&-15&20&25 \end{bmatrix}$

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29	`GvW[[k`	$\begin{bmatrix} 0&1&1&1&0&0&1&1\\ 1&0&0&1&1&0&0&0\\ 1&0&0&1&1&0&0&0\\ 1&1&1&0&0&1&0&1\\ 0&1&1&0&0&1&1&0\\ 0&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&0&0&1&0&1&1&0 \end{bmatrix}$	$\begin{bmatrix} -3&-1&-1&-1&0&0&1&-4\\ -1&0&0&1&1&0&0&0\\ -1&0&0&1&1&0&0&0\\ -1&1&1&6&0&1&0&5\\ 0&1&1&0&19&-4&-5&0\\ 0&0&0&1&-4&1&1&1\\ 1&0&0&0&-5&1&1&1\\ -4&0&0&5&0&1&1&1 \end{bmatrix}$
30	`GrY[[k`	$\begin{bmatrix} 0&1&1&0&0&1&1&1\\ 1&0&0&1&1&0&0&0\\ 1&0&0&1&1&0&0&0\\ 0&1&1&0&0&1&0&1\\ 0&1&1&0&0&1&1&0\\ 1&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&0&0&1&0&1&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&3&3&0&0&2&5&4\\ 3&0&0&4&-5&0&0&0\\ 3&0&0&4&-5&0&0&0\\ 0&4&4&1&0&1&0&-3\\ 0&-5&-5&0&-1&-2&-3&0\\ 2&0&0&1&-2&1&4&5\\ 5&0&0&0&-3&4&16&20\\ 4&0&0&-3&0&5&20&25 \end{bmatrix}$
31	`Gv][[k`	$\begin{bmatrix} 0&1&1&1&0&1&1&1\\ 1&0&0&1&1&0&0&0\\ 1&0&0&1&1&0&0&0\\ 1&1&1&0&1&1&0&1\\ 0&1&1&1&0&1&1&0\\ 1&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&0&0&1&0&1&1&0 \end{bmatrix}$	$\begin{bmatrix} 13&-3&-3&4&0&4&-5&-2\\ -3&0&0&-1&-1&0&0&0\\ -3&0&0&-1&-1&0&0&0\\ 4&-1&-1&4&1&3&0&1\\ 0&-1&-1&1&-1&2&-1&0\\ 4&0&0&3&2&1&1&1\\ -5&0&0&0&-1&1&1&1\\ -2&0&0&1&0&1&1&1 \end{bmatrix}$

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