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MAPS PRESERVING GENERAL MEANS OF POSITIVE OPERATORS*

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Abstract. Under some mild conditions, the general form of bijective transformations of the set of all positive linear operators on a Hilbert space which preserve a symmetric mean in the sense of Kubo-Ando theory is described.

Key words. Positive operators, Mean, Automorphisms, Preservers.

AMS subject classifications. 47B49, 47A64.

1. Introduction. Let H be a complex Hilbert space and denote by $B(H)$ the algebra of all bounded linear operators on H. An operator $A \in B(H)$ is said to be positive if $\langle Ax, x\rangle \geq 0$ holds for every vector $x \in H$. The cone of all positive operators on H is denoted by $B(H)^+$.

In our recent papers [5, 6], we have described the structure of all bijective maps ϕ on $B(H)^+$ (assuming $\dim H \geq 2$) which preserve either the geometric mean $\#$ or the harmonic mean ! of operators. For example, in the case of the geometric mean, this means that $\phi(A\#B) = \phi(A)\#\phi(B)$ holds for any $A, B \in B(H)^+$. We have proved that all those transformations are of the form $\phi(A) = TAT^*$, $A \in B(H)^+$ with an invertible bounded linear or conjugate-linear operator $T : H \to H$.

However, there is a general abstract theory of operators means due to Kubo and Ando [2] which has important applications in several areas, from operator theory to quantum information theory (cf. the theory of monotone metrics due to Petz [8]). In this paper, we extend our previous investigations and study the structure of all bijective transformations of $B(H)^+$ which preserve general operator means. Unfortunately, we do not have a result in the most general sense, i.e., relating to arbitrary means on $B(H)^+$ and arbitrary bijective transformations on $B(H)^+$ with the mean-preserving property. In fact, we have to pose a mild condition on the means

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ELA

(symmetry) and some mild regularity assumption (e.g. a sort of continuity) on the transformations. As for the main idea of the proof, it is based on the knowledge of the structure of the order automorphisms of $B(H)^+$ that was determined in our paper [3] (alternatively, see Section 2.5 in the book [4]). Some recent applications of that result to other problems can be found in the paper [7].

Now, let us summarize the basic notions and results of Kubo-Ando theory that we shall need in our investigations. In what follows, H denotes a complex Hilbert space with $\dim H \geq 2$. A binary operation σ on $B(H)^+$ is said to be a connection if the following requirements are fulfilled (from (i) to (iii) all operators are supposed to belong to $B(H)^+$).

(i) If $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$.
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(iii) If $A_n \downarrow A$ and $B_n \downarrow B$ strongly, then $A_n\sigma B_n \downarrow A\sigma B$ strongly.

If $I\sigma I = I$ holds too, then the connection σ is called a mean. A mean σ is called symmetric if $A\sigma B = B\sigma A$ holds for all $A, B \in B(H)^+$. Operations like convex combination and order among connections are defined in a natural way.

By the fundamental result Theorem 3.2 in [2], there is an affine order-isomorphism from the class of connections σ onto the class of operator monotone functions $f :]0,\infty[\to [0,\infty[$ given by the formula $f(t) = I\sigma tI$ $(t > 0)$. For invertible $A, B \in B(H)^+$, we have

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}. \tag{1.1}$$

By property (iii), we obtain that the formula (1.1) extends to any invertible $A \in B(H)^+$ and arbitrary $B \in B(H)^+$.

We shall need the following so-called transfer property of connections. By (ii), we easily deduce that $C(A\sigma B)C = (CAC)\sigma(CBC)$ holds for all invertible $C \in B(H)^+$. Now, using polar decomposition, the formula (1.1) and the continuity property (iii), it follows that we have

$$T(A\sigma B)T^* = (TAT^*)\sigma(TBT^*)$$

for all invertible bounded linear or conjugate-linear operator T on H. This trivially implies that $(tA)\sigma(tB) = t(A\sigma B)$ for any $t > 0$ and $A, B \in B(H)^+$.

We shall also use Theorem 3.3 in [2] which says that for every mean σ, we have $A\sigma A = A$, $A \in B(H)^+$.

Relating to operator monotone functions, it is well-known that they have a certain

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ELA

integral representation. Namely, by Lemma 3.1 in [2], the formula

$$f(s) = \int_{[0,\infty]} \frac{s(1+t)}{s+t} dm(t), \qquad s > 0 \tag{1.2}$$

provides an affine isomorphism from the class of all positive Radon measures m on $[0,\infty]$ onto the set of all operator monotone functions $f\ :]0,\infty[\to [0,\infty[$. Remark that in the above representation, we have $f(0) \doteq \lim_{s\to 0} f(s) = m(\{0\})$ and $\lim_{s\to\infty} f(s)/s = m(\{\infty\})$.

Using the formula (1.2), we then obtain an integral representation of any connection σ. In fact, Theorem 3.4 in [2] tells us that for each connection σ on $B(H)^+$, there exists a unique positive Radon measure on $[0,\infty]$ such that with $a = m(\{0\})$ and $b = m(\{\infty\})$ we have

$$A\sigma B = aA + bB + \int_{]0,\infty[} \frac{1+t}{t}\{(tA : B)\} dm(t) \tag{1.3}$$

for all $A, B \in B(H)^+$ (here and below : stands for the parallel sum of positive operators which is the half of the harmonic mean). Moreover, the correspondence $\sigma \mapsto m$ is an affine isomorphism from the class of all connections onto the class of all positive Radon measures on $[0,\infty]$.

Below whenever we write σ, f, m we always mean that σ is a connection, f is its representing operator monotone function and m is its representing positive Radon measure.

2. Results and proofs. In what follows, σ denotes a symmetric mean with the property that $I\sigma 0 = 0\sigma I = 0$. This means that in (1.3) we have $a = m(\{0\}) = f(0) = 0$. Taking into account that symmetry of σ means that $f(s) = sf(1/s)$ holds for all $s > 0$ (see Corollary 4.2. in [2]), we obtain that $b = m(\{\infty\}) = \lim_{s\to\infty} f(s)/s = \lim_{s\to 0} f(s) = f(0) = 0$. Consequently, the integral representation of σ is

$$A\sigma B = \int_{]0,\infty[} \frac{1+t}{t}\{(tA : B)\} dm(t), \qquad A, B \in B(H)^+. \tag{2.1}$$

There is still another fact that we shall need: for any operator $A \in B(H)^+$ we have $I\sigma A = f(A)$ (see (3.7) in [2]).

THEOREM 2.1. *Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map that satisfies*

$$\phi(A\sigma B) = \phi(A)\sigma\phi(B) \tag{2.2}$$

for all $A, B \in B(H)^+$. If there exists an invertible operator $A \in B(H)^+$ such that either ϕ is continuous on the set of scalar multiples tA, $t \geq 0$ of A, or it maps any

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scalar multiple tA, $t \geq 0$ of A to a scalar multiple of $\phi(A)$, then it follows that ϕ is of the form

$$\phi(A) = TAT^*, \qquad A \in B(H)^+$$

with some invertible bounded linear or conjugate-linear operator $T : H \to H$.

Observe that any transformation ϕ of the form $\phi(A) = TAT^*$, $A \in B(H)^+$ with an invertible bounded linear or conjugate-linear operator $T : H \to H$ satisfies (2.2) and also has both of the regularity properties appearing in the theorem above as assumptions.

For the proof of the theorem, we need some auxiliary results.

Lemma 2.2. *For any $A \in B(H)^+$, we have that A is a projection if and only if $I\sigma A = A$.*

Proof. First we point out that Lemma 5.1 in [2] tells us that for f we have $s < f(s) < 1$ for $0 < s < 1$ and $1 < f(s) < s$ for $1 < s < \infty$. Now, $f(A) = A$ holds if and only if $f(s) = s$ holds on the spectrum of A. This is equivalent to the spectrum of A is in $\{0, 1\}$, which means exactly that A is a projection. □

Theorem 3.7 in [2] tells us that for any projections P, Q on H we have $P\sigma Q = P \wedge Q$. It follows that $P \leq Q$ if and only if $P\sigma Q = P$.

Lemma 2.3. *We have that f is injective.*

Proof. In fact, assuming on the contrary that f is non-injective, it follows that f is constant on some closed interval either before or after the point 1. In the latter case, there are two more possibilities: the interval can be of finite or infinite length. Using Lemma 5.1 in [2], the concavity of $f(s)$ and the convexity of $sf(s)$ (Lemma 5.2 in [2]) elementary considerations show that we would arrive at contradictions in all cases. This gives us the injectivity of f. □

Lemma 2.4. *Suppose that f is unbounded. Then $A \in B(H)^+$ is invertible if and only if the equation $A\sigma X = Y$ has a solution $X \in B(H)^+$ for any given $Y \in B(H)^+$.*

Proof. Since f is injective and unbounded, we have $f^{-1} : [0, \infty[\to [0, \infty[$. Suppose that A is invertible. We have learnt from (1.1) that

$$A\sigma X = A^{1/2} f(A^{-1/2} X A^{-1/2}) A^{1/2} \tag{2.3}$$

holds for any $X \in B(H)^+$. It requires only easy computation to see that for a given Y, defining $X = A^{1/2} f^{-1}(A^{-1/2} Y A^{-1/2}) A^{1/2}$, gives a solution of the equation $A\sigma X = Y$. Assume now that A is not invertible. In that case, for any invertible $X \in B(H)^+$, we have

$$A\sigma X = X\sigma A = X^{1/2} f(X^{-1/2} A X^{-1/2}) X^{1/2}.$$

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ELA

Since $f(0) = 0$, by the spectral mapping theorem we deduce that the right-hand side of this equality and hence $A\sigma X$, too, are non-invertible. If $X \in B(H)^+$ is arbitrary, it follows from $A\sigma X \leq A\sigma(X + I)$ that $A\sigma X$ is also not invertible. □

LEMMA 2.5. *Suppose that f is bounded. The operator $A \in B(H)^+$ is invertible if and only if the set of all operators of the form*

$$(\dots((A\sigma T_1)\sigma T_2)\dots)\sigma T_n, \quad n \in \mathbb{N}, T_1, \dots, T_n \in B(H)^+$$

coincides with $B(H)^+$.

Proof. To prove the necessity, assume that A is invertible. By the transfer property it is easy to see that there is no serious loss of generality in assuming that $A = I$. Let $\lim_{s\to\infty} f(s) = r$. Clearly, we have $1 < r < \infty$. Pick an s with $1 < s$ and let $Y \in B(H)^+$. Pick $n \in \mathbb{N}$ such that $\|Y\|/f(s)^n < r$. Choosing $T_1 = sI$, $T_2 = f(s)sI$, $T_3 = f(s)^2 sI$, $\dots$, $T_n = f(s)^{n-1}sI$, $T_{n+1} = f(s)^n X$, we see that

$$(\dots((I\sigma T_1)\sigma T_2)\dots)\sigma T_{n+1} = f(s)^n f(X).$$

The equation $f(s)^n f(X) = Y$ clearly has solution $X = f^{-1}(Y/f(s)^n)$. As for the sufficiency, if A is non-invertible then we obtain that all $(\dots((A\sigma T_1)\sigma T_2)\dots)\sigma T_n$ are non-invertible. □

In what follows, we compute $A\sigma P$ for an arbitrary positive operator $A \in B(H)^+$ and rank-one projection P on H. To do so, we recall the notion of the strength of a positive operator along a ray represented by a unit vector. This concept was introduced by Busch and Gudder in [1]. Let $A \in B(H)^+$ be a positive operator, consider a unit vector φ in H and denote by P_φ the rank-one projection onto the subspace generated by φ. The quantity

$$\lambda(A, P_\varphi) = \sup\{\lambda \in \mathbb{R}_+ : \lambda P_\varphi \leq A\}$$

is called the strength of A along the ray represented by φ. According to [1, Theorem 4], we have the following formula for the strength:

$$\lambda(A, P_\varphi) = \begin{cases} \|A^{-1/2}\varphi\|^{-2}, & \text{if } \varphi \in \operatorname{rng}(A^{1/2}); \\ 0, & \text{else.} \end{cases} \tag{2.4}$$

(The symbol rng denotes the range of operators and $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range.)

LEMMA 2.6. *Let $A \in B(H)^+$ and P be a rank-one projection on H. We have $A\sigma P = P\sigma A = f(\lambda(A, P))P$.*

Proof. First assume that $\lambda(A, P) > 0$. By (2.1), we have

$$A\sigma P = \int_{]0,\infty[} \frac{1+t}{t}\{(tA : P)\}dm(t). \tag{2.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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The parallel sum : is known to be the half of the harmonic mean !. In Lemma 2, [6] we proved that for an arbitrary positive operator $T \in B(H)^+$ and rank-one projection P on H, we have

$$2(T : P) = T!P = \frac{2\lambda(T,P)}{\lambda(T,P)+1}P.$$

Therefore, denoting $s = \lambda(A,P)$, we can continue (2.5) as follows

$$A\sigma P = \int_{]0,\infty[} \frac{1+t}{t}\frac{\lambda(tA,P)}{\lambda(tA,P)+1}Pdm(t)$$

$$= \int_{]0,\infty[} \frac{1+t}{t}\frac{t\lambda(A,P)}{t\lambda(A,P)+1}Pdm(t) = \int_{]0,\infty[} \frac{1+t}{t}\frac{ts}{ts+1}dm(t)P$$

$$= s\int_{]0,\infty[} \frac{(1/s)(t+1)}{(1/s)+t}dm(t)P = sf(1/s)P = f(s)P = f(\lambda(A,P))P.$$

If $\lambda(A,P) = 0$, then in a similar fashion we see $A\sigma P = 0 = f(\lambda(A,P))P$. ◻

Lemma 2.7. *For any $A, B \in B(H)^+$, we have $A\sigma B \neq 0$ if and only if* $\operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2} \neq \{0\}$.

Proof. To see the sufficiency, by the formula (2.4) it follows from $\operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2} \neq \{0\}$ that there is a rank-one projection P on H and a number $t > 0$ such that $tP \leq A, B$. This implies $0 \neq tP = (tP)\sigma(tP) \leq A\sigma B$.

Conversely, assume $A\sigma B \neq 0$. By (2.1), we deduce that $(tA) : B \neq 0$ and hence that $(tA)!B \neq 0$ holds for some $t > 0$. It then follows that some positive scalar multiple of a rank-one projection P is less than or equal to $(tA)!B$. Since B is less than or equal to a scalar multiple of the identity, by the monotonicity property (i) of means this further implies that for some $s > 1$, we have $P \leq (sA)!(sI) = s(A!I)$. Therefore, $2P/(I+P) = P \leq 2sA/(I+A)$. The inverse function of $h : t \mapsto 2st/(1+t)$, $t \geq 0$ is $k : r \mapsto r/(2s-r)$, $0 \leq r < 2s$, which is easily seen to be operator monotone on the interval $[0, 2s[$. It follows that

$$k(2P/(I+P)) \leq k(2sA/(I+A)) = A.$$

One can check that the operator on the left hand side is $k(2P/(I+P)) = (1/(2s-1))P$. This gives us that a positive scalar multiple of P is less than or equal to A implying that the range of P is included in $\operatorname{rng} A^{1/2}$. We obtain in a similar fashion that $\operatorname{rng} P \subset \operatorname{rng} B^{1/2}$ holds, too. This completes the proof of the lemma. ◻

After these preliminaries, we are now in a position to present the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemmas 2.4 and 2.5, we see that our transformation ϕ preserves the invertible operators in both directions. This means that for any $A \in B(H)^+$, we have A is invertible if and only if $\phi(A)$ is invertible.

Electronic Journal of Linear Algebra ISSN 1081-3810
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It follows that $\phi(I)$ is invertible. By the transfer property, the transformation $\phi(I)^{-1/2}\phi(.)\phi(I)^{-1/2}$ is a bijective map on $B(H)^+$ which fulfils (2.2) and sends I to I. Therefore, we may and do assume that already our original map ϕ satisfies $\phi(I) = I$.

By the characterization of projections (Lemma 2.2) and the order among them (see the sentence before Lemma 2.3), it follows that ϕ preserves the projections in both directions as well as the order among them. Consequently, $\phi(0) = 0$.

Now, by Lemma 2.7, we see that for any $A, B \in B(H)^+$ we have $\operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2} \neq \{0\}$ if and only if $\operatorname{rng} \phi(A)^{1/2} \cap \operatorname{rng} \phi(B)^{1/2} \neq \{0\}$. One can easily verify that it implies

$$\operatorname{rng} A^{1/2} \subset \operatorname{rng} B^{1/2} \Longleftrightarrow \operatorname{rng} \phi(A)^{1/2} \subset \operatorname{rng} \phi(B)^{1/2}.$$

From this property, we infer that ϕ preserves the (finite) rank of the elements of $B(H)^+$.

Let P be a rank-one projection. For every $t > 0$, the operator $\phi(tP)$ is of rank one. Since $\phi(tP)\sigma\phi(P) = \phi((tP)\sigma P) \neq 0$, it follows that the range of $\phi(tP)$ has non-trivial intersection with the range of $\phi(P)$. This gives us that $\phi(tP)$ is a scalar multiple of $\phi(P)$. Next we deduce that there exists a bijective function $g_P : [0, \infty[\to [0, \infty[$ such that

$$\phi(tP) = g_P(t)\phi(P)$$

holds for every $t \geq 0$.

Assume now that ϕ is continuous on the scalar multiples of an invertible operator $A \in B(H)^+$. Considering the transformation

$$X \longmapsto \phi(A)^{-1/2}\phi(A^{1/2}XA^{1/2})\phi(A)^{-1/2},$$

we obtain a bijective map which satisfies (2.2), sends I to I and is continuous on the set of nonnegative scalar multiples of the identity. Hence, there is no real loss of generality in assuming that this particular operator A equals I. In what follows, we do use this assumption.

For any $t \geq 0$, we have

$$\phi(tI)\sigma\phi(P) = \phi((tI)\sigma P) = \phi(f(t)P) = g_P(f(t))\phi(P).$$

By the formula (2.3), for an arbitrary $B \in B(H)^+$, the transformation $C \mapsto C\sigma B$ is norm continuous on the set of all invertible elements of $B(H)^+$. We deduce that $g_P(f(t)) \to g_P(f(t_0))$ whenever $t_0 > 0$ and $t \to t_0$. Applying (iii), the same follows for $t_0 = 0$, too. Therefore, we obtain that g_p is continuous on the range of f. Since

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 864-874, September 2011

ELA

g_P is a bijection of the nonnegative reals and $g_P(0) = 0$, $g_P(1) = 1$ obviously hold, it follows that g_P is strictly increasing on the range of f.

We show that ϕ preserves the order of positive operators in both directions. We compute on one hand

$$\phi(A\sigma P) = \phi(f(\lambda(A,P))P) = g_P(f(\lambda(A,P)))\phi(P),$$

while on the other hand we have

$$\phi(A\sigma P) = \phi(A)\sigma\phi(P) = \lambda(\phi(A),\phi(P))\phi(P).$$

We obtain that $g_P(f(\lambda(A,P))) = \lambda(\phi(A),\phi(P))$ holds for any $A \in B(H)^+$ and rank-one projection P. Now, for given $A, B \in B(H)^+$ and arbitrary rank-one projection P on H, we have

$$\begin{aligned}\lambda(A,P) \le \lambda(B,P) &\Leftrightarrow g_P(f(\lambda(A,P))) \le g_P(f(\lambda(B,P))) \\ &\Leftrightarrow \lambda(\phi(A),\phi(P)) \le \lambda(\phi(B),\phi(P)).\end{aligned}$$

We learn from [1] that $A \le B$ holds if and only if $\lambda(A,P) \le \lambda(B,P)$ holds for every rank-one projection P on H. Therefore, it follows that we have $A \le B$ if and only if $\phi(A) \le \phi(B)$. This means that ϕ is an order automorphism of $B(H)^+$. The structure of such transformations was described in [3]. We proved there that every such map is implemented by an invertible bounded linear or conjugate-linear operator. Consequently, it follows that ϕ is of the form

$$\phi(A) = TAT^*, \qquad A \in B(H)^+$$

with an invertible bounded linear or conjugate-linear operator T on H, and this completes the proof of the theorem in the present case.

Suppose now that ϕ maps the scalar multiples of an invertible operator A into scalar multiples of $\phi(A)$. Considering the transformation

$$T \longmapsto \phi(A)^{-1/2}\phi(A^{1/2}TA^{1/2})\phi(A)^{-1/2}$$

just as above, we see that there is no loss of generality in assuming that this particular operator A equals the identity. In what follows, we assume that it is really the case. It means that there is an injective function $g : [0,\infty[\to [0,\infty[$ such that $\phi(tI) = g(t)I$, $t \ge 0$.

We compute

$$\phi(f(t)P) = \phi((tI)\sigma P) = (g(t)I)\sigma\phi(P) = f(g(t))\phi(P).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Now, for an arbitrary $B \in B(H)^+$, we compute $B\sigma P = f(\lambda(B,P))P$ implying $\phi(B\sigma P) = \phi(f(\lambda(B,P))P) = f(g(\lambda(B,P)))\phi(P)$. On the other hand, we have

$$\phi(B\sigma P) = \phi(B)\sigma\phi(P) = f(\lambda(\phi(B),\phi(P))).$$

Therefore, by the injectivity of f, it follows that

$$g(\lambda(B,P)) = \lambda(\phi(B),\phi(P)) \tag{2.6}$$

holds for all $B \in B(H)^+$ and rank-one projection P on H. Pick an invertible $B \in B(H)^+$ and consider the set of all $\lambda(B,P)$ where P runs through the connected set of all rank-one projections. By the formula (2.4), it is easy to see that this set is an interval. Taking into account the equality (2.6), it follows that the injective function $g : [0,\infty[\to [0,\infty[$ maps every interval $[\alpha,\beta]$ with $0 < \alpha < \beta < \infty$ onto an interval. It apparently yields that the restriction of g onto the open interval $]0,\infty[$ is strictly monotone. Assume for a moment that this function is strictly monotone decreasing. Let $A, B \in B(H)^+$ be invertible and P be an arbitrary rank-one projection on H. Then we infer

$$\begin{aligned}\lambda(A,P) \le \lambda(B,P) &\Leftrightarrow g(\lambda(B,P)) \le g(\lambda(A,P))\\ &\Leftrightarrow \lambda(\phi(B),\phi(P)) \le \lambda(\phi(A),\phi(P)).\end{aligned}$$

This means that ϕ, when restricted onto the set of invertible elements of $B(H)^+$, is an order reversing automorphism. Considering the transformation $A \mapsto \phi(A)^{-1}$ we obviously obtain an order automorphism of that set. By a result in [7], the structure of those transformations is just the same as that of the order automorphisms of the whole set $B(H)^+$. That is, they are implemented by invertible bounded linear or conjugate-linear operators on H. Since here we also have that the identity is sent to the identity, it follows easily that ϕ is of the form $\phi(A) = UA^{-1}U^*$ with a unitary or antiunitary operator U on H. It implies that the inverse operation satisfies (2.2), i.e., we have $(A\sigma B)^{-1} = A^{-1}\sigma B^{-1}$ for all invertible $A, B \in B(H)^+$. Putting $A = I$ and $B = sI$, this immediately gives us that $1/f(s) = f(1/s)$, $s > 0$. But the mean σ is symmetric implying that $sf(1/s) = f(s)$ holds for all $s > 0$. It follows trivially that we necessarily have $f(s) = \sqrt{s}$, $s > 0$, i.e., σ is the geometric mean. But the bijective maps on $B(H)^+$ preserving the geometric mean have been described in [5]. The result presented there shows that those maps just coincide with the order automorphisms of $B(H)^+$. But this contradicts the fact above that ϕ is an order reversing automorphism of the set of all invertible elements of $B(H)^+$. It yields that the case where the restriction of g onto the open interval $]0,\infty[$ is strictly monotone decreasing is untenable. It remains that g is strictly monotone increasing on $]0,\infty[$ and hence also on $[0,\infty[$. Using (2.6), we can see just as before that ϕ is an order automorphism of $B(H)^+$ and then complete the proof as in the first case. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
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We conclude the paper with a result describing the bijective transformations of $B(H)^+$ which preserve the norm of means of operators. The theorem below shows that every such map originates from an isometric linear or conjugate-linear *-algebra automorphism of $B(H)$.

THEOREM 2.8. *Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map with the property that*

$$\|\phi(A)\sigma\phi(B)\| = \|A\sigma B\|$$

holds for every $A, B \in B(H)^+$. Then there exists either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^*, \qquad A \in B(H)^+.$$

Proof. First observe that $\phi(0) = 0$. Indeed, it follows from

$$\|\phi(A)\| = \|\phi(A)\sigma\phi(A)\| = \|A\sigma A\| = \|A\|$$

meaning that ϕ is norm-preserving.

The assumptions in the theorem imply that for any $A, B \in B(H)^+$, we have $A\sigma B \neq 0$ if and only if $\phi(A)\sigma\phi(B) \neq 0$. Just as in the proof of Theorem 2.1, we infer that $\operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2} \neq \{0\}$ holds if and only if $\operatorname{rng}\phi(A)^{1/2} \cap \operatorname{rng}\phi(B)^{1/2} \neq \{0\}$ which then implies that

$$\operatorname{rng} A^{1/2} \subset \operatorname{rng} B^{1/2} \iff \operatorname{rng}\phi(A)^{1/2} \subset \operatorname{rng}\phi(B)^{1/2}.$$

We can proceed showing that ϕ necessarily preserves the rank-one operators in both directions. As ϕ preserves the norm, too, it follows that ϕ preserves the rank-one projections in both directions. For any rank-one projection P on H, we have

$$f(\lambda(\phi(A), \phi(P))) = \|\phi(A)\sigma\phi(P)\| = \|A\sigma P\| = f(\lambda(A, P))$$

implying $\lambda(\phi(A), \phi(P)) = \lambda(A, P)$. Just as in the proof of Theorem 2.1, this gives us that ϕ is an order automorphism of $B(H)^+$ and hence it is of the form

$$\phi(A) = TAT^*, \qquad A \in B(H)^+$$

with an invertible bounded linear or conjugate-linear operator T on H. Referring again to the property that ϕ preserves the norm, one can easily deduce that T is in fact either a unitary or an antiunitary operator. This completes the proof of the theorem. ∎

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