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GROUP RECONSTRUCTION SYSTEMS*

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Abstract. We consider classes of reconstruction systems (RS's) for finite dimensional real or complex Hilbert spaces $\mathcal{H}$, called group reconstruction systems (GRS's), that are associated with representations of finite groups $\mathcal{G}$. These GRS's generalize frames with high degree of symmetry, such as harmonic or geometrically uniform ones. Their canonical dual and canonical Parseval are shown to be GRS's. We establish simple conditions for one-erasure robustness. Projective GRS's, that can be viewed as fusion frames, are also considered. We characterize the Gram matrix of a GRS in terms of block group matrices. Unitary equivalences and unitary symmetries of RS's are studied. The relation between the irreducibility of the representation and the tightness of the GRS is established. Taking into account these results, we consider the construction of Parseval, projective and one-erasure robust GRS's.

Key words. Reconstruction systems, Fusion frames, g-frames, Group representation, Robustness, Gram matrix.

AMS subject classifications. 42C15, 42C40, 20C15, 05B20, 15A60.

1. Introduction.

A *frame* is a set of vectors in a real or complex Hilbert space $\mathcal{H}$ such that each $f \in \mathcal{H}$ has representations in terms of the elements of the frame (see, e.g., [3, 6, 14]). Frames are more flexible than bases permitting us to construct them them with special properties and select one representation of $f \in \mathcal{H}$ between several options, according to the problem at hand. Some of the application areas of frames include signal processing, coding theory, communication theory, sampling theory and the development of fast algorithms.

In many applications such as distributing sensing, parallel processing and packet encoding, a distributed processing by combining locally data vectors has to be implemented. *Fusion frames* (or *frames for subspaces*) [4, 5] are a generalization of frames and provide a mathematical framework suitable for these applications. They are collections of weights and orthogonal projections, and permit us to recover an element $f \in \mathcal{H}$ from packets of linear coefficients. For finite dimensional Hilbert spaces $\mathcal{H}$, *reconstruction systems* (RS's) [15, 16], that are collections of operators that provide a

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resolution of an invertible operator, generalize fusion frames. RS's are more suitable than fusion frames for smooth perturbations and duality considerations. RS's include *coordinate operators* [1] that provide a resolution of the identity on $\mathcal{H}$ generalizing *Parseval fusion frames*.

Frames with a high degree of symmetry, such as harmonic or geometrically uniform ones, are particularly useful in some of the applications [10, 12, 23]. Here we study two types of RS's associated with representations of finite groups, called *group reconstruction systems* (GRS's), that generalize these frames. The obtained results are in line with [10, 23, 25]. One of these types of RS can be viewed as the operator factors of the mixed quantum states with a broad class of symmetries considered in [11, 8] (see also [9]) in problems of quantum detection. We briefly discuss this topic in Remark 3.11. They are also considered in [17] as an example of RS's that have 1-loss optimal canonical dual for the mean square error and for the worst-case error (see Remark 3.10).

We begin Section 2 introducing some notation. Then we present the definition of RS (Definition 2.1) and some properties. We introduce the associated RS operator, synthesis operator, analysis operator and Grammian operator. We consider the relation of RS's with frames (Remark 2.3) and fusion frames (Remark 2.4). The lower and upper bounds of a RS are defined and then we introduce the most important class of RS's, *tight RS's* and, in particular, *Parseval RS's*. The Parseval RS nearest to a given RS in the Hilbert-Schmidt norm, called the *canonical Parseval RS*, is introduced. Taking into account that any RS can be obtained by grouping the elements of a frame (Remark 2.3(2)), we characterize RS's with the same Gram matrix (Lemma 2.5). We also characterize Parseval RS's as those with an orthogonal projection Gram matrix (Theorem 2.6). At the end of Section 2, we recall the definition of *dual RS* (Definition 2.8) and, in particular, of the *canonical dual* of a RS's.

In Section 3, we introduce two types of GRS's along with some of their properties. The relation between the two types of GRS's is established (Remark 3.3) and it is shown that they have the same RS operator (Proposition 3.4). We also consider the relation of GRS's with frames associated with finite groups (Remark 3.5). GRS's corresponding to unitary equivalent representations are characterized (Lemma 3.6 and Lemma 3.7). We show that the canonical dual and the canonical Parseval of a GRS are GRS's (Proposition 3.8(2) and Proposition 3.12(2)). We establish necessary and sufficient conditions for a GRS to be robust under one-erasure, i.e., to continue being a reconstruction system after one-erasure (Proposition 3.8(4) and Proposition 3.12(4)). Projective GRS's are characterized (Proposition 3.8(5) and Proposition 3.12(5)). We finish this section presenting a brief discussion about GRS's in the determination of an optimal dual (Remark 3.10) and in problems of quantum detection (Remark 3.11).

In Section 4, we characterize the Gram matrices of GRS's as group block matrices

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(Theorem 4.5 and Theorem 4.10). We illustrate how we can obtain a Parseval GRS from a projective group block matrix using its eigendecomposition (Example 4.8). An upper bound for the maximal number of distinct entries of the Gram matrix of a real GRS is given (Corollary 4.15 and Corollary 4.16). We establish that the representation is faithful if and only if the Gram matrix is elementary (Theorem 4.21 and Theorem 4.22).

In Section 5, two types of unitary equivalences and unitary symmetries of RS's are defined (Definition 5.1 and Definition 5.4). Then equivalent and transitive RS's are characterized in terms of their Gram matrices (Theorem 5.3, Theorem 5.9 and Theorem 5.12).

The relation between the irreducibility of the representation and the tightness of the GRS is considered in Section 6. We prove that the GRS generated by an irreducible representation is tight (Theorem 6.2, Theorem 6.3 and Proposition 6.4). Conditions that assure the tightness of a GRS generated by a reducible representation are established (Theorem 6.7, Theorem 6.8, Theorem 6.9 and Theorem 6.10). We also present some examples that illustrate the results (Example 6.6 and Example 6.12).

Finally, in Section 7, some conclusions and future lines of investigation are presented.

2. Reconstruction systems. Let $\mathcal{H}, \mathcal{K}$ be finite dimensional Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $L(\mathcal{H}, \mathcal{K})$ be the space of linear transformations from $\mathcal{H}$ to $\mathcal{K}$. Given $T \in L(\mathcal{H}, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$, $N(T) \subseteq \mathcal{H}$ and T^* denote the image, the null space and the adjoint of T, respectively. $GL(\mathcal{H})$ and $U(\mathcal{H})$ denote the group of invertible and unitary operators in $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$, respectively. If $V \subset \mathcal{H}$ is a subspace, $P_V \in L(\mathcal{H})$ denotes the orthogonal projection onto V. The inner product and the norm in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively.

Let $m, n, d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$. In the sequel, $\mathcal{H}$ will be a finite dimensional Hilbert space over $\mathbb{F}$ of dimension d. If $T \in L(\mathcal{H}, \mathcal{K})$, then $\|T\|_{HS}$ and $\|T\|_{sp}$ denote the Hilbert-Schmidt and the spectral norms of T, respectively. Given $T \in L(\mathcal{H})$, $\lambda(T)$ denotes an eigenvalue of T, whereas $\lambda_{\min}(T)$ and $\lambda_{\max}(T)$ denote the smallest and largest eigenvalues of T, respectively.

$\mathbb{F}^{d \times n}$ denotes the set of matrices of order $d \times n$ with entries in $\mathbb{F}$. If $M \in \mathbb{F}^{d \times n}$, then M^* denotes the conjugate transpose of M. The elements of $\mathbb{F}^n$ will be considered as column vectors, i.e., we identify $\mathbb{F}^n$ with $\mathbb{F}^{n \times 1}$, and if $f \in \mathbb{F}^n$ then $f(i)$ denotes the ith component of f. Given two elements $f, g \in \mathbb{F}^n$, we consider the product $\langle f, g \rangle_{\mathbb{F}^n} = g^* f$. The standard basis of $\mathbb{F}^n$ will be denoted by $\left\{\delta_j^n\right\}_{j=1}^n$. Let $M \in \mathbb{F}^{d \times n}$. We denote the entry i, j, the ith row and the jth column of M with $M(i, j)$, $M(i, :)$ and $M(:, j)$, respectively. We also denote the matrices consisting of the rows from

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i_1 to i_2 and the columns from j_1 to j_2 of M with $M(i_1 : i_2, :)$ and $M(:, j_1 : j_2)$, respectively. $M(i_1 : i_2, j_1 : j_2)$ denotes the matrix consisting of the rows from i_1 to i_2 and the columns from j_1 to j_2 of M. We say that $M \in \mathbb{F}^{d\times n}$ is diagonal if $M(i,j) = 0$ for $i \neq j$, $i = 1, \ldots, d$, $j = 1, \ldots, n$. $M(i,i)$ is a diagonal entry. We will identify linear transformations from $\mathbb{F}^d$ to $\mathbb{F}^n$ with their matrix representation with respect to the standard bases of $\mathbb{F}^d$ and $\mathbb{F}^n$.

We denote the set of permutations (i.e., bijections) on $\{1, \ldots, m\}$ with $\mathcal{S}_m$.

Definition 2.1. A sequence $(T_i)_{i=1}^m$ where $T_i \in L(\mathbb{F}^{n_i}, \mathcal{H})$ is an $(m, \mathbf{n}, \mathcal{H})$-reconstruction system (RS) if

$$S = \sum_{i=1}^m T_i T_i^* \in GL(\mathcal{H}).$$

S is called the *RS operator* of $(T_i)_{i=1}^m$. If $n_1 = \cdots = n_m = n$, we write $(m, n, \mathcal{H})$-RS. The concept of $(m, n, \mathcal{H})$-RS (with $\mathbb{F}^n$ replaced by any Hilbert space $\mathcal{K}$ of dimension n) was introduced in [15] and $\left(m, \mathbf{n}, \mathbb{F}^d\right)$-RS's are considered in [16]. In [20], RS's for non necessarily finite dimensional Hilbert spaces are called g-frames and are shown to be equivalent to stable space splittings of Hilbert spaces [18]. g-frames with $S = I_{\mathcal{H}}$ are considered in [1] (see also [2]) under the name of coordinate operators.

The set of $(m, \mathbf{n}, \mathcal{H})$-RS's will be denoted with $\mathcal{RS}\,(m, \mathbf{n}, \mathcal{H})$. A necessary condition of being an $(m, \mathbf{n}, \mathcal{H})$-RS is $\operatorname{tr}(\mathbf{n}) := \sum_{i=1}^m n_i \geq d$. $(T_i)_{i=1}^m$ where $T_i : \mathbb{F}^{n_i} \to \mathcal{H}$ is an $(m, \mathbf{n}, \mathcal{H})$-RS for $R(T)$, so it is an $(m, \mathbf{n}, \mathcal{H})$-RS if and only if $R(T) = \mathcal{H}$.

Remark 2.2 (Relation between $\mathcal{RS}\,(m, n, \mathcal{H})$ and $\mathcal{RS}\,(n, m, \mathcal{H})$). Let $\sigma \in \mathcal{S}_n$ and $\pi \in \mathcal{S}_m$. Given $(T_i)_{i=1}^m \in \mathcal{RS}\,(m, n, \mathcal{H})$ there exists a unique $\left(\widetilde{T}_j\right)_{j=1}^n \in \mathcal{RS}\,(n, m, \mathcal{H})$ such that $\widetilde{T}_j \delta_i^m = T_{\pi(i)} \delta_{\sigma(j)}^n$. Reciprocally, given $(T_j)_{j=1}^n \in \mathcal{RS}\,(n, m, \mathcal{H})$ there exists a unique $\left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}\,(m, n, \mathcal{H})$ such that $\widetilde{T}_i \delta_j^n = T_{\sigma(j)} \delta_{\pi(i)}^m$.

Associated with $(T_i)_{i=1}^m$, where $T_i \in L(\mathbb{F}^{n_i}, \mathcal{H})$, is the *synthesis operator*

$$T : \bigoplus_{i=1}^m \mathbb{F}^{n_i} \cong \mathbb{F}^{\operatorname{tr}(\mathbf{n})} \to \mathcal{H}, \ \ T\left((x_i)_{i=1}^m\right) = \sum_{i=1}^m T_i x_i,$$

the *analysis operator*

$$T^* : \mathcal{H} \to \bigoplus_{i=1}^m \mathbb{F}^{n_i}, \ \ T^* f = \left(T_i^* f\right)_{i=1}^m,$$

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and the *Grammian operator*

$$G = T^*T : \bigoplus_{i=1}^{m} \mathbb{F}^{n_i} \to \bigoplus_{i=1}^{m} \mathbb{F}^{n_i}, \;\; G\left((x_i)_{i=1}^m\right) = \left(T_i^*\left(\sum_{j=1}^{m} T_j x_j\right)\right)_{i=1}^{m}.$$

In matrix form, $G \in \mathbb{F}^{\mathrm{tr}(\mathbf{n})\times \mathrm{tr}(\mathbf{n})}$ is a block matrix with blocks $T_i^*T_j \in \mathbb{F}^{n_i\times n_j}$.

In the sequel, we suppose that none of the T_i is a null operator.

REMARK 2.3 (Relation between RS's and frames).

1. The sequence of vectors $(T_i(1))_{i=1}^m$ where $(T_i)_{i=1}^m \in \mathcal{RS}(m,1,d)$ or the sequence of vectors $(T\delta_i^n)_{i=1}^n$ where $(T) \in \mathcal{RS}(1,n,\mathcal{H})$, are frames [3, 6].
2. $(T_i)_{i=1}^m \in \mathcal{RS}(m,\mathbf{n},\mathcal{H})$ if and only if $\forall \pi \in \mathcal{S}_{\mathrm{tr}(\mathbf{n})}$, $(f_i)_{i=1}^{\mathrm{tr}(\mathbf{n})}$ with
$$f_{\pi\left(l_i+\sum_{k=1}^{i-1} n_k\right)} := T_i\delta_{l_i}^{n_i}, \;\; l_i = 1,\dots,n_i, \; i=1,\dots,m,$$
is a frame for $\mathcal{H}$.

As was noted in Remark 2.3(2), any RS is obtained from a frame by grouping its elements, so we can carry over properties of frames to properties of RS's.

REMARK 2.4 (Relation between RS's and fusion frames). $(T_i)_{i=1}^m \in \mathcal{RS}(m,\mathbf{n},\mathcal{H})$ is said to be *projective* if there exists a sequence of weights $\mathbf{v} = (v_i)_{i=1}^m \in \mathbb{R}_+^m$ such that

$$T_i^*T_i = v_i^2 I_{\mathbb{F}^{n_i}}, \;\; i=1,\dots,m.$$

If $(T_i)_{i=1}^m \in \mathcal{RS}(m,\mathbf{n},\mathcal{H})$ is projective, then $v_i = \|T_i\|_{sp}$, $T_iT_i^* = v_i^2 P_{R(T_i)}$ and $S = \sum_{i=1}^m v_i^2 P_{R(T_i)}$. Thus, $(T_i)_{i=1}^m$ can be seen as a fusion frame [4] with the identification $T_i \simeq \left(\|T_i\|_{sp}, R(T_i)\right)$, $i=1,\dots,m$.

Observe that

$$\langle Sf,f\rangle_{\mathcal{H}} = \sum_{k=1}^m \langle T_kT_k^*f,f\rangle_{\mathcal{H}} = \sum_{k=1}^m \|T_k^*f\|_{\mathbb{F}^{n_k}}^2, \; \forall f\in\mathcal{H},$$

and

$$\lambda_{\min}(S)\,\|f\|_{\mathcal{H}}^2 \le \sum_{k=1}^m \|T_k^*f\|_{\mathbb{F}^{n_k}}^2 \le \lambda_{\max}(S)\,\|f\|_{\mathcal{H}}^2, \; \forall f\in\mathcal{H}. \tag{2.1}$$

$(T_k)_{k=1}^m$ is an $(m,\mathbf{n},\mathcal{H})$-RS for $\mathcal{H}$ if and only if $\lambda_{\min}(S) > 0$, or if and only if there exist constants $\alpha,\beta > 0$ such that

$$\alpha\,\|f\|_{\mathcal{H}}^2 \le \sum_{k=1}^m \|T_k^*f\|_{\mathbb{F}^{n_k}}^2 \le \beta\,\|f\|_{\mathcal{H}}^2, \; \forall f\in\mathcal{H}. \tag{2.2}$$

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The numbers α and β in (2.2) are called a *lower RS bound* and an *upper RS bound* of the RS, respectively. The *optimal lower RS bound* is the supremum over all lower RS bounds and the *optimal upper RS bound* is the infimum over all upper RS bounds. By (2.1), the optimal lower frame bound is $\lambda_{\min}(S)$ and the optimal upper frame bound is $\lambda_{\max}(S)$. The RS is *tight* if $\lambda_{\min}(S) = \lambda_{\max}(S)$, i.e., $S = \lambda_{\min}(S) I_{\mathcal{H}}$, and it is a *Parseval RS* if $\lambda_{\min}(S) = \lambda_{\max}(S) = 1$, i.e., $S = I_{\mathcal{H}}$. If $S = \alpha I_{\mathcal{H}}$, $\alpha > 0$, we say that it is an α-tight RS.

$\left(S^{-1/2}T_i\right)_{i=1}^m$ is called the *canonical Parseval RS* of $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. $S^{-1/2}T$ is the solution to the following optimization problem: minimize $\left\|\widetilde{T} - T\right\|_{HS}^2$ subject to $\widetilde{T}\widetilde{T}^* = I_{\mathcal{H}}$.

Using the relation between RS's and frames stated in Remark 2.3(2) and an argument similar to that used to prove Lemma 2.7 in [23], it is easy to see that for RS's we have the following result.

LEMMA 2.5. *Let* $(T_i)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$, $\left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{K})$, *with* $\dim(\mathcal{K}) = \dim(\mathcal{H})$. *Then there exists a (unique) unitary* $Q : \mathcal{K} \to \mathcal{H}$ *such that* $T_i = Q\widetilde{T}_i$, $i = 1, \ldots, m$, *if and only if* $T_i^* T_j = \widetilde{T}_i^* \widetilde{T}_j$, $i, j = 1, \ldots, m$, *i.e.,* $(T_i)_{i=1}^m$ *and* $\left(\widetilde{T}_i\right)_{i=1}^m$ *have the same Gram matrix.*

The next theorem is related with Theorem 2.5 in [25]. The proof of the direct implication is similar, but the proof of the converse implication is different. We use the eigendecomposition of G instead of the columns of G to construct the RS. We prefer the eigendecomposition argument because it permits us to obtain a RS for $\mathbb{F}^{\operatorname{rank}(G)}$ directly, rather than obtaining it first for $R(G)$. This is used in Example 4.8 bellow.

THEOREM 2.6. $G \in \mathbb{F}^{\operatorname{tr}(\mathbf{n}) \times \operatorname{tr}(\mathbf{n})}$ *is the Gram matrix of a Parseval* $(m, \mathbf{n}, \mathcal{H})$-*RS if and only if* G *is an orthogonal projection matrix, i.e.,* $G = G^* = G^2$, *with* $\operatorname{rank}(G) = d$.

Proof. Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ be Parseval with synthesis operator T and $G = T^*T$. Clearly, G is Hermitian and since $S \in GL(\mathcal{H})$, $\operatorname{rank}(G) = d$. We also have

$$I_{\mathcal{H}} = S = \sum_{k=1}^m T_k T_k^*,$$

then

$$T_i^* T_j = \sum_{k=1}^m T_i^* T_k T_k^* T_j.$$

Thus, $G = G^2$. Therefore, G is an orthogonal projection matrix.

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Conversely, suppose that $G = G^* = G^2$, with $\operatorname{rank}(G) = d$. Then G has eigendecomposition,

$$G = V^*\Sigma^*\Sigma V, \tag{2.3}$$

where $V \in \mathbb{F}^{\operatorname{tr}(\mathbf{n})\times\operatorname{tr}(\mathbf{n})}$ is unitary and $\Sigma \in \mathbb{R}^{d\times\operatorname{tr}(\mathbf{n})}$ is diagonal with diagonal entries equal to one. Let

$$V_i = V\left(1:d, \sum_{k=1}^{i-1} n_k + 1 : \sum_{k=1}^{i} n_k\right) \in \mathbb{F}^{d\times n_i}, \quad i = 1,\ldots,m.$$

Take the standard basis $\left\{\delta_i^d\right\}_{i=1}^d$ for $\mathbb{F}^d$ and any orthonormal basis $\{v_i\}_{i=1}^d$ for $\mathcal{H}$, and define a unitary operator $U : \mathbb{F}^d \to \mathcal{H}$ by $U\delta_i^d = v_i$. We have $(UV_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ with Gram matrix G and

$$\begin{aligned} S &= \sum_{i=1}^m (UV_i)(UV_i)^* \\ &= U\left(\sum_{i=1}^m V_iV_i^*\right)U^* \\ &= U\left(V(1:d,:)\,(V(1:d,:))^*\right)U^* \\ &= I_{\mathcal{H}}. \quad \square \end{aligned}$$

Remark 2.7. Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ with synthesis operator T such that $G = T^*T$ is an orthogonal projection matrix. Then G (and consequently, S) has rank d with 1 as its unique nonzero eigenvalue. Thus, $S = TT^* = I_{\mathcal{H}}$, i.e., $(T_i)_{i=1}^m$ is Parseval. By Lemma 2.5, there exists a unique $Q \in U(\mathcal{H})$ such that $QT_i = UV_i$, $i = 1,\ldots,m$, where $(UV_i)_{i=1}^m$ is as in the proof of Theorem 2.6.

Definition 2.8 (cf. [16], Definition 2.5). Let $(T_i)_{i=1}^m, \left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. Then $(T_i)_{i=1}^m$ and $\left(\widetilde{T}_i\right)_{i=1}^m$ are dual if $\widetilde{T}T^* = I_{\mathcal{H}}$, or equivalently,

$$f = \sum_{i=1}^m \widetilde{T}_iT_i^*f = \sum_{i=1}^m T_i\widetilde{T}_i^*f.$$

Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ with bounds α, β and RS operator S. We denote the set of dual RSs of $(T_i)_{i=1}^m$ with $\mathcal{D}\left((T_i)_{i=1}^m\right)$. We have $\left(S^{-1}T_i\right)_{i=1}^m \in \mathcal{D}\left((T_i)_{i=1}^m\right)$. $\left(S^{-1}T_i\right)_{i=1}^m$ is called the *canonical dual* RS of $(T_i)_{i=1}^m$, has RS bounds β^{-1}, α^{-1} and its RS operator is S^{-1}. Let $\widetilde{T}$ be the corresponding synthesis operator, then $\widetilde{T} = S^{-1}T = (T^*)^\dagger$ the Moore-Penrose pseudo-inverse of T^*.

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Since $SS^{-1} = S^{-1}S = I_{\mathcal{H}}$,

$$f = \sum_{k=1}^{m} S^{-1}T_kT_k^*f = \sum_{k=1}^{m} T_kT_k^*S^{-1}f,\ \forall f \in \mathcal{H}. \tag{2.4}$$

If $f = \sum_{k=1}^{m} T_kc_k$ for some $(c_k)_{k=1}^{m} \in \bigoplus_{k=1}^{m} \mathbb{F}^{n_k}$ then $\left(c_k - T_k^*S^{-1}f\right)_{k=1}^{m} \in \mathcal{N}(T)$ and $\left(T_k^*S^{-1}f\right)_{k=1}^{m} \in \mathcal{R}(T^*)$. Thus,

$$\sum_{k=1}^{m} \|c_k\|_{\mathbb{F}^{n_k}}^2 = \sum_{k=1}^{m} \left\|T_k^*S^{-1}f\right\|_{\mathbb{F}^{n_k}}^2 + \sum_{k=1}^{m} \left\|c_k - T_k^*S^{-1}f\right\|_{\mathbb{F}^{n_k}}^2. \tag{2.5}$$

If $N(T) = \{0\}$ then each $f \in \mathcal{H}$ can be expressed uniquely as in (2.4) and $T_k^*S^{-1}T_l = \delta_k^m(l)\, I_{\mathbb{F}^m}$. If $N(T) \neq \{0\}$, each $f \in H$ can be expressed as in (2.4) but this representation is not unique. By (2.5), $\left(T_k^*S^{-1}f\right)_{k=1}^{m}$ has minimal ℓ^2-norm among all $(c_k)_{k=1}^{m} \in \bigoplus_{k=1}^{m} \mathbb{F}^{n_k}$ such that $f = \sum_{k=1}^{m} T_kc_k$.

3. Group reconstruction systems. Let $\mathcal{G}$ be a group of order n. We recall that a *representation* of $\mathcal{G}$ is a group homomorphism $\rho : \mathcal{G} \to GL(V)$ where V is a finite dimensional vector space. Since any representation of $\mathcal{G}$ is equivalent to a unitary representation, we are going to use only representations $\rho : \mathcal{G} \to U(\mathcal{H})$.

Next we define RS's associated with representations of $\mathcal{G}$. We consider $(n, m, \mathcal{H})$-RS's of the form $\left(T_g\right)_{g\in\mathcal{G}}$, i.e., indexed by $\mathcal{G}$, and the vectors of the standard basis δ_i^n, $i = 1, \ldots, n$, are written as δ_g^n, $g \in \mathcal{G}$. In both cases we are implicitly assuming a defined bijection $\mathcal{G} \to \{1, \ldots, n\}$.

Definition 3.1. $\left(T_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(n, m, \mathcal{H})$ is a $(\mathcal{G}, m, \mathcal{H})$-RS if there exists a representation $\rho : \mathcal{G} \to U(\mathcal{H})$ such that

$$\rho(g)T_h = T_{gh},\ \forall g, h \in \mathcal{G}.$$

Definition 3.2. $(T_i)_{i=1}^{m} \in \mathcal{RS}(m, n, \mathcal{H})$ is an $(m, \mathcal{G}, \mathcal{H})$-RS if there exists a representation $\rho : \mathcal{G} \to U(\mathcal{H})$ such that for $i = 1, \ldots, m$,

$$\rho(g)T_i\delta_h^n = T_i\delta_{gh}^n,\ \forall g, h \in \mathcal{G}.$$

We generically refer to any of the RS's defined previously as a *group reconstruction system* (GRS). The set of $(\mathcal{G}, m, \mathcal{H})$-RS's will be denoted by $\mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ and the set of $(m, \mathcal{G}, \mathcal{H})$-RS's will be denoted by $\mathcal{RS}(m, \mathcal{G}, \mathcal{H})$. Sometimes we mention explicitly the used representation writing $(\mathcal{G}, \rho, m, \mathcal{H})$ or $(m, \mathcal{G}, \rho, \mathcal{H})$.

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By Definition 3.1, $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ consists of the N elements of $\mathcal{G}$-orbits of T_g for any $g \in \mathcal{G}$ each repeated $\mathcal{G}/|N|$ times. A similar observation can be made if $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ for each $\left(T_i\delta_g^n\right)_{g\in\mathcal{G}}$ as $\mathcal{G}$-orbits of $T_i\delta_g^n$ for any $g \in \mathcal{G}$.

Remark 3.3 (Relation between $\mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H})$ and $\mathcal{RS}(m, \mathcal{G}, \rho, \mathcal{H})$). Let $\sigma : \mathcal{G} \to \mathcal{G}$ be an isomorphism and $\pi \in \mathcal{S}_m$. The function given by

$$(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H}) \mapsto (T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \rho\circ\sigma, \mathcal{H}), T_i\delta_g^n = T_{\sigma(g)}\delta_{\pi(i)}^m,$$

is a bijection with inverse

$$(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \rho\circ\sigma, \mathcal{H}) \mapsto (T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H}), T_g\delta_i^m = T_{\pi^{-1}(i)}\delta_{\sigma^{-1}(g)}^n.$$

Proposition 3.4. *Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H})$ and $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \rho, \mathcal{H})$ be such that there exists an isomorphism $\sigma : \mathcal{G} \to \mathcal{G}$ and $\pi \in \mathcal{S}_m$ such that $T_i\delta_g^n = T_{\sigma(g)}\delta_{\pi(i)}^m$. Then $(T_g)_{g\in\mathcal{G}}$ and $(T_i)_{i=1}^m$ have the same RS operator.*

Proof. Let $f \in \mathcal{H}$. We have,

$$\begin{aligned}\sum_{i=1}^m T_iT_i^* f &= \sum_{i=1}^m \sum_{g\in\mathcal{G}} \left\langle f, T_i\delta_g^n \right\rangle_{\mathcal{H}} T_i\delta_g^n \\ &= \sum_{g\in\mathcal{G}} \sum_{i=1}^m \left\langle f, T_{\sigma(g)}\delta_{\pi(i)}^m \right\rangle_{\mathcal{H}} T_{\sigma(g)}\delta_{\pi(i)}^m \\ &= \sum_{g\in\mathcal{G}} T_gT_g^* f. \quad \square\end{aligned}$$

By Remark 3.3, a property of any of the sets $\mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ or $\mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ has an analog in the other set. Moreover, we can use Remark 3.3 to prove any result about $\mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ starting from the analog result about $\mathcal{RS}(m, \mathcal{G}, \mathcal{H})$, and vice versa.

A *compound* $(\mathcal{G}, \rho, m, \mathcal{H})$*-frame* for $\mathcal{H}$ consists of vectors $v_{g,i} \in \mathcal{H}$, $g \in \mathcal{G}$, $i = 1, \ldots, m$, such that $\rho(g)v_{h,i} = v_{gh,i}$. For $\mathcal{G}$ abelian, compound $(\mathcal{G}, m, \mathcal{H})$-frames are considered, e.g., in [10] under the name of *compound geometrically uniform frames* and in [21] where $\mathcal{H}$ is any separable Hilbert space and G is denumerable. For $m = 1$ and $\mathcal{G}$ abelian compound $(\mathcal{G}, m, \mathcal{H})$-frame are called *geometrically uniform frames* and Parseval $(\mathcal{G}, m, \mathcal{H})$-frames are known as *harmonic*. The set of compound $(\mathcal{G}, m, \mathcal{H})$-frames for $\mathcal{H}$ with m elements associated with a representation ρ of $\mathcal{G}$ will be denoted with $\mathcal{CF}(\mathcal{G}, \rho, m, \mathcal{H})$.

Remark 3.5 (Relation between compound $(\mathcal{G}, m, \mathcal{H})$-frames and GRS's). Let $\sigma : \mathcal{G} \to \mathcal{G}$ be an isomorphism and $\pi \in \mathcal{S}_m$. Given a compound $(\mathcal{G}, \rho, m, \mathcal{H})$-frame

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consisting of vectors $v_{g,i} \in \mathcal{H}$, $g \in \mathcal{G}$, $i = 1, \ldots, m$, we can define a unique $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho \circ \sigma, m, \mathcal{H})$ such that

$$T_g \delta_i^m = v_{\sigma(g),\pi(i)},$$

and a unique $(T_i)_{i=1}^m \in \mathcal{RS}(\mathcal{G}, \rho \circ \sigma, m, \mathcal{H})$ where

$$T_i \delta_g^n = v_{\sigma(g),\pi(i)}.$$

Given $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H})$, there is a unique compound $(\mathcal{G}, \rho \circ \sigma, m, \mathcal{H})$-frame given by

$$v_{g,i} = T_{\sigma(g)} \delta_{\pi(i)}^m.$$

A similar consideration can be made for $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \rho, \mathcal{H})$, defining

$$v_{g,i} = T_{\pi(i)} \delta_{\sigma(g)}^n.$$

Lemma 3.6. *Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H})$ and $\left(\widetilde{T}_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \widetilde{\rho}, m, \mathcal{K})$ with $\dim(\mathcal{K}) = \dim(\mathcal{H})$. If there exists a (unique) unitary $Q : \mathcal{K} \to \mathcal{H}$ such that $T_g = Q\widetilde{T}_g$, $g \in \mathcal{G}$, then $\widetilde{\rho}$ is Q-equivalent to ρ, i.e., $\widetilde{\rho}(g) = Q^* \rho(g) Q$, $g \in \mathcal{G}$. Conversely, if $\widetilde{\rho}$ is Q-equivalent to ρ and $T_1 = Q\widetilde{T}_1$ then $T_g = Q\widetilde{T}_g$, $g \in \mathcal{G}$.*

Proof. Suppose that there exists a (unique) unitary $Q : \mathcal{K} \to \mathcal{H}$ such that $\widetilde{T}_g = QT_g$, $g \in \mathcal{G}$. Then

$$T_{gh} = \rho(g) T_h = \rho(g) Q\widetilde{T}_h,$$

and

$$T_{gh} = Q\widetilde{T}_{gh} = Q\widetilde{\rho}(g) \widetilde{T}_h.$$

Thus,

$$\widetilde{\rho}(g) \widetilde{T}_h = Q^* \rho(g) Q\widetilde{T}_h.$$

Since, by Remark 2.3(2), $\left\{\widetilde{T}_h \delta_j^m : h \in \mathcal{G},\, j = 1, \ldots, m\right\}$ is a frame for $\mathcal{K}$,

$$\widetilde{\rho}(g) = Q^* \rho(g) Q.$$

The converse implication is immediate. ∎

As a consequence of Remark 3.3 and Lemma 3.6, we have the following.

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LEMMA 3.7. *Let* $(T_i)_{i=1}^m \in \mathcal{RS}(m,\mathcal{G},\rho,\mathcal{H})$ *and* $\left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}(m,\mathcal{G},\widetilde{\rho},\mathcal{H})$ *with* $\dim(\mathcal{K}) = \dim(\mathcal{H})$*. If there exists a (unique) unitary* $Q : \mathcal{K} \to \mathcal{H}$ *such that* $T_i = Q\widetilde{T}_i$*,* $i = 1,\dots,m$*, then* $\widetilde{\rho}$ *is* Q*-equivalent to* ρ*, i.e.,* $\widetilde{\rho}(g) = Q^*\rho(g)Q$*,* $g \in \mathcal{G}$*. Conversely, if* $\widetilde{\rho}$ *is* Q*-equivalent to* ρ *and* $T_i\delta_1^n = Q\widetilde{T}_i\delta_1^n$*,* $i = 1,\dots,m$*, then* $T_i = Q\widetilde{T}_i$*,* $i = 1,\dots,m$*.*

The next proposition states some basic properties of the elements of $\mathcal{RS}(\mathcal{G},m,\mathcal{H})$. In particular, parts (a)-(d) are generalizations of properties of $(\mathcal{G},m,\mathcal{H})$-frames for $\mathcal{G}$ abelian (see [10]). Part (b) asserts that the canonical dual and the canonical Parseval of a GRS is a GRS. In part (c) we give necessary and sufficient conditions for a GRS in $\mathcal{RS}(\mathcal{G},m,\mathcal{H})$ to remain a RS after the erasure of one operator. We note that for frames this gives a more general result than Corollary 2 in [10]. Projective GRS's are characterized in part (d).

PROPOSITION 3.8. *Let* $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G},m,\mathcal{H})$*.*

1. S *(and therefore,* S^{-1} *and* $S^{-1/2}$*) commutes with* $\rho(g)$*,* $\forall g \in \mathcal{G}$*.*
2. $\left(S^{-1}T_g\right)_{g\in\mathcal{G}}, \left(S^{-1/2}T_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G},m,\mathcal{H})$*.*
3. $\|T_g\|_{sp} = \|T_1\|_{sp}$*,* $\|T_g\|_{HS} = \|T_1\|_{HS}$*,* $g \in \mathcal{G}$*, and*
$$\lambda_{\min}(S) \le \frac{n}{d}\|T_1\|_{HS}^2 \le \lambda_{\max}(S).$$
4. *Suppose that* $(T_g)_{g\in\mathcal{G}}$ *is tight. Let* $g_0 \in \mathcal{G}$ *and* $\mathcal{G}' = \mathcal{G}\setminus\{g_0\}$*. Then* $(T_g)_{g\in\mathcal{G}'} \in \mathcal{RS}(n-1,m,\mathcal{H})$ *if and only if*
$$\|T_1\|_{sp}^2 < \frac{n}{d}\|T_1\|_{HS}^2, \tag{3.1}$$
and its optimal RS bounds are $\frac{n}{d}\|T_1\|_{HS}^2 - \lambda_{max}(T_1T_1^*)$ *and* $\frac{n}{d}\|T_1\|_{HS}^2 - \lambda_{min}(T_1T_1^*)$*. In particular, if* $n > d$ *then condition (3.1) holds.*
5. $(T_g)_{g\in\mathcal{G}}$ *is projective if and only if there exists* $h \in \mathcal{G}$ *such that* $T_h^*T_h = v^2 I_{\mathbb{F}^m}$ *for some* $v > 0$*. In this case, all the weights* v_i *are equal to* $v = \|T_1\|_{sp}$*.*

Proof. Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G},m,\mathcal{H})$.

(1) We have
$$S = \sum_{g\in\mathcal{G}} T_gT_g^* = \sum_{g\in\mathcal{G}} \rho(g)T_1T_1^*\rho(g^{-1}).$$

Then
$$\begin{aligned}S\rho(g) &= \sum_{h\in\mathcal{G}} \rho(h)T_1T_1^*\rho(h^{-1})\rho(g) = \rho(g)\sum_{h\in\mathcal{G}} \rho(g^{-1})\rho(h)T_1T_1^*\rho(h^{-1})\rho(g)\\ &= \rho(g)\sum_{h\in\mathcal{G}} \rho(g^{-1}h)T_1T_1^*\rho\left(\left(g^{-1}h\right)^{-1}\right) = \rho(g)S.\end{aligned}$$

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(2) Taking into account part (1) of this theorem,

$$S^{-1}T_{gh} = S^{-1}\rho(g)T_h = \rho(g)S^{-1}T_h,$$

and similarly,

$$S^{-1/2}T_{gh} = \rho(g)S^{-1/2}T_h.$$

(3) Since

$$T_g^*T_g = T_1^*\rho(g)^*\rho(g)T_1 = T_1^*\rho(h)^*\rho(h)T_1 = T_h^*T_h, \tag{3.2}$$

the operators T_g have the same spectral and Hilbert-Schmidt norms. The rest follows from

$$\begin{aligned}\sum_{k=1}^{d}\lambda_k(S) = \mathrm{tr}\,(S) &= \sum_{g\in\mathcal{G}}\mathrm{tr}\left(T_gT_g^*\right) = \sum_{g\in\mathcal{G}}\mathrm{tr}\left(T_g^*T_g\right) \\ &= \sum_{g\in\mathcal{G}}\mathrm{tr}\left(T_1^*\rho\,(g)^*\,\rho\,(g)\,T_1\right) = \sum_{g\in\mathcal{G}}\mathrm{tr}\,(T_1^*T_1) = n\,\|T_1\|_{HS}^2\,,\end{aligned}$$

and

$$\lambda_{\min}(S) \le \frac{1}{d}\sum_{k=1}^{d}\lambda_k(S) \le \lambda_{\max}(S).$$

(4) By part (3) of the present theorem, $S = \frac{n}{d}\,\|T_1\|_{HS}^2\,I_{\mathcal{H}}$. Let S' be the RS operator of $(T_g)_{g\in\mathcal{G}'}$. We have,

$$\begin{aligned}S' &= \sum_{g\in\mathcal{G}'}T_gT_g^* = \sum_{g\in\mathcal{G}}T_gT_g^* - T_{g_0}T_{g_0}^* = \frac{n}{d}\,\|T_1\|_{HS}^2\,I_{\mathcal{H}} - \rho(g_0)T_1T_1^*\rho(g_0)^* \\ &= \rho(g_0)\left(\frac{n}{d}\,\|T_1\|_{HS}^2\,I_{\mathcal{H}} - T_1T_1^*\right)\rho(g_0)^*.\end{aligned}$$

Thus, S' and $\frac{n}{d}\,\|T_1\|_{HS}^2\,I_{\mathcal{H}} - T_1T_1^*$ have the same eigenvalues. So, S' has minimal and maximal eigenvalues $\frac{n}{d}\,\|T_1\|_{HS}^2 - \lambda_{max}\,(T_1T_1^*)$ and $\frac{n}{d}\,\|T_1\|_{HS}^2 - \lambda_{min}\,(T_1T_1^*)$, respectively. Moreover, S' is positive definite if and only if (3.1) is satisfied. Finally, since $\|T_1\|_{sp} \le \|T_1\|_{HS}$, if $n > d$ then (3.1) holds.

(5) It is a consequence of (3.2). □

Remark 3.9. In applications, given $(T_i)_{i=1}^m \in \mathcal{RS}(n,m,\mathcal{H})$ an element $f \in \mathcal{H}$ (e.g., a signal) is converted into the data vectors T_i^*f, $i = 1,\ldots,m$. These vectors are transmitted and f is reconstructed by the receiver using (2.4). Sometimes some of the data vectors are lost, and it is necessary to reconstruct f with the partial information

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at hand. Proposition 3.8(4) asserts that if a GRS $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ is used and one data vector is lost, say $T^*_{g_0} f$, we can reconstruct f from the remaining information using $(T_g)_{g\in\mathcal{G}\setminus\{g_0\}} \in \mathcal{RS}(n-1, m, \mathcal{H})$. The property of remaining a RS under erasures is known as *robustness*.

Remark 3.10 (Optimal dual for GRS's). Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$. In [17] optimal dual RS's for a fixed RS are studied. It is proved that if $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ is projective, then $\left\{S^{-1}T_g\right\}_{g\in\mathcal{G}}$ is the 1-loss optimal dual for the mean square error and it is the unique 1-loss optimal dual for the worst-case error. Also, if $T_1 \in L(\mathbb{F}^m, \mathcal{H})$ is injective, the projective RS nearest to $\left\{S^{-1}T_g\right\}_{g\in\mathcal{G}}$ is computed and it is shown that it is a $(\mathcal{G}, m, \mathcal{H})$-RS.

Remark 3.11 (GRS's in quantum detection). Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. In the context of quantum information and communication $(T_iT_i^*)_{i=1}^m$ can be interpreted as a set of quantum states onto which the transmission party encodes the message with prior probabilities $\{p_i\}_{i=1}^m$ where $p_i > 0$ and $\sum_{i=1}^m p_i = 1$. The receiving party must find the measurement

$$\left(\widetilde{T}_i\widetilde{T}_i^*\right)_{i=1}^m,$$

satisfying

$$\sum_{i=1}^m \widetilde{T}_i\widetilde{T}_i^* = I_{\mathcal{H}},$$

that maximizes the probability of correct detection

$$P_d = \sum_{i=1}^m p_i \mathrm{tr}\left(T_iT_i^*\widetilde{T}_i\widetilde{T}_i^*\right).$$

The least square measurement (LSM) is

$$\left\{S_p^{-1/2}\left(\sqrt{p_i}T_i\right)\left(\sqrt{p_i}T_i\right)^* S_p^{-1/2}\right\}_{i=1}^m,$$

where S_p is the RS operator corresponding to $\left\{\sqrt{p_i}T_i\right\}_{i=1}^m$. The LSM is equivalent to the square-root measurement, it has many desirable properties and in many cases it is optimal, i.e., it minimizes the probability of the detection error.

Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ with equal prior probabilities $\frac{1}{n}$. Note that $S_p = \frac{1}{n}S$, and then $S_p^{-1/2}\left(\frac{1}{\sqrt{n}}T_g\right) = S^{-1/2}T_g$. As a consequence of Proposition 3.8:

1. The LSM is $\left\{S^{-1/2}T_gT_g^*S^{-1/2}\right\}_{g\in\mathcal{G}}$ with $\left\{S^{-1/2}T_g\right\}_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$.
2. The probability of correctly detecting each of the states is the same

$$\mathrm{tr}\left(S^{-1/2}T_gT_g^*S^{-1/2}\left(\frac{1}{\sqrt{n}}T_g\right)\left(\frac{1}{\sqrt{n}}T_g\right)^*\right) = \frac{1}{n}\mathrm{tr}\left(S^{-1/2}T_1T_1^*S^{-1/2}T_1T_1^*\right).$$

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In this case, we also have [11]:

3. If $T_1^* S^{-1/2} T_1 = \alpha I_{\mathbb{F}^m}$, then the LSM minimizes the probability of a detection error. In particular, this condition is satisfied if $\{\rho(g) T\}_{g\in\mathcal{G}}$ is projective and tight, since by Proposition 3.8(3)(5), $T_1^* T_1 = \|T_1\|_{sp}^2 I_{\mathbb{R}^m}$ and $S = \frac{n}{d}\|T_1\|_{HS}^2 I_{\mathcal{H}}$. Therefore,
$$T_1^* S^{-1/2} T_1 = \sqrt{\frac{d}{n}} \frac{\|T_1\|_{sp}^2}{\|T_1\|_{HS}} I_{\mathbb{R}^m}.$$
4. There exists an optimal measurement corresponding to some $\left(\widetilde{T}_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ such that $\widetilde{T}_1$ maximizes $\operatorname{tr}\left(\widetilde{T}_1 \widetilde{T}_1^* T_1 T_1^*\right)$ subject to $\sum_{g\in\mathcal{G}} \widetilde{T}_g \widetilde{T}_g^* = I_{\mathcal{H}}$.

Mixed quantum states obtained from elements of $\mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ were also considered in other problems of quantum detection [8].

The next result is the analogous of Proposition 3.8 for $\mathcal{RS}(m, \mathcal{G}, \mathcal{H})$.

Proposition 3.12. *Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$.*

1. *S (and therefore, S^{-1} and $S^{-1/2}$) commutes with $\rho(g)$, $\forall g \in \mathcal{G}$.*
2. *$\left(S^{-1}T_i\right)_{i=1}^m, \left(S^{-1/2}T_i\right)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$.*
3. *$\left\|T_i \delta_g^n\right\|_{\mathcal{H}} = \|T_i \delta_1^n\|_{\mathcal{H}}$, $i = 1, \ldots, m$, $g \in \mathcal{G}$ and*
$$\lambda_{\min}(S) \le \frac{n}{d} \sum_{i=1}^m \|T_i \delta_1^m\|_{\mathcal{H}}^2 \le \lambda_{\max}(S).$$
4. *Suppose that $(T_i)_{i=1}^m$ is tight. Let $g_0 \in \mathcal{G}$ and $\mathcal{G}' = \mathcal{G} \setminus \{g_0\}$. Let $T_i^{'} : \operatorname{span}\left\{\delta_g^n\right\}_{g\in\mathcal{G}'} \cong \mathbb{F}^{n-1} \to \mathcal{H}$ be defined by $T_i^{'}\delta_g^n = T_i \delta_g^n$, $g \in \mathcal{G}'$. Let $\widetilde{T} : \mathbb{F}^m \to \mathcal{H}$ given by $\widetilde{T}(c) = \sum_{i=1}^m c(i) T_i \delta_1^n$. Then $\left(T_i^{'}\right)_{i=1}^m \in \mathcal{RS}(m, n-1, \mathcal{H})$ if and only if*
$$\left\|\widetilde{T}\right\|_{sp}^2 < \frac{n}{d}\left\|\widetilde{T}\right\|_{HS}^2, \tag{3.3}$$
and its optimal RS bounds are $\frac{n}{d}\left\|\widetilde{T}\right\|_{HS}^2 - \lambda_{max}\left(\widetilde{T}\widetilde{T}^*\right)$ and $\frac{n}{d}\left\|\widetilde{T}\right\|_{HS}^2 - \lambda_{min}\left(\widetilde{T}\widetilde{T}^*\right)$. In particular, if $n > d$ then (3.3) holds.
5. *$(T_i)_{i=1}^m$ is projective if and only if $\left(T_i\delta_g^n\right)_{g\in\mathcal{G}}$ is orthogonal, $i = 1, \ldots, m$. In this case, the weight v_i is equal to $\|T_i\delta_1^n\|_{\mathcal{H}}$.*

Proof. We note that $T_i^* T_i = v_i^2 I_{\mathbb{F}^n}$ if and only if
$$\left\langle T_i \delta_h^n, T_i \delta_g^n\right\rangle_{\mathcal{H}} = \left\langle T_i^* T_i \delta_h^n, \delta_g^n\right\rangle_{\mathbb{F}^n} = v_i^2 \left\langle \delta_h^n, \delta_g^n\right\rangle_{\mathbb{F}^n},$$

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and

$$\left\|T_i\delta_g^n\right\|_{\mathcal{H}} = \left\|\rho(g)T_i\delta_1^n\right\|_{\mathcal{H}} = \left\|T_i\delta_1^n\right\|_{\mathcal{H}}.$$

This prove (5) and the first part of (3). Defining $T_g \in L\left(\mathbb{F}^m, \mathcal{H}\right)$ by $T_g\delta_i^m = T_i\delta_g^n$, $g \in \mathcal{G}$, $i = 1, \ldots, m$, the rest follows from Remark 3.3, Proposition 3.4 and Proposition 3.8. ◻

4. Group reconstructions systems and group matrices. Theorem 4.1 in [25] asserts that a set of vectors in $\mathcal{H}$ is a $(\mathcal{G}, 1, \mathcal{H})$-frame if and only if its Gram matrix is a group matrix. In the first part of this section we generalize this result for GRS's. We begin recalling some definitions.

Definition 4.1. By a block matrix of type (n, m) is meant an $nm \times nm$ block matrix A with $n \times n$ blocks in $\mathbb{F}^{m\times m}$.

The sets of block matrices of type (n, m) will be denoted with $\mathcal{B}(\mathbb{F}^{nm\times nm})$ and if $A \in \mathcal{B}(\mathbb{F}^{nm\times nm})$ we denote the block (k, l) of A with $A_{k,l}$.

Definition 4.2. $A \in \mathcal{B}(\mathbb{F}^{nm\times nm})$ is a block $\mathcal{G}$-matrix of type (n, m) if there exists a function $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$ such that $A_{g,h} = \nu(g^{-1}h)$, $g, h \in \mathcal{G}$.

Definition 4.3. $A \in \mathcal{B}(\mathbb{F}^{mn\times mn})$ is a block matrix of type (m, n) with $\mathcal{G}$-blocks if there exist functions $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$ such that $A_{i,j}(g, h) = \nu_{i,j}(g^{-1}h)$, $i, j = 1, \ldots, m$, $g, h \in \mathcal{G}$.

The set of block $\mathcal{G}$-matrices of type (n, m) will be denoted with $\mathcal{BG}(\mathbb{F}^{nm\times nm})$ and the set of block matrices of type (m, n) with $\mathcal{G}$-blocks will be denoted with $\mathcal{GB}(\mathbb{F}^{mn\times mn})$. To express the relation between A and ν or $\{\nu_{i,j}\}_{i,j=1}^m$ described in the above definitions we write $A(\nu)$ or $A(\{\nu_{i,j}\}_{i,j=1}^m)$, respectively. If $\mathcal{G}$ is a cyclic group then $A \in B\mathcal{G}(\mathbb{F}^{nm\times nm})$ is a block circulant matrix of type (n, m) and $A \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$ is a block matrix of type (m, n) with circulant blocks (see [7]). The notation $\mathcal{BG}$ and $\mathcal{GB}$ used here is analogous to the notation $\mathcal{BC}$ and $\mathcal{CB}$ used in [7] in the circulant case.

The following lemma is immediate from Definition 4.2.

Lemma 4.4. $A \in B\mathcal{G}(\mathbb{F}^{nm\times nm})$ *if and only if* $A_{h_1,h_2} = A_{1,h_1^{-1}h_2} = A_{gh_1,gh_2}$ *for all* $g, h_1, h_2 \in \mathcal{G}$.

The next theorem generalizes Theorem 4.1 in [25].

Theorem 4.5. $\left(T_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ *if and only if* $G \in \mathcal{BG}(\mathbb{F}^{nm\times nm})$ *and* $\operatorname{rank}(G) = d$.

Proof. If $\left(T_g\right)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ then G has rank d, and it is a block matrix with

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$n \times n$ blocks

$$T_g^* T_h = (\rho(g) T_1)^* (\rho(h) T_1) = T_1^* \left(\rho(g)^{-1} \rho(h) T_1\right) = T_1^* T_{g^{-1}h} \in \mathbb{F}^{m\times m}.$$

Thus, by Lemma 4.4, $G \in \mathcal{BG}(\mathbb{F}^{nm\times nm})$.

Conversely, suppose $G \in \mathcal{BG}(\mathbb{F}^{nm\times nm})$ has rank d. Then $S \in GL(\mathcal{H})$. Let

$$U_g = \sum_{h\in\mathcal{G}} T_{gh} T_h^* S^{-1}. \tag{4.1}$$

Using Lemma 4.4, we obtain

$$\begin{aligned}
U_g^* U_g &= S^{-1} \left(\sum_{h_1\in\mathcal{G}} \sum_{h_2\in\mathcal{G}} T_{h_1} T_{gh_1}^* T_{gh_2} T_{h_2}^* \right) S^{-1} \\
&= S^{-1} \left(\sum_{h_1\in\mathcal{G}} \sum_{h_2\in\mathcal{G}} T_{h_1} T_{h_1}^* T_{h_2} T_{h_2}^* \right) S^{-1} \\
&= S^{-1} \left(\sum_{h_1\in\mathcal{G}} T_{h_1} T_{h_1}^* \right) \left(\sum_{h_2\in\mathcal{G}} T_{h_2} T_{h_2}^* \right) S^{-1} \\
&= S^{-1} S S S^{-1} = I,
\end{aligned}$$

and thus, $U_g \in U(\mathcal{H})$. We also have

$$\begin{aligned}
U_g^* T_{gh} &= S^{-1} \sum_{h_1\in\mathcal{G}} T_{h_1} T_{gh_1}^* T_{gh} \\
&= S^{-1} \sum_{h_1\in\mathcal{G}} T_{h_1} T_{h_1}^* T_h \\
&= S^{-1} S T_h = T_h,
\end{aligned}$$

and then

$$U_{g_1 g_2} T_h = T_{g_1 g_2 h} = U_{g_1} T_{g_2 h} = U_{g_1} U_{g_2} T_h.$$

Since $\mathcal{H} = \operatorname{span}\{T_h \delta_i^m : h \in \mathcal{G},\, i = 1, \ldots, m\}$ this last equality implies

$$U_{g_1 g_2} = U_{g_1} U_{g_2}.$$

Therefore, $\rho : \mathcal{G} \to U(\mathcal{H})$, given by $\rho(g) = U_g$, is a representation of $\mathcal{G}$ such that $\rho(g) T_h = T_{gh}$. Thus, $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$. □

Corollary 4.6. *$(T_g)_{g\in\mathcal{G}}$ is a Parseval $(\mathcal{G}, m, \mathcal{H})$-RS if and only if G is an orthogonal projection with rank d in $\mathcal{BG}(\mathbb{F}^{nm\times nm})$.*

Proof. It follows from Theorem 4.5 and Remark 2.7. □

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COROLLARY 4.7. *If* $\mathcal{G} = \{g_1, \ldots, g_n\}$, $(v_{g_1,1}, \ldots, v_{g_1,m}, \ldots, v_{g_n,1}, \ldots, v_{g_n,m}) \in \mathcal{CF}(\mathcal{G}, \rho, \mathcal{H})$ *if and only if* $G \in \mathcal{BG}(\mathbb{F}^{nm\times nm})$.

Proof. It follows from Theorem 4.5 and Remark 3.5. ∎

EXAMPLE 4.8. Consider the smallest nonabelian group, i.e., the dihedral group of order 6,

$$\mathcal{G} = \left\langle a, b : a^3 = b^2 = 1, b^{-1}ab = a^{-1} \right\rangle.$$

If the elements of $\mathcal{G}$ are ordered as

$$g_1 = 1,\ g_2 = a,\ g_3 = a^2,\ g_4 = b,\ g_5 = ab,\ g_6 = a^2b,$$

by Definition 4.2, $G \in \mathcal{BG}(\mathbb{F}^{6m\times 6m})$ if and only if

$$G(\nu) = \begin{pmatrix} \nu(1) & \nu(a) & \nu\left(a^2\right) & \nu(b) & \nu(ab) & \nu\left(a^2b\right) \\ \nu\left(a^2\right) & \nu(1) & \nu(a) & \nu\left(a^2b\right) & \nu(b) & \nu(ab) \\ \nu(a) & \nu\left(a^2\right) & \nu(1) & \nu(ab) & \nu\left(a^2b\right) & \nu(b) \\ \nu(b) & \nu\left(a^2b\right) & \nu(ab) & \nu(1) & \nu\left(a^2\right) & \nu(a) \\ \nu(ab) & \nu(b) & \nu\left(a^2b\right) & \nu(a) & \nu(1) & \nu\left(a^2\right) \\ \nu\left(a^2b\right) & \nu(ab) & \nu(b) & \nu\left(a^2\right) & \nu(a) & \nu(1) \end{pmatrix},$$

for certain $\nu : \mathcal{G} \to \mathbb{R}^{m\times m}$. Suppose that G is projective with $\operatorname{rank}(G) = d$. Consider the eigendecomposition (2.3) and $V_{g_i} \in \mathbb{F}^{d\times m}$ as in the proof of Theorem 2.6. We have $G_{g_i,g_j} = V_{g_i}^* V_{g_j}$ and by Theorem 2.6 and Theorem 4.5, $(V_{g_i})_{i=1}^n$ is a Parseval $(\mathcal{G}, m, \mathbb{F}^d)$-RS.

Taking into account the proof of Theorem 4.5 (see (4.1)),

$$\rho : \mathcal{G} \to \mathcal{U}(\mathbb{F}^m), \quad \rho(g_i) = \sum_{j=1}^{6} V_{g_i g_j} V_{g_j}^*$$

is a representation of $\mathcal{G}$ such that $V_{g_i} = \rho(g_i) V_{g_1}$. For concreteness, consider

$$\nu(1) = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \nu(a) = \begin{pmatrix} -\frac{1}{12} & -\frac{1}{4\sqrt{3}} \\ \frac{1}{4\sqrt{3}} & -\frac{1}{12} \end{pmatrix}, \nu\left(a^2\right) = \begin{pmatrix} -\frac{1}{12} & \frac{1}{4\sqrt{3}} \\ -\frac{1}{4\sqrt{3}} & -\frac{1}{12} \end{pmatrix},$$

$$\nu(b) = \begin{pmatrix} \frac{1}{4\sqrt{3}} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{1}{4\sqrt{3}} \end{pmatrix}, \nu(ab) = \begin{pmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{pmatrix}, \nu\left(a^2b\right) = \begin{pmatrix} -\frac{1}{4\sqrt{3}} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{4\sqrt{3}} \end{pmatrix}.$$

In this case, $\operatorname{rank}(G(\nu)) = 2$,

$$V_1 = \begin{pmatrix} -\frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{6}} \end{pmatrix}, V_a = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{6}} \end{pmatrix}, V_{a^2} = \begin{pmatrix} -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{6}} \end{pmatrix},$$

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$$V_b = \begin{pmatrix} -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}} \\ \frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \end{pmatrix}, V_{ab} = \begin{pmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}} \\ \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \end{pmatrix}, V_{a^2b} = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 \end{pmatrix},$$

and

$$\rho\,(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho\,(a) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \rho\,(a^2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\rho\,(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \rho\,(ab) = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \rho\,(a^2b) = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Since $n = 6 > d = 2$, by Proposition 3.8(4), $(V_{g_i})_{i=1}^6$ is one-erasure robust. Moreover, since $V_{g_i}^* V_{g_i} = \nu\,(1) = \frac{1}{6} I_{\mathbb{R}^2}$, $(V_{g_i})_{i=1}^6$ is projective with weight $\frac{1}{6}$.

In Example 6.6 below, we consider another representation of $\mathcal{G}$ and

$$\left(\rho\,(g_i)\,T_{2,\pi/12}\right)_{i=1}^6 \in \mathcal{RS}(\mathcal{G}, 2, \mathbb{F}^2),\;\; T_{2,\pi/12} = \sqrt{\frac{1}{6}} \begin{pmatrix} \cos\pi/12 & -\sin\pi/12 \\ \sin\pi/12 & \cos\pi/12 \end{pmatrix},$$

with the same $G\,(\nu)$. By Lemma 2.5 and Lemma 3.6, these representations are unitarily equivalent.

Now we prove similar results for $\mathcal{G}B(\mathbb{F}^{mn\times mn})$.

Lemma 4.9. *$A \in \mathcal{G}B(\mathbb{F}^{mn\times mn})$ if and only if $A_{i,j}(h_1, h_2) = A_{i,j}(1, h_1^{-1}h_2) = A_{i,j}(gh_1, gh_2)$, for all $i, j = 1, \ldots, m$ and all $g, h_1, h_2 \in \mathcal{G}$.*

Theorem 4.10. *$(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ if and only if $G \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$ and* $\operatorname{rank}(G) = d$.

Proof. If $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ then G has rank d, it is a block matrix with blocks $T_i^* T_j$ and

$$\begin{aligned}\left\langle T_i^* T_j \delta_h^n, \delta_g^n \right\rangle_{\mathbb{F}^n} &= \left\langle T_j \delta_h^n, T_i \delta_g^n \right\rangle_{\mathcal{H}} = \left\langle \rho\,(h)\, T_j \delta_1^n, \rho\,(g)\, T_i \delta_1^n \right\rangle_{\mathcal{H}} \\ &= \left\langle \rho\left(g^{-1}h\right) T_j \delta_1^n, T_i \delta_1^n \right\rangle_{\mathcal{H}} = \left\langle T_j \delta_{g^{-1}h}^n, T_i \delta_1^n \right\rangle_{\mathcal{H}} \\ &= \left\langle T_i^* T_j \delta_{g^{-1}h}^n, \delta_1^n \right\rangle_{\mathbb{F}^n}.\end{aligned}$$

Thus, by Lemma 4.10, $G \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$.

Conversely, suppose $G \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$ with $\operatorname{rank}(G) = d$. Then $S \in GL\,(\mathcal{H})$. Let $T_i^g \in L\,(\mathbb{F}^n, \mathcal{H})$ be defined by $T_i^g \delta_h^n = T_i \delta_{gh}^n$ and let

$$U_g = \sum_{i=1}^m T_i^g T_i^* S^{-1}.$$

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Using Lemma 4.9, we obtain

$$\begin{aligned}\left\langle (T_i^g)^* T_j^g \delta_{h_2}^n, \delta_{h_1}^n \right\rangle_{\mathbb{F}^n} &= \left\langle T_j^g \delta_{h_2}^n, T_i^g \delta_{h_2}^n \right\rangle_{\mathcal{H}} = \left\langle T_j \delta_{gh_2}^n, T_i \delta_{gh_1}^n \right\rangle_{\mathcal{H}} \\ &= \left\langle T_i^* T_j \delta_{gh_2}^n, \delta_{gh_1}^n \right\rangle_{\mathbb{F}^n} = \left\langle T_i^* T_j \delta_{h_2}^n, \delta_{h_1}^n \right\rangle_{\mathbb{F}^n},\end{aligned}$$

and thus

$$(T_i^g)^* T_j^g = T_i^* T_j.$$

Then

$$\begin{aligned}U_g^* U_g &= S^{-1} \left(\sum_{i,j=1}^m T_i (T_i^g)^* T_j^g T_j^* \right) S^{-1} \\ &= S^{-1} \left(\sum_{i,j=1}^m T_i T_i^* T_j T_j^* \right) S^{-1} \\ &= S^{-1} \left(\sum_{i=1}^m T_i T_i^* \right) \left(\sum_{j=1}^m T_j T_j^* \right) S^{-1} \\ &= S^{-1} S S S^{-1} = I,\end{aligned}$$

and thus $U_g \in U(\mathcal{H})$. Since, by Lemma 4.9,

$$\begin{aligned}\left\langle (T_i^g)^* T_j \delta_{gh}^n, \delta_{h_1}^n \right\rangle_{\mathbb{F}^n} &= \left\langle T_j \delta_{gh}^n, T_i^g \delta_{h_1}^n \right\rangle_{\mathcal{H}} = \left\langle T_j \delta_{gh}^n, T_i \delta_{gh_1}^n \right\rangle_{\mathcal{H}} \\ &= \left\langle T_i^* T_j \delta_{gh}^n, \delta_{gh_1}^n \right\rangle_{\mathbb{F}^n} = \left\langle T_i^* T_j \delta_h^n, \delta_{h_1}^n \right\rangle_{\mathbb{F}^n},\end{aligned}$$

we also have

$$\begin{aligned}U_g^* T_j \delta_{gh}^n &= \sum_{i=1}^m S^{-1} T_i (T_i^g)^* T_j \delta_{gh}^n \\ &= S^{-1} \sum_{i=1}^m T_i T_i^* T_j \delta_h^n \\ &= S^{-1} S T_j \delta_h^n = T_j \delta_h^n,\end{aligned}$$

and then

$$U_{g_1 g_2} T_i \delta_h^n = T_i \delta_{g_1 g_2 h}^n = U_{g_1} T_i \delta_{g_2 h}^n = U_{g_1} U_{g_2} T_i \delta_h^n.$$

Since $\mathcal{H} = \operatorname{span}\{T_i \delta_h^n : i = 1, \ldots, m,\ g \in \mathcal{G}\}$, this last equality implies

$$U_{g_1 g_2} U_{g_1} U_{g_2}.$$

Therefore, $\rho : \mathcal{G} \to U(\mathcal{H})$, given by $\rho(g) = U_g$, is a representation of $\mathcal{G}$ such that $\rho(g) T_i \delta_h^n = T_i \delta_{gh}^n$. Thus, $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$. $\square$

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COROLLARY 4.11. *$(T_i)_{i=1}^m$ is a Parseval $(m, \mathcal{G}, \mathcal{H})$-RS if and only if G is an orthogonal projection with rank d in $\mathcal{GB}(\mathbb{F}^{mn\times mn})$.*

Proof. It follows from Theorem 4.10 and Remark 2.7. $\square$

COROLLARY 4.12. *Let $\mathcal{G} = \{g_1, \dots, g_n\}$. $(v_{1,g_1}, \dots, v_{1,g_1}, \dots, v_{m,g_1}, \dots, v_{m,g_n}) \in \mathcal{CF}(\mathcal{G}, \rho, \mathcal{H})$ if and only if $G \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$.*

Proof. It follows from Theorem 4.10 and Remark 3.5. $\square$

DEFINITION 4.13. $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ is said to be real if $\mathbb{F} = \mathbb{R}$ or there exists a unitary operator $U : \mathcal{H} \to \mathbb{R}^d$ such that $UT : \bigoplus_{i=1}^m \mathbb{F}^{n_i} \to \mathbb{R}^d$.

As a consequence of Remark 2.3 and Lemma 2.12 in [23], we have the following.

THEOREM 4.14. *$(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ is real if and only if $G \in \mathbb{R}^{\mathrm{tr}(\mathbf{n})\times\mathrm{tr}(\mathbf{n})}$.*

With a similar proof, the next corollaries generalize for GRS's Corollary 4.11 in [23] enunciated for real isometric tight frames.

COROLLARY 4.15. *Let $A = \{g \in \mathcal{G} : g^2 = 1\}$. Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ be real and $N_e(G)$ be the number of distinct entries of G. Then $N_e(G) \le m^2\left(\frac{|\mathcal{G}|+|A|}{2}\right)$.*

Proof. By Theorem 4.5, there exists $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$ such that $G_{g,h} = \nu(g^{-1}h)$. If $g \in \mathcal{G}$, by Theorem 4.14 we have,

$$\nu\left(g^{-1}\right) = G_{g,1} = (G_{1,g})^t = \nu(g)^t.$$

Therefore, since

$$g \in A \Leftrightarrow g^{-1} \in A \Leftrightarrow g^{-1} = g, \tag{4.2}$$

then the number of distinct entries of G is at most $m^2\left(|A| + \frac{|\mathcal{G}\setminus A|}{2}\right) = m^2\left(\frac{|\mathcal{G}|+|A|}{2}\right)$. $\square$

COROLLARY 4.16. *Let $A = \{g \in \mathcal{G} : g^2 = 1\}$. Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ be real and $N_e(G)$ be the number of distinct entries of each block of G. Then $N_e(G) \le (|\mathcal{G}| + |A|)/2$.*

Proof. By Theorem 4.10, the entries of the blocks of G are the values of $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$ where $\nu_{i,j}(g^{-1}h) = G_{i,j}(g,h)$. If $g \in \mathcal{G}$, by Theorem 4.14 we have,

$$\nu_{i,j}\left(g^{-1}\right) = G_{i,j}(g,1) = G_{i,j}(1,g) = \nu_{i,j}(g).$$

Therefore, by (4.2) $\nu_{i,j}$ can take at most $\frac{|\mathcal{G}|+|A|}{2}$ distinct values. $\square$

Theorem 4.21 and Theorem 4.22 below generalize Theorem 4.2 in [23]. In order to state them, we first introduce the following definitions.

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DEFINITION 4.17. $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$ is elementary if it cannot be factored $\nu = \theta \circ \eta$ where $\theta : G \to \mathcal{K}$ is a group homomorphism with $|\mathcal{K}| < |\mathcal{G}|$ and $\eta : \mathcal{K} \to \mathbb{F}^{m\times m}$, or equivalently, if

$$N = \{g \in \mathcal{G} : \nu(h_1 g h_2) = \nu(h_1 h_2),\ \forall h_1, h_2 \in \mathcal{G}\} = \{1\}.$$

DEFINITION 4.18. $A(\nu) \in B\mathcal{G}(\mathbb{F}^{nm\times nm})$ is elementary if ν is elementary.

DEFINITION 4.19. $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$, $i, j = 1, \ldots, m$, are jointly elementary if $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$ defined by $\nu(g)(i,j) = \nu_{i,j}(g)$ is elementary.

DEFINITION 4.20. $A(\{\nu_{i,j}\}_{i,j=1}^{m}) \in \mathcal{G}B(\mathbb{F}^{mn\times mn})$ is elementary if $\nu_{i,j}$, $i, j = 1, \ldots, m$, are jointly elementary.

Injective representations are called *faithful*.

THEOREM 4.21. *Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, \rho, m, \mathcal{H})$. The following are equivalent:*

1. *ρ is faithful.*
2. *$G(\nu)$ is elementary.*
3. *$T_{gh} = T_h$, $\forall h \in \mathcal{G}$, implies that $g = 1$.*

Proof. Since $\{T_g \delta_i^m : g \in \mathcal{G},\ i = 1, \ldots, m\}$ is a frame for $\mathcal{H}$ we have,

(1) $\Leftrightarrow$ (3). ρ is faithful if and only if

$$T_{gh}\delta_i^m = \rho(g) T_h \delta_i^m = T_h \delta_i^m,\ \forall h \in \mathcal{G},\ i = 1, \ldots, m, \text{ implies that } g = 1,$$

if and only if

$$T_{gh} = T_h,\ \forall h \in \mathcal{G}, \text{ implies that } g = 1.$$

(2) $\Leftrightarrow$ (3).

$$\begin{aligned} N &= \{g \in \mathcal{G} : \upsilon(h_1 g h_2) = \upsilon(h_1 h_2),\ \forall h_1, h_2 \in \mathcal{G}\} \\ &= \left\{g \in \mathcal{G} : T^*_{h_1^{-1}} T_{gh_2} = T^*_{h_1^{-1}} T_{h_2},\ \forall h_1, h_2 \in \mathcal{G}\right\}. \end{aligned}$$

Now, $T^*_{h_1^{-1}} T_{gh_2} = T^*_{h_1^{-1}} T_{h_2}$ is equivalent to

$$\left\langle T_{gh_2}\delta_i^m, T_{h_1^{-1}}\delta_j^m \right\rangle_{\mathcal{H}} = \left\langle T_{h_2}\delta_i^m, T_{h_1^{-1}}\delta_j^m \right\rangle_{\mathcal{H}},\quad i, j = 1, \ldots, m,$$

then

$$N = \{g \in \mathcal{G} : T_{gh} = T_h,\ \forall h \in \mathcal{G}\}.$$

Therefore, $N = \{1\}$ is equivalent to (3). $\square$

THEOREM 4.22. *Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \rho, \mathcal{H})$. The following are equivalent:*

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1. *ρ is faithful.*
2. *$G(\{\nu_{i,j}\}_{i,j=1}^m)$ is elementary.*
3. *$T_i\delta^n_{gh} = T_i\delta^n_h$, $\forall h \in \mathcal{G}$, $i = 1, \ldots, m$, implies that $g = 1$.*

Proof. Using that $\left\{T_i\delta^n_g : g \in \mathcal{G},\, i = 1, \ldots, m\right\}$ is a frame for $\mathcal{H}$ we have,

(1) $\Leftrightarrow$ (3). ρ is faithful if and only if

$$T_i\delta^n_{gh} = \rho(g)T_i\delta^n_h = T_i\delta^n_h,\ \forall h \in \mathcal{G},\ i = 1, \ldots, m, \text{ implies that } g = 1.$$

(2) $\Leftrightarrow$ (3).

$$\begin{aligned} N &= \{g \in \mathcal{G} : \upsilon_{i,j}\left(h_1 g h_2\right) = \upsilon_{i,j}\left(h_1 h_2\right),\ \forall h_1, h_2 \in \mathcal{G},\ i, j = 1, \ldots, m\} \\ &= \left\{g \in \mathcal{G} : \left\langle T_j\delta^n_{gh_2}, T_i\delta^n_{h_1^{-1}}\right\rangle_{\mathcal{H}} = \left\langle T_j\delta^n_{h_2}, T_i\delta^n_{h_1^{-1}}\right\rangle_{\mathcal{H}},\ \forall h_1, h_2 \in \mathcal{G},\ i, j = 1, \ldots, m\right\}. \end{aligned}$$

Then

$$N = \left\{g \in \mathcal{G} : T_j\delta^n_{gh} = T_j\delta^n_h,\ \forall h \in \mathcal{G},\ j = 1, \ldots, m\right\}.$$

Therefore, $N = \{1\}$ is equivalent to (3). □

5. Unitary symmetries of reconstruction systems. In [23, 24] symmetries of frames are studied. In [23] tight frames are considered and it is shown that any symmetry of a tight frame, i.e., any invertible linear transformation on $\mathcal{H}$ which maps the set of the elements of a tight frame onto itself, is unitary. In [24] any frame is considered and symmetries are permutations on the index set. If the frame is tight the symmetry groups considered in [23] and [24] are isomorphic. The aim of this section is to consider RS's associated with unitary symmetries, i.e., unitary operators on $\mathcal{H}$ that permute the elements of a (non necessarily tight) RS.

Given $\pi \in \mathcal{S}_m$, P_π denotes the permutation matrix of order m associated with π (i.e., $P_\pi(i,j) = 1$ if and only if $\pi(i) = j$). Given $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{p\times q}$ we consider the tensor product

$$A \otimes B = \begin{pmatrix} A(1,1)B & \cdots & A(1,n)B \\ \vdots & \ddots & \vdots \\ A(m,1)B & \cdots & A(m,n)B \end{pmatrix} \in \mathbb{F}^{mp\times nq}.$$

Definition 5.1. $(T_i)_{i=1}^m, \left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$ are:

1. Type I equivalent if there exist $U \in U(\mathcal{H})$ and $\pi \in \mathcal{S}_m$ such that

$$UT_i = \widetilde{T}_{\pi(i)},\ \ i = 1, \ldots, m.$$

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2. Type II equivalent if there exist $U \in U(\mathcal{H})$ and $\pi \in \mathcal{S}_n$ such that

$$UT_i\delta_j^n = \widetilde{T}_i\delta_{\pi(j)}^n, \;\; i = 1, \ldots, m, \; j = 1, \ldots, n.$$

REMARK 5.2 (Relation between type I and type II equivalence relations). Let $\sigma \in \mathcal{S}_n$ and $\pi \in \mathcal{S}_m$.

Suppose that $(T_{1,i})_{i=1}^m, (T_{2,i})_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$ are type I equivalent with

$$UT_{1,i} = T_{2,\upsilon(i)}, \;\; i = 1, \ldots, m,$$

for some $U \in U(\mathcal{H})$ and $\upsilon \in \mathcal{S}_m$. Let $\left(\widetilde{T}_{1,j}\right)_{j=1}^n, \left(\widetilde{T}_{2,j}\right)_{j=1}^n \in \mathcal{RS}(n, m, \mathcal{H})$ be such that

$$\widetilde{T}_{1,j}\delta_i^m = T_{1,\pi(i)}\delta_{\sigma(j)}^n,$$

and

$$\widetilde{T}_{2,j}\delta_i^m = T_{2,\pi(i)}\delta_{\sigma(j)}^n.$$

Then

$$U\widetilde{T}_{1,j}\delta_i^m = UT_{1,\pi(i)}\delta_{\sigma(j)}^n = T_{2,(\upsilon\circ\pi)(i)}\delta_{\sigma(j)}^n = \widetilde{T}_{2,j}\delta_{(\pi^{-1}\circ\upsilon\circ\pi)(i)}^m.$$

Thus, $\left(\widetilde{T}_{1,j}\right)_{j=1}^n$ and $\left(\widetilde{T}_{2,j}\right)_{j=1}^n$ are type II equivalent.

Conversely, suppose that $\left(\widetilde{T}_{1,j}\right)_{j=1}^n, \left(\widetilde{T}_{2,j}\right)_{j=1}^n \in \mathcal{RS}(n, m, \mathcal{H})$ are type II equivalent. Let $(T_{1,i})_{i=1}^m, (T_{2,i})_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$ be such that

$$T_{1,i}\delta_j^n = \widetilde{T}_{1,\sigma(j)}\delta_{\pi(i)}^m,$$

and

$$T_{2,i}\delta_j^n = \widetilde{T}_{2,\sigma(j)}\delta_{\pi(i)}^m.$$

In a similar manner, it can be proved that $(T_{1,i})_{i=1}^m$ and $(T_{2,i})_{i=1}^m$ are type I equivalent.

As a consequence of Theorem 2.9 in [23] and its proof, we have the following.

THEOREM 5.3. $(T_i)_{i=1}^m, \left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$ *are:*

1. *Type I equivalent if and only if there exists* $\pi \in \mathcal{S}_m$ *such that*

$$G = (P_\pi \otimes I_n)\, \widetilde{G}\, (P_\pi \otimes I_n).$$

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2. *Type II equivalent if and only if there exists* $\pi \in \mathcal{S}_n$ *such that*

$$G = (I_m \otimes P_\pi)\, \widetilde{G}\, (I_m \otimes P_\pi)\,.$$

DEFINITION 5.4. Let $T = (T_i)_{i=1}^m \in \mathcal{RS}(m,n,\mathcal{H})$. The **type I symmetry group** of T is

$$\mathrm{Sym}_{\mathrm{I}}(T) := \left\{U \in U(\mathcal{H}) : \exists \pi \in \mathcal{S}_m,\ \forall i = 1,\ldots,m,\ UT_i = T_{\pi(i)}\right\},$$

and the **type II symmetry group** of T is

$$\mathrm{Sym}_{\mathrm{II}}(T) := \left\{U \in U(\mathcal{H}) : \exists \pi \in \mathcal{S}_n,\ \forall i = 1,\ldots,m,\ j = 1,\ldots,n,\ UT_i\delta_j^n = T_i\delta_{\pi(j)}^n\right\}.$$

Note that $\mathrm{Sym}_{\mathrm{I}}(T)$ and $\mathrm{Sym}_{\mathrm{II}}(T)$ are subgroups of $U(\mathcal{H})$.

REMARK 5.5. Suppose that $T = (T_i)_{i=1}^m, \widetilde{T} = \left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}\,(m,n,\mathcal{H})$ are type I equivalent with $UT_i = \widetilde{T}_{\upsilon(i)}$, $i = 1,\ldots,m$, for some $U \in U(\mathcal{H})$ and $\upsilon \in \mathcal{S}_m$. Then $\mathrm{Sym}_{\mathrm{I}}(\widetilde{T}) = U\mathrm{Sym}_{\mathrm{I}}(T)U^*$. Similarly, if $T = (T_i)_{i=1}^m, \widetilde{T} = \left(\widetilde{T}_i\right)_{i=1}^m \in \mathcal{RS}\,(m,n,\mathcal{H})$ are type II equivalent with $UT_i\delta_j^n = \widetilde{T}_i\delta_{\upsilon(j)}^n$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, for some $U \in U(\mathcal{H})$ and $\upsilon \in \mathcal{S}_n$, then $\mathrm{Sym}_{\mathrm{II}}(\widetilde{T}) = U\mathrm{Sym}_{\mathrm{II}}(T)U^*$.

REMARK 5.6. Let $T = (T_j)_{j=1}^m, \in \mathcal{RS}\,(m,n,\mathcal{H})$ and $\left(\widetilde{T}_i\right)_{i=1}^n \in \mathcal{RS}\,(n,m,\mathcal{H})$ such that $\widetilde{T}_i\delta_j^m = T_{\pi(j)}\delta_{\sigma(i)}^n$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, for some $\sigma \in \widetilde{S}_n$ and $\pi \in \widetilde{S}_m$. On the same lines as in Remark 5.2, it is easy to see that $\mathrm{Sym}_{\mathrm{I}}(T) = \mathrm{Sym}_{\mathrm{II}}(\widetilde{T})$.

We denote the set obtained with the elements of $(T_i)_{i=1}^n$ with $\mathcal{T}$.

DEFINITION 5.7. $T = (T_i)_{i=1}^n \in \mathcal{RS}\,(n,m,\mathcal{H})$ is said to be **I-transitive** if $\mathrm{Sym}_{\mathrm{I}}(T)$ acts transitively on $\mathcal{T}$, i.e.,

$$\mathrm{Sym}_{\mathrm{I}}(T)T_i = \mathcal{T},\ \ i = 1,\ldots,n.$$

DEFINITION 5.8. $A \in \mathbb{F}^{\widehat{n}m\times\widehat{n}m}$ is **type I $\mathcal{G}$-augmented** if there exist a function $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$, a subgroup $\mathcal{K}$ of $\mathcal{G}$ and a set $\{g_1,\ldots,g_{\widehat{n}}\}$ of left coset representatives of $\mathcal{K}$ in $\mathcal{G}$ such that

$$\nu\,(h_1gh_2) = \nu\,(g)\,,\ g \in \mathcal{G},\ h_1,h_2 \in \mathcal{K},$$

and

$$A_{i,j} = \nu\left(g_i^{-1}g_j\right),\ \ i,j = 1,\ldots,\widehat{n}.$$

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In this case, we write $A(\nu)$.

Let $A \in \mathbb{F}^{nm \times nm}$. $A \in B\mathcal{G}(\mathbb{F}^{nm\times nm})$ if and only if A is $\mathcal{G}$-augmented with $\mathcal{K} = \{1\}$.

Theorem 5.9 and Theorem 5.12 below are in line with Theorem 4.8 in [23].

THEOREM 5.9. *Let $T = (T_i)_{i=1}^{\widehat{n}} \in \mathcal{RS}(n, m, \mathcal{H})$ without multiplicities. The following are equivalent:*

1. *T is I-transitive.*
2. *G is type I $\mathcal{G}$-augmented for some group $\mathcal{G}$.*
3. *There exist a group $\mathcal{G}$ and a surjection $\eta : \mathcal{G} \to \mathcal{T}$, $\eta(g) = T_g$, for which all the values $\left|\eta^{-1}(T_g)\right|$ are equal, and*

$$T^*_{gg_1} T_{gg_2} = T^*_{g_1} T_{g_2},\ \forall g, g_1, g_2 \in \mathcal{G}. \tag{5.1}$$

Proof. (1) $\Rightarrow$ (2). Let $T_0 \in \mathcal{T}$. Since T is I-transitive, $\mathrm{Sym}_{\mathrm{I}}(T)T_0 = \mathcal{T}$. Let

$$\mathcal{K} = \{U \in \mathrm{Sym}_{\mathrm{I}}(T) : UT_0 = T_0\}$$

and $\{U_1, \ldots, U_{\widehat{n}}\}$ be a set of left representatives of the subgroup $\mathcal{K}$ in $\mathrm{Sym}_{\mathrm{I}}(T)$ such that

$$T = (U_1 T_0, \ldots, U_{\widehat{n}} T_0).$$

So G is type I $\mathrm{Sym}_{\mathrm{I}}(T)$-augmented with the subgroup $\mathcal{K}$ and $\nu : \mathrm{Sym}_{\mathrm{I}}(T) \to \mathbb{F}^{m\times m}$ given by $\nu(U) = T_0^* U T_0$.

(2) $\Rightarrow$ (3). Suppose G is type I $\mathcal{G}$-augmented for some group $\mathcal{G}$, with a function $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$, a subgroup $\mathcal{K}$ of $\mathcal{G}$ and a set $\{g_1, \ldots, g_{\widehat{n}}\}$ of left coset representatives of $\mathcal{K}$ in $\mathcal{G}$. Consider the bijection $\{g_1, \ldots, g_{\widehat{n}}\} \to T$ given by $g_j \mapsto T_j$ where

$$\nu\left(g_i^{-1} g_j\right) = G_{i,j} = T_i^* T_j.$$

Then (3) is satisfied with $\eta : \mathcal{G} \to \mathcal{T}$ defined by $\eta(g) = T_i$ where i is such that $g \sim_{\mathcal{K}} g_i$.

(3) $\Rightarrow$ (1). Define an equivalence relation on $\mathcal{G}$ by

$$g_1 \sim_\eta g_2 \Leftrightarrow \eta(g_1) = \eta(g_2).$$

Let $\{g_1, \ldots, g_{\widehat{n}}\}$ be a set of representatives such that $\eta(g_i) = T_{g_i} = T_i$ and $r := \left|\eta^{-1}(T_g)\right|$. Note that

$$\sum_{g\in\mathcal{G}} T_g T_g^* = \sum_{i=1}^{\widehat{n}} \sum_{g\in[g_i]_\eta} T_g T_g^* = r \sum_{i=1}^{\widehat{n}} T_{g_i} T_{g_i}^* = rS.$$

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Thus, by (5.1) and an argument similar to that used in the proof of Theorem 4.5,

$$U_g := \frac{1}{r}\sum_{h\in\mathcal{G}} T_{gh}T_g^* S^{-1} \in U(\mathcal{H}),$$

$$U_g T_h = T_{gh},$$

and

$$U_{gh} = U_g U_h.$$

If $T_{gg_i} = T_{gg_j}$ then

$$\forall h \in \mathcal{G},\ T_{g_i}^* T_h = T_{gg_i}^* T_{gh} = T_{gg_j}^* T_{gh} = T_{g_j}^* T_h.$$

Since $\left\{T_h\delta_j^m : g \in \mathcal{G},\ j = 1,\dots,m\right\}$ is a frame for $\mathcal{H}$, this implies $T_{g_i} = T_{g_j}$. Therefore, $U_g \in \mathrm{Sym}_{\mathrm{I}}(T)$. Since the elements of $\mathcal{T}$ are of the form $U_g T_{g_i} = T_{gg_i}$, $g \in \mathcal{G}$, and

$$U_g T_{g_i} \in \mathrm{Sym}_{\mathrm{I}}(T) T_{g_i} \subseteq \mathcal{T},\ \ g \in \mathcal{G},$$

it results that T is I-transitive. □

We denote the set obtained with the elements of $\left(T_i\delta_j^n\right)_{j=1}^n$ with $\mathcal{T}_i$.

Definition 5.10. $(T_i)_{i=1}^m \in \mathcal{RS}(m,n,\mathcal{H})$ is said to be **II-transitive** if $\mathrm{Sym}_{\mathrm{II}}(T)$ acts transitively on each $\mathcal{T}_i$, i.e.,

$$\mathrm{Sym}_{\mathrm{II}}(T) T_i \delta_j^n = \mathcal{T}_i,\ \ i = 1,\dots,m,\ j = 1,\dots,n.$$

Definition 5.11. $A \in \mathbb{F}^{m\widehat{n}\times m\widehat{n}}$ is type II $\mathcal{G}$-augmented if there exist functions $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$, $i,j = 1,\dots,m$, subgroups $\mathcal{K}_i$, $i = 1,\dots,m$ of $\mathcal{G}$ and a set $\{g_{i1},\dots,g_{i\widehat{n}}\}$ of left coset representatives of $\mathcal{K}_i$ in $\mathcal{G}$ such that

$$\nu_{i,j}(h_i g h_j) = \nu_{i,j}(g),\ g \in \mathcal{G},\ h_i \in \mathcal{K}_i,$$

and

$$A_{i,j}(k,l) = \nu_{i,j}\left(g_{ik}^{-1} g_{jl}\right),\ \ k,l = 1,\dots,\widehat{n}.$$

In this case, we write $A(\{\nu_{i,j}\}_{i,j=1}^m)$.

Let $A \in \mathbb{F}^{mn\times mn}$. $A \in \mathcal{G}B(\mathbb{F}^{mn\times mn})$ if and only if A is $\mathcal{G}$-augmented with $\mathcal{K}_i = \{1\}$, $i = 1,\dots,m$.

Theorem 5.12. *Let $T = (T_i)_{i=1}^m \in \mathcal{RS}(m,n,\mathcal{H})$. The following are equivalent:*

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1. T is II-transitive with $\widehat{n} := |\mathcal{T}_i|$ independent of i.
2. G is type II $\mathcal{G}$-augmented for some group $\mathcal{G}$.
3. There exist a group $\mathcal{G}$ and surjections $\eta_i : \mathcal{G} \to \mathcal{T}_i$ for which all the values $\left|\eta_i^{-1}\left(T_i\delta_k^n\right)\right|$ are equal, and

$$(5.2)\ \langle \eta_i\left(gg_1\right), \eta_j\left(gg_2\right)\rangle = \langle \eta_i\left(g_1\right), \eta_j\left(g_2\right)\rangle,\ \forall g, g_1, g_2 \in \mathcal{G},\ i, j = 1, \ldots, m.$$

Proof. (1) $\Rightarrow$ (2). Let $1 \le i \le m$, and for each such a i, set $1 \le j_i \le n$. Since T is II-transitive,

$$\mathrm{Sym}_{\mathrm{II}}(T)T_i\delta_{j_i}^n = \mathcal{T}_i.$$

Let

$$\mathcal{K}_i = \left\{U \in \mathrm{Sym}_{\mathrm{II}}(T) : UT_i\delta_{j_i}^n = T_i\delta_{j_i}^n\right\},$$

and $\{U_{i1}, \ldots, U_{i\widehat{n}_i}\}$ a set of left representatives of the subgroup $\mathcal{K}_i$ in $\mathrm{Sym}_{\mathrm{II}}(T)$. Then

$$\mathcal{T}_i = \left\{U_{i1}T_i\delta_{j_i}^n, \ldots, U_{i\widehat{n}_i}T_i\delta_{j_i}^n\right\}.$$

Since $\widehat{n} = |\mathcal{T}_i|$ is independent of i, $\widehat{n} = \widehat{n}_1 = \cdots = \widehat{n}_n$ and G is type II $\mathrm{Sym}_{\mathrm{II}}(T)$-augmented with the subgroups $\mathcal{K}_i$ and the functions $\nu_{i,j} : \mathrm{Sym}_{\mathrm{I}}(T) \to \mathbb{F}$ given by

$$\nu_{i,j}\left(U\right) = \left\langle T_i^*UT_j\delta_{l_i}^n, \delta_{k_i}^n\right\rangle.$$

(2) $\Rightarrow$ (3). Suppose G is type II $\mathcal{G}$-augmented for some group $\mathcal{G}$, with functions $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$, subgroups $\mathcal{K}_i$ of $\mathcal{G}$ and sets $\{g_{i1}, \ldots, g_{i\widehat{n}}\}$ of left coset representatives of $\mathcal{K}_i$ in $\mathcal{G}$. Consider the bijection $\{g_{i1}, \ldots, g_{i\widehat{n}}\} \to \mathcal{T}_i$ given by $g_{ik} \mapsto T_i\delta_{i_k}$ where

$$\nu_{i,j}\left(g_{ik}^{-1}g_{jl}\right) = G_{i,j}(k,l) = \left\langle T_j\delta_{j_l}, T_i\delta_{i_k}\right\rangle.$$

Then

$$\mathcal{T}_i = \left\{T_i\delta_{i_k}\right\}_{k=1}^{\widehat{n}},$$

and (3) is satisfied with $\eta_i : \mathcal{G} \to \{T_i\delta_{i_k}\}_{k=1}^{\widehat{n}}$ defined by $\eta_i\left(g\right) = T_i\delta_{i_k}$ where k is such that $g \sim_{\mathcal{K}_i} g_{ik}$.

(3) $\Rightarrow$ (1). Define equivalence relations on $\mathcal{G}$ by

$$g_1 \sim_{\eta_i} g_1 \Leftrightarrow \eta_i\left(g_1\right) = \eta_i\left(g_2\right).$$

Let $\{g_{i1}, \ldots, g_{i\widehat{n}_i}\}$ be a set of representatives. Since $\left|\eta_i^{-1}\left(T_i\delta_k^n\right)\right|$ is independent of i and k, $\widehat{n} := \widehat{n}_1 = \cdots = \widehat{n}_m$. We have

$$\mathcal{T}_i = \left\{\eta_i\left(g_{i1}\right), \ldots, \eta_i\left(g_{i\widehat{n}}\right)\right\}.$$

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Let $T_i^g \in L(\mathbb{F}^n, \mathcal{H})$ be defined by $T_i^g \delta_k^n = \eta_i(gg_{ir})$ where $T_i \delta_k^n = \eta_i(g_{ir})$. We have

$$\begin{aligned}\left\langle (T_i^g)^* T_j^g \delta_l^n, \delta_k^n \right\rangle &= \left\langle T_j^g \delta_l^n, T_i^g \delta_k^n \right\rangle = \left\langle \eta_j(gg_{ir}), \eta_i(gg_{is}) \right\rangle \\ &= \left\langle \eta_j(g_{ir}), \eta_i(g_{is}) \right\rangle = \left\langle T_j \delta_l^n, T_i \delta_k^n \right\rangle = \left\langle T_i^* T_j \delta_l^n, \delta_k^n \right\rangle.\end{aligned}$$

Thus, by (5.2) and an argument similar to that used in the proof of Theorem 4.10,

$$U_g := \sum_{h \in \mathcal{G}} T_i^g T_i^* S^{-1} \in U(\mathcal{H}),$$

and, since

$$\begin{aligned}\left\langle \left(T_j^g\right)^* \eta_i(gh), \delta_k \right\rangle &= \left\langle \eta_i(gh), T_j^g \delta_k \right\rangle = \left\langle \eta_i(gh), \eta_j(gg_{is}) \right\rangle \\ &= \left\langle \eta_i(h), \eta_j(g_{is}) \right\rangle = \left\langle \eta_i(h), T_j \delta_k^n \right\rangle = \left\langle T_j^* \eta_i(h), \delta_k^n \right\rangle,\end{aligned}$$

we also have

$$U_g \eta_i(h) = \eta_i(gh).$$

If $\eta_i(gg_{ir}) = \eta_i(gg_{is})$ then for $h \in \mathcal{G}$, $j = 1, \ldots, m$,

$$\left\langle \eta_i(g_{ir}), \eta_j(h) \right\rangle = \left\langle \eta_i(gg_{ir}), \eta_j(gh) \right\rangle = \left\langle \eta_i(gg_{is}), \eta_j(gh) \right\rangle = \left\langle \eta_i(g_{is}), \eta_j(h) \right\rangle. \tag{5.3}$$

Since T is a RS for $\mathcal{H}$ then $\{\eta_j(h) : j = 1, \ldots, m, h \in \mathcal{G}\}$ is a frame for $\mathcal{H}$, and thus (5.3) implies $\eta_i(g_{ir}) = \eta_i(g_{is})$. Therefore, $U_g \in \mathrm{Sym}_{\mathrm{II}}(T)$. Given $h \in \mathcal{G}$, any element of $\mathcal{T}_i$ is of the form $U_g \eta_i(h)$ for some $g \in \mathcal{G}$, and

$$\{U_g \eta_i(h) : g \in \mathcal{G}\} \subseteq \mathrm{Sym}_{\mathrm{II}}(T) \eta_i(h) \subseteq \mathcal{T}_i,$$

it results that T is II-transitive. $\square$

6. Tightness and irreducibility. We introduce now a definition from [22] (see also [23]).

Definition 6.1. A finite subgroup $\mathcal{W} \subseteq U(\mathcal{H})$ is irreducible if $\mathcal{W}\phi$ spans $\mathcal{H}$ for every nonzero $\phi \in \mathcal{H}$.

Recall that $\phi \in \mathcal{H}$ is called cyclic if $\rho(\mathcal{G})\phi$ spans $\mathcal{H}$. Theorem 6.2 and Theorem 6.3 are a generalization of Theorem 2.2 in [22] (see also Theorem 6.3 in [23]) with an identical proof.

Theorem 6.2. *Let $\rho(\mathcal{G})$ be irreducible, $T \in L(\mathbb{F}^m, \mathcal{H})$ and $T \neq 0$. Then $(\rho(g) T)_{g \in \mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ is tight.*

Proof. Since $\rho(\mathcal{G})$ is irreducible, then $(\rho(g) T)_{g \in \mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$ and it has a positive definite RS operator S. Let $\lambda > 0$ be an eigenvalue of S with λ-eigenvector f_λ. Let $h \in \mathcal{G}$. Using Proposition 3.8(1),

$$S\rho(h) f_\lambda = \rho(h) S f_\lambda = \rho(h) \lambda f_\lambda = \lambda \rho(h) f_\lambda,$$

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then $\rho(h) f_\lambda$ is an λ-eigenvector of S. Thus, $Sf = \lambda f$ for all $f \in \operatorname{span}\{\rho(h) f_\lambda\}_{h\in\mathcal{G}}$. Since $\rho(\mathcal{G})$ is irreducible, then $\operatorname{span}\{\rho(h) f_\lambda\}_{h\in\mathcal{G}} = \mathcal{H}$, and hence $S = \lambda I_{\mathcal{H}}$. ◻

As noted in [22] the argument used in the previous proof is essentially Schur's Lemma for irreducible modules.

Note that if $\rho(\mathcal{G})$ is irreducible, then $n \geq d$. In the following theorem we drop the assumption $T_i \neq 0$, $i = 1, \ldots, m$.

Theorem 6.3. *Let $\rho(\mathcal{G})$ be irreducible. If $\{f_i\}_{1=1}^m \subset \mathcal{H}$, where $f_i \neq 0$ for some i, and $T_i \in L(\mathbb{F}^n, \mathcal{H})$ is given by $T_i\delta_g^n = \rho(g) f_i$, then $\{T_i\}_{1=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ is tight.*

Proof. It follows from Remark 3.3, Proposition 3.4 and Proposition 6.2. ◻

A representation $\rho : \mathcal{G} \to U(\mathcal{H})$ is called *irreducible* if the only invariant subspaces are $\{0\}$ and $\mathcal{H}$, i.e., if V is a subspace of $\mathcal{H}$ such that for all $g \in \mathcal{G}$, $\rho(g)V \subseteq V$ then $V = \{0\}$ or $V = \mathcal{H}$. The above theorems can be stated in terms of irreducible representations as a consequence of the following well known result.

Proposition 6.4. *Let $\rho : \mathcal{G} \to U(\mathcal{H})$ be a representation. $\rho(\mathcal{G})$ is irreducible if and only if ρ is irreducible.*

By Proposition 6.4, Theorem 6.2 and Theorem 6.3 are more general results than Corollary 3.15 in [19].

Remark 6.5. Suppose that $n > d$ and $T \in \mathbb{F}^{d\times m}$, $T \neq 0$. Let $\rho : \mathcal{G} \to \mathcal{U}(\mathbb{F}^d)$ be an irreducible representation of $\mathcal{G}$. By Theorem 6.2 and Proposition 3.8(4), $(\rho(g) T)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathbb{F}^d)$ is tight and one-erasure robust. Moreover, by Proposition 3.8(3),

$$S = \frac{n}{d} \|T\|_{HS}^2 I_{\mathbb{F}^{d\times d}}.$$

By Proposition 3.8(5), $(\rho(g) T)_{g\in\mathcal{G}}$ is projective if and only if T has orthogonal columns of equal norm. In this case, $m \leq d$. Therefore, $(\rho(g) T)_{g\in\mathcal{G}}$ is Parseval, projective and one-erasure robust if and only if T has orthogonal columns of norm $\sqrt{\frac{d}{nm}}$.

Example 6.6. Consider as in Example 4.8 the dihedral group of order 6, $\mathcal{G} = \langle a, b : a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and the orthogonal matrices

$$A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that act on $\mathbb{R}^2$ as a rotation through $2\pi/3$ and a reflection in the x-axis, respectively.

The function $\rho : \mathcal{G} \to \mathcal{U}(\mathbb{R}^2)$, $\rho(a^j b^j) = A^j B^j$ is a faithful irreducible representation of $\mathcal{G}$. Next some concrete examples of Parseval, projective and one-erasure

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robust GRS's appear. We consider

$$\rho(g)\,T_{i,\theta},\ g=1,a,a^2,b,ab,a^2b,\ \theta=\pi/6,\pi/12,0,\ i=1,2,$$

$$T_{1,\theta}=\sqrt{\frac{1}{3}}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},$$

and

$$T_{2,\theta}=\sqrt{\frac{1}{6}}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

Following we list the elements of each obtained GRS.

- $T_{1,\pi/6}=\sqrt{\frac{1}{3}}\begin{pmatrix}\cos\pi/6\\ \sin\pi/6\end{pmatrix}$:

$$\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2\sqrt{3}}\end{pmatrix},\begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2\sqrt{3}}\end{pmatrix},\begin{pmatrix}0\\ -\frac{1}{\sqrt{3}}\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2\sqrt{3}}\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{\sqrt{3}}\end{pmatrix},\begin{pmatrix}-\frac{1}{2}\\ -\frac{1}{2\sqrt{3}}\end{pmatrix}.$$

- $T_{2,\pi/6}=\sqrt{\frac{1}{6}}\begin{pmatrix}\cos\pi/6 & -\sin\pi/6\\ \sin\pi/6 & \cos\pi/6\end{pmatrix}$:

$$\begin{pmatrix}\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}}\\ \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}}\end{pmatrix},\begin{pmatrix}-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}}\\ \frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}}\end{pmatrix},\begin{pmatrix}0 & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & 0\end{pmatrix},$$

$$\begin{pmatrix}\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}}\\ -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}}\end{pmatrix},\begin{pmatrix}0 & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}} & 0\end{pmatrix},\begin{pmatrix}-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}}\\ -\frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}}\end{pmatrix}.$$

- $T_{1,\pi/12}=\sqrt{\frac{1}{3}}\begin{pmatrix}\cos\pi/12\\ \sin\pi/12\end{pmatrix}$:

$$\begin{pmatrix}\frac{\sqrt{3}+1}{2\sqrt{6}}\\ \frac{\sqrt{3}-1}{2\sqrt{6}}\end{pmatrix},\begin{pmatrix}-\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{pmatrix},\begin{pmatrix}\frac{1-\sqrt{3}}{2\sqrt{6}}\\ -\frac{\sqrt{3}+1}{2\sqrt{6}}\end{pmatrix},\begin{pmatrix}\frac{\sqrt{3}+1}{2\sqrt{6}}\\ \frac{1-\sqrt{3}}{2\sqrt{6}}\end{pmatrix},\begin{pmatrix}\frac{1-\sqrt{3}}{2\sqrt{6}}\\ \frac{\sqrt{3}+1}{2\sqrt{6}}\end{pmatrix},\begin{pmatrix}-\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\end{pmatrix}.$$

- $T_{2,\pi/12}=\sqrt{\frac{1}{6}}\begin{pmatrix}\cos\pi/12 & -\sin\pi/12\\ \sin\pi/12 & \cos\pi/12\end{pmatrix}$:

$$\begin{pmatrix}\frac{\sqrt{3}+1}{4\sqrt{3}} & \frac{1-\sqrt{3}}{4\sqrt{3}}\\ \frac{\sqrt{3}-1}{4\sqrt{3}} & \frac{\sqrt{3}+1}{4\sqrt{3}}\end{pmatrix},\begin{pmatrix}-\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}}\\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}}\end{pmatrix},\begin{pmatrix}\frac{1-\sqrt{3}}{4\sqrt{3}} & \frac{\sqrt{3}+1}{4\sqrt{3}}\\ -\frac{\sqrt{3}+1}{4\sqrt{3}} & \frac{1-\sqrt{3}}{4\sqrt{3}}\end{pmatrix},$$

$$\begin{pmatrix}\frac{\sqrt{3}+1}{4\sqrt{3}} & \frac{1-\sqrt{3}}{4\sqrt{3}}\\ \frac{1-\sqrt{3}}{4\sqrt{3}} & -\frac{\sqrt{3}+1}{4\sqrt{3}}\end{pmatrix},\begin{pmatrix}\frac{1-\sqrt{3}}{4\sqrt{3}} & \frac{\sqrt{3}+1}{4\sqrt{3}}\\ \frac{\sqrt{3}+1}{4\sqrt{3}} & \frac{\sqrt{3}-1}{4\sqrt{3}}\end{pmatrix},\begin{pmatrix}-\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}}\\ -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}}\end{pmatrix}.$$

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- $T_{1,0} = \sqrt{\frac{1}{3}} \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix}$:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \end{pmatrix}.$$

- $T_{2,0} = \sqrt{\frac{1}{6}} \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$:

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{6}} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{6}} \end{pmatrix}.$$

Note that in these cases although the associated $(\mathcal{G}, m, \mathcal{H})$-frames have repeated elements, each GRS consists of distinct operators.

$\left(\rho\left(g\right) T_{2,\pi/12} : g = 1, a, a^2, b, ab, a^2b\right)$ has the same Gram matrix than the GRS considered in Example 4.8. As was noted in that example, the representation is unitarily equivalent to the representation considered in the present example. Specifically, if

$$Q = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix},$$

then

$$Q \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} Q^* = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

and

$$Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^* = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Using Lemma 6.6. in [23] (a version of Maschke's theorem that can be restated in terms of unitary representations), for a reducible representation $\rho : \mathcal{G} \to U\left(\mathcal{H}\right)$ we have the orthogonal direct decomposition

$$\mathcal{H} = \bigoplus_{i=1}^{r} \mathcal{H}_i, \tag{6.1}$$

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where $\mathcal{H}_i$ is an irreducible $\mathbb{F}\mathcal{G}$-submodule of $\mathcal{H}$. The summands are unique up to ordering and $\mathbb{F}\mathcal{G}$-isomorphism. Correspondingly, there exist irreducible representations

$$\rho_i : \mathcal{G} \to U(\mathcal{H}_i)\,, \;\; \rho_i(g) = \rho(g)_{|\mathcal{H}_i}\,, \;\; i = 1, \ldots, r,$$

such that

$$\rho(g)\,h = \sum_{i=1}^{r} \rho_i(g)\,h_i,\; h = \oplus_{i=1}^{r} h_i,\; h_i \in \mathcal{H}_i.$$

We consider the projections

$$P_i : \mathcal{H} \to \mathcal{H}_i,$$

given by

$$P_i(h) = h_i,\; h = \oplus_{i=1}^{r} h_i,\; h_i \in \mathcal{H}_i.$$

Theorem 6.7 and Theorem 6.8 below generalize Lemma 6.7 in [23].

THEOREM 6.7. *Let $T \in L(\mathbb{F}^m, \mathcal{H})$ be such that $P_i T \neq 0$, $i = 1, \ldots, r$. Then $(\rho(g)\,T)_{g\in\mathcal{G}}$ is an α-tight $(\mathcal{G}, m, \mathcal{H})$-RS if and only if*

$$\frac{n\,\|P_i T\|_{HS}^2}{\dim(\mathcal{H}_i)} = \alpha, \;\; i = 1, \ldots, r, \tag{6.2}$$

$$\sum_{g\in\mathcal{G}} \rho_j(g)\,P_j T T^* P_i \rho_i(g)^*\,P_i T = 0, \;\; i, j = 1, \ldots, r,\; j \neq i. \tag{6.3}$$

Proof. We have,

$$S = \sum_{g\in\mathcal{G}} T_g T_g^* = \sum_{g\in\mathcal{G}} \rho(g)\,T T^* \rho(g)^* = \sum_{j=1}^{r} \sum_{g\in\mathcal{G}} \rho_j(g)\,P_j T T^* \rho(g)^*\,,$$

and if $f_i \in \mathcal{H}_i$,

$$T^* \rho(g)^*\,f_i = \sum_{j=1}^{r} T^* P_j \rho_i(g)^*\,f_i = T^* P_i \rho_i(g)^*\,f_i.$$

Thus, $S = \alpha I_{\mathcal{H}}$ if and only if

$$\sum_{j=1}^{r} \sum_{g\in\mathcal{G}} \rho_j(g)\,P_j T T^* P_i \rho_i(g)^*\,f_i = \alpha f_i,\; f_i \in \mathcal{H}_i, \;\; i = 1, \ldots, r. \tag{6.4}$$

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Equating terms in the above orthogonal sums it results that (6.4) is equivalent to

$$\sum_{g\in\mathcal{G}} \rho_i(g) P_i TT^* P_i \rho_i(g)^* f_i = \alpha f_i,\ f_i \in \mathcal{H}_i,\ i = 1,\dots,r, \tag{6.5}$$

and

$$\sum_{g\in\mathcal{G}} \rho_j(g) P_j TT^* P_i \rho_i(g)^* f_i = 0,\ f_i \in \mathcal{H}_i,\ i,j = 1,\dots,r,\ j\neq i. \tag{6.6}$$

Since ρ_j is irreducible and $P_iT\neq 0$, $i=1,\dots,r$, by Theorem 6.2, Proposition 6.4 and Proposition3.8(3),

$$\sum_{g\in\mathcal{G}} \rho_i(g) P_i TT^* P_i \rho_i(g)^* f_i = \frac{n\left\|P_iT\right\|_{HS}^2}{\dim(\mathcal{H}_i)} f_i,\quad i=1,\dots,r.$$

Hence, (6.5) holds if and only if (6.2) holds. By Theorem 6.2, Proposition 6.4 and Remark 2.3, $\left\{\rho_i(g) P_i T\delta_j^m : g\in\mathcal{G}, j\in\{1,\dots,m\}\right\}$ is a (tight) frame for $\mathcal{H}_i$, so (6.6) is equivalent to

$$\sum_{g\in\mathcal{G}} \rho_j(g) P_j TT^* P_i \rho_i(g)^* \rho_i(h) P_i T = 0,\ h\in\mathcal{G},\ i,j=1,\dots,r,\ j\neq i,$$

or

$$\rho_j(h)\sum_{g\in\mathcal{G}} \rho_j\left(h^{-1}g\right) P_j TT^* P_i \rho_i\left(h^{-1}g\right)^* P_i T = 0,\ h\in\mathcal{G},\ i,j=1,\dots,r,\ j\neq i.$$

This last condition is equivalent to (6.3). ◻

Theorem 6.8. *Let $\{f_i\}_{1=1}^m \subset \mathcal{H}$ where for all $j=1,\dots,r$, $P_jf_i\neq 0$ for some i, and $T_i\in L(\mathbb{F}^n,\mathcal{H})$ given by $T_i\delta_g^n = \rho(g) f_i$. Then $\{T_i\}_{1=1}^m \in \mathcal{RS}(m,\mathcal{G},\mathcal{H})$ is an α-tight $(m,\mathcal{G},\mathcal{H})$-RS if and only if*

$$\frac{n}{\dim(\mathcal{H}_j)}\sum_{i=1}^m \left\|P_jf_i\right\|_{\mathcal{H}}^2 = \alpha,\quad j=1,\dots,r, \tag{6.7}$$

$$\sum_{i=1}^m\sum_{g\in\mathcal{G}} \langle P_jf_{i'}, \rho(g) P_jf_i\rangle_{\mathcal{H}}\, \rho(g) P_l f_i = 0,\ i'=1,\dots,m,\ j,l=1,\dots,r,\ l\neq j. \tag{6.8}$$

Proof. Let $T\in L(\mathbb{F}^m,\mathcal{H})$ by $T\delta_i^m = T_i\delta_1^n$ (where 1 is the unit in $\mathcal{G}$), $i=1,\dots,m$.

For $i'=1,\dots,m$, $j,l=1,\dots,r$ we have,

$$\begin{aligned}\sum_{i=1}^m\sum_{g\in\mathcal{G}} \langle P_jf_{i'}, \rho(g) P_jf_i\rangle_{\mathcal{H}}\, \rho(g) P_l f_i &= \sum_{g\in\mathcal{G}}\sum_{i=1}^m \langle P_jT\delta_{i'}^m, \rho(g) P_jT\delta_i^m\rangle_{\mathcal{H}}\, \rho(g) P_l T\delta_i^m \\ &= \sum_{g\in\mathcal{G}} \rho_j(g) P_l TT^* P_j \rho_j(g)^* P_j T\delta_{i'}^m.\end{aligned}$$

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Thus, the proof follows from Proposition 3.4 and Theorem 6.7. ☐

Theorem 6.9 and Theorem 6.10 below generalize Theorem 6.18 in [23].

THEOREM 6.9. *Suppose that in decomposition (6.1) the summands that occur more than once are absolutely irreducible. Let* $\{f_i\}_{1=1}^{m} \subset \mathcal{H}$ *where for all* $j = 1, \ldots, r$, $P_j f_i \neq 0$ *for some* i, *and* $T_i \in L(\mathbb{F}^n, \mathcal{H})$ *given by* $T_i \delta_g^n = \rho(g) f_i$. *Then* $\{T_i\}_{1=1}^{m} \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ *is an* α*-tight* $(m, \mathcal{G}, \mathcal{H})$*-RS if and only if*

$$\frac{n}{\dim(\mathcal{H}_j)} \sum_{i=1}^{m} \|P_j f_i\|_{\mathcal{H}}^2 = \alpha, \quad j = 1, \ldots, r, \tag{6.9}$$

and if there exists an $\mathbb{F}\mathcal{G}$*-isomorphism* $\sigma : \mathcal{H}_j \to \mathcal{H}_l$ *with* $l \neq j$,

$$\sum_{i \in \mathcal{I}} \frac{\|P_j f_i\|_{\mathcal{H}}^2}{\|\sigma P_j f_i\|_{\mathcal{H}}^2} \langle P_l f_i, \sigma P_j f_i \rangle_{\mathcal{H}} = 0, \tag{6.10}$$

where $\mathcal{I} = \{i \in \{1, \ldots, m\} : P_j f_i \neq 0, P_l f_i \neq 0\}$.

Proof. Condition (6.9) is condition (6.7). Let

$$S_i : \mathcal{H}_j \to \mathcal{H}_l, \;\; S_i f = \sum_{g \in \mathcal{G}} \langle f, \rho(g) P_j f_i \rangle_{\mathcal{H}} \rho(g) P_l f_i, \;\; j, l = 1, \ldots, r, \; l \neq j.$$

We have,

$$\rho(h) S_i f = \sum_{g \in \mathcal{G}} \langle \rho(h) f, \rho(hg) P_j f_i \rangle_{\mathcal{H}} \rho(hg) P_l f_i = S_i \rho(h) f,$$

thus S_i is an $\mathbb{F}\mathcal{G}$-homomorphism and (6.8) holds if and only if

$$\sum_{i=1}^{m} S_i \rho(g) P_j f_{i'} = 0, \; g \in \mathcal{G}, \;\; i' = 1, \ldots, m.$$

By Theorem 6.3, Proposition 6.4 and Remark 2.3, $\{\rho(g) P_j f_{i'} : g \in \mathcal{G}, i' = 1, \ldots, m\}$ is a (tight) frame for $\mathcal{H}_j$, so (6.8) is equivalent to

$$\sum_{i=1}^{m} S_i = 0.$$

By Schur's lemma (see, e.g., [13]), if $\mathcal{H}_j$ and $\mathcal{H}_l$ are not $\mathbb{F}\mathcal{G}$-isomorphic, $S_i = 0$, $i = 1, \ldots, m$, and then (6.8) holds. Otherwise, if $i \in \mathcal{I}$ and $\sigma : \mathcal{H}_j \to \mathcal{H}_l$ is an $\mathbb{F}\mathcal{G}$-isomorphism, by Lemma 6.14 in [23],

$$S_i = \frac{n \|P_j f_i\|_{\mathcal{H}}^2}{\dim(\mathcal{H}_j) \|\sigma P_j f_i\|_{\mathcal{H}}^2} \langle P_l f_i, \sigma P_j f_i \rangle_{\mathcal{H}} \sigma.$$

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Thus, condition (6.10) is equivalent to $\sum_{i\in\mathcal{I}} S_i = 0$, and thus, also equivalent to (6.8). ☐

Theorem 6.10. *Suppose that in decomposition (6.1) the summands that occur more than once are absolutely irreducible. Let $T \in L(\mathbb{F}^m, \mathcal{H})$ be such that $P_iT \neq 0$, $i = 1, \ldots, r$. Then $(\rho(g)T)_{g\in\mathcal{G}}$ is an α-tight $(\mathcal{G}, m, \mathcal{H})$-RS if and only if*

$$\frac{n\,\|P_iT\|_{HS}^2}{\dim(\mathcal{H}_i)} = \alpha, \quad i = 1, \ldots, r,$$

and if there exists an $\mathbb{F}\mathcal{G}$-isomorphism $\sigma : \mathcal{H}_j \to \mathcal{H}_l$ with $l \neq j$,

$$\sum_{i\in\mathcal{I}} \frac{\|P_jT\delta_i^m\|_{\mathcal{H}}^2}{\|\sigma P_jT\delta_i^m\|_{\mathcal{H}}^2} \langle P_lT\delta_i^m, \sigma P_jT\delta_i^m\rangle_{\mathcal{H}} = 0,$$

where $\mathcal{I} = \{i \in \{1, \ldots, m\} : P_jT\delta_i^m \neq 0, P_lT\delta_i^m \neq 0\}$.

Proof. Let $f_i = T\delta_i^m \in \mathcal{H}$, $i = 1, \ldots, m$ and $T_i \in L(\mathbb{F}^n, \mathcal{H})$ given by $T_i\delta_g^n = \rho(g) f_i$. By Proposition 3.4 and Theorem 6.9, the proof follows. ☐

Remark 6.11. Let $n > d \geq m$ and $T \in \mathbb{F}^{d\times m}$, $T \neq 0$. Let $\rho : \mathcal{G} \to \mathcal{U}(\mathbb{F}^d)$ be a reducible representation of $\mathcal{G}$ with

$$\rho(g) = \begin{pmatrix} \rho_1(g) & & \\ & \ddots & \\ & & \rho_r(g) \end{pmatrix},$$

where $\rho_i(g) \in \mathcal{U}(\mathbb{F}^{d_i})$ and ρ_i is an irreducible representation of $\mathcal{G}$, $i = 1, \ldots, m$.

By Proposition 3.8(4)(5), $(\rho(g)T)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathbb{F}^d)$ is projective and one-erasure robust if and only if T has orthogonal columns of equal norm. By Theorem 6.10 and Proposition 3.8(3), $(\rho(g)T)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathbb{F}^d)$ is tight with

$$S = \frac{n}{d}\,\|T\|_{HS}^2\, I_{\mathbb{F}^{d\times d}},$$

if

$$\left\| T\left(\sum_{s=1}^{i-1} d_s + 1 : \sum_{s=1}^{i} d_s, :\right)\right\|_{HS} = \sqrt{\frac{d_i}{d}}\,\|T\|_{HS}, \quad i = 1, \ldots, r, \tag{6.11}$$

and in the case there exists an $\mathbb{F}\mathcal{G}$-isomorphism $\sigma : \mathbb{F}^{d_{i_1}} \to \mathbb{F}^{d_{i_2}}$ with $i_2 \neq i_1$,

$$\left\langle T\left(\sum_{s=1}^{i_1-1} d_s + 1 : \sum_{s=1}^{i_1} d_s, j\right), \sigma T\left(\sum_{s=1}^{i_2-1} d_s + 1 : \sum_{s=1}^{i_2} d_s, j\right)\right\rangle_{\mathbb{F}^{d_{i_1}}} = 0,$$

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for $j = 1, \ldots, m$. In particular, we obtain a Parseval GRS replacing (6.11) by

$$\left\| T\left(\sum_{s=1}^{i-1} d_s + 1 : \sum_{s=1}^{i} d_s, :\right)\right\|_{HS} = \sqrt{\frac{d_i}{n}}, \ \ i = 1, \ldots, r.$$

EXAMPLE 6.12. Let $\rho : \mathcal{G} \to \mathcal{U}(\mathbb{R}^4)$ be a reducible representation of $\mathcal{G}$ with $\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$, where $\rho_1(g), \rho_2(g) \in \mathcal{U}(\mathbb{R}^2)$ and $\rho_1 = \rho_2$ is an irreducible representation of $\mathcal{G}$. Some possibilities for T to obtain a Parseval, projective and one-erasure robust GRS's, $(\rho(g)T)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$, are

$$\sqrt{\frac{1}{3}}\begin{pmatrix} \cos\theta \\ \sin\theta \\ -\sin\theta \\ \cos\theta \end{pmatrix}, \sqrt{\frac{1}{6}}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \\ -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix},$$

$$\sqrt{\frac{1}{9}}\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & \cos\theta_2 \\ \sin\theta_1 & -\cos\theta_1 & \sin\theta_2 \\ -\sin\theta_1 & \cos\theta_1 & \sin\theta_2 \\ \cos\theta_1 & \sin\theta_1 & -\cos\theta_2 \end{pmatrix},$$

$$\sqrt{\frac{1}{12}}\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_1 & -\cos\theta_1 & \sin\theta_2 & \cos\theta_2 \\ -\sin\theta_1 & \cos\theta_1 & \sin\theta_2 & \cos\theta_2 \\ \cos\theta_1 & \sin\theta_1 & -\cos\theta_2 & \sin\theta_2 \end{pmatrix}.$$

7. Concluding remarks. In this paper, we have established results about GRS's associated with a finite group. Their Gram matrices were characterized and simple conditions for one-erasure robustness were established. The relation between tightness and irreducibility of the representation was studied. The obtained results were used to construct Parseval, projective and one-erasure robust GRS's. An important case, not considered here, is the GRS's associated with an abelian finite group, and in particular, with a finite cyclic group. Also, symmetries of a RS can be further studied in line with [24]. We can also consider the more general case of a non finite Hilbert space and a non finite group. All these questions are currently under investigation.

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