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IS THE BOUNDED RANK PERTURBATIONS PROBLEM FOR MATRIX PENCILS JUST A COMPLETION PROBLEM?*

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Abstract. In this paper, we study a direct link between the bounded rank perturbations problem and the completion problem for matrix pencils. We conjecture that the bounded rank perturbations problem is, in fact, equivalent to a completion problem. We prove the conjecture in three cases: when the rank bound is one, when the involved pencils are of full row rank, and when the rank bound equals the rank distance of the involved matrix pencils.

Key words. Matrix pencils, Bounded rank perturbations, Completion of matrix pencils.

AMS subject classifications. 15A22, 05A17, 15A83, 47A55.

1. Introduction. Let $\mathbb{F}$ be an algebraically closed field. Let $B(\lambda), B'(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be matrix pencils. We say that they are strictly equivalent, $B(\lambda) \sim B'(\lambda)$, if and only if there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$B'(\lambda) = PB(\lambda)Q.$$

The aim of this paper is to give a direct and explicit link between two fundamental Linear Algebra Problems for matrix pencils: *the bounded rank perturbations problem and the general completion problem.*

PROBLEM 1. *[The bounded rank perturbations problem for matrix pencils]*
Let $x \geq 0$, and $B(\lambda)$, and $C(\lambda)$ be matrix pencils of the same size. Find necessary and sufficient conditions for the existence of matrix pencils $B'(\lambda)$ and $C'(\lambda)$, strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq x.$$

Note that since strict equivalence operations preserve rank, Problem 1 is equivalent to finding necessary and sufficient conditions for the existence of a matrix pencil $B'(\lambda) \sim B(\lambda)$ such that $\operatorname{rank}(B'(\lambda)-C(\lambda)) \leq x$. Therefore, Problem 1 can be naturally rephrased as finding necessary and sufficient conditions for the existence of a matrix pencil $X(\lambda)$, with $\operatorname{rank} X(\lambda) \leq x$, such that $B(\lambda) \sim C(\lambda) + X(\lambda)$, i.e., determining when we can perturbe the pencil $C(\lambda)$ by a matrix pencil of bounded rank x, in order to obtain a matrix pencil strictly equivalent to $B(\lambda)$. The rank of a matrix pencil is the number of its linearly independent rows or columns.

PROBLEM 2. *[The General Matrix Pencil Completion Problem]*
Find necessary and sufficient conditions for the existence of a matrix pencil with a prescribed subpencil and strict equivalence class.

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Both Problems 1 and 2 are well studied and have many particular solutions and important applications. For most important references for Problem 1 see, e.g. [4–8, 19, 25, 27–29], and for Problem 2 see, e.g. [3, 10, 11, 17, 18, 23, 31–36].

In [2, 12, 14, 15, 22], it has been shown that Problem 1 is equivalent to the existence of a nonnegative integer $x_1 \leq x$, a matrix pencil $A(\lambda)$, and matrix pencils $X(\lambda)$, $X'(\lambda)$, $Y(\lambda)$, $Y'(\lambda)$, $Z(\lambda)$, and $Z'(\lambda)$, where $Z(\lambda), Z'(\lambda) \in \mathbb{F}[\lambda]^{x_1 \times (x - x_1)}$, such that

$$\left[\begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim B(\lambda) \quad \text{and} \quad \left[\begin{array}{c|c} A(\lambda) & X'(\lambda) \\ \hline Y'(\lambda) & Z'(\lambda) \end{array}\right] \sim C(\lambda). \tag{1.1}$$

The connection (1.1) shows that Problem 1 can be written as two simultaneous matrix pencil completion problems, with a common prescribed subpencil. This way of expressing Problem 1 led to its solution in several cases: when $x = 1$ [2, 12]; when one of the involved pencils is regular [16]; when both involved pencils are quasi-regular (i.e., have full row or column rank) [14]; and also when both involved pencils have no nontrivial homogeneous invariant factors [22]. However, applying (1.1) is very technical due to existence of two related completion problems, and so the potential of this approach for a general solution of Problem 1 is limited.

In this paper, we study a novel, more direct approach, consisting in expressing Problem 1 as a special case of Problem 2. This novel approach has been introduced in [15], where we have studied and solved Problem 1 in the limit case $x = |\operatorname{rank} B(\lambda) - \operatorname{rank} C(\lambda)|$. There we have discovered that in that critical case, there exists a single completion problem that is equivalent to the perturbations problem. Hence, if $x = |\operatorname{rank} B(\lambda) - \operatorname{rank} C(\lambda)|$, the perturbations problem is just a subcase of Problem 2. This was an unexpected discovery that we are following in this paper.

In fact, in this paper, we show that this was no accidental example and that there exists a deeper relation between two apparently different problems (Problems 1 and 2). There are three important results in this paper.

The first result is given in Theorem 3.4 in Section 3. It consists of finding a completion problem that is equivalent to Problem 1 in the case $x = 1$.

The second result is given in Theorem 4.7 in Section 4. It consists of finding a completion problem that is equivalent to Problem 1 in the case when both pencils $B(\lambda)$ and $C(\lambda)$ are quasi-regular.

Theorems 3.4 and 4.7 prove a remarkable direct link between Problems 1 and 2 in specific cases. We expect them to be milestones toward finding the completion problem that corresponds to Problem 1 in the general case.

In fact, in this paper, we give a conjecture of how to express Problem 1 as a case of Problem 2. This is the third result of the paper, and it is given as Conjecture 1 in Section 5. In Section 5, by using Theorems 3.4, 4.7 and 2.4, we prove Conjecture 1 in three cases: if $x = 1$; if both pencils $B(\lambda)$ and $C(\lambda)$ are quasi-regular; and if $x = |\operatorname{rank} B(\lambda) - \operatorname{rank} C(\lambda)|$.

Proving Conjecture 1 in full generality would show that Problem 1 is just a case of Problem 2, which is a highly nontrivial observation. Also, this would open new directions in solving the specific case of Problem 2 given in Conjecture 1, which in turn should lead to a complete solution to Problem 1. This paper gives a solid base for future results and opens a path in proving Conjecture 1, and consequently putting an end to Problem 1.

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2. Notation and auxiliary results. Throughout the paper, all polynomials will be homogeneous and monic. A polynomial $p \in \mathbb{F}[\lambda, \mu]$ is called homogeneous of degree k, if $p(c\lambda, c\mu) = c^k p(\lambda, \mu)$, for every $c \in \mathbb{F}$. We denote the degree of a homogeneous polynomial p by $d(p)$. A homogeneous polynomial $p \in \mathbb{F}[\lambda, \mu]$ is called monic if the coefficient of the monomial with the largest exponent of λ is equal to 1. For any chain of homogeneous polynomials $\alpha_1 \mid \cdots \mid \alpha_n$ by convention, we set $\alpha_i := 1$, for $i \leq 0$; and $\alpha_i := 0$, for $i \geq n+1$. For any polynomial p, we have $1 \mid p$ and $p \mid 0$.

By a partition we mean a nonincreasing sequence of integers. For a nonincreasing sequence of integers $a_1 \geq \cdots \geq a_s$, we denote by $\mathbf{a}$ the corresponding partition, i.e., $\mathbf{a} := (a_1, \dots, a_s)$. For a partition $\mathbf{a} = (a_1, \dots, a_s)$, we assume $a_0 = +\infty$ and $a_{s+1} = a_{s+2} = \cdots = -\infty$. Also, we put $\sum_{i=p}^{q} a_i = 0$ whenever $p > q$.

The complete set of strict equivalence invariants (also called the *Kronecker invariants*) of a matrix pencil consists of its invariant factors, infinite elementary divisors, and column and row minimal indices [24]. For more compact presentation of the conditions and involved combinatorics, in this paper, we consider invariant factors and infinite elementary divisors of a pencil unified as *homogeneous invariant factors*. The sum of the degrees of the homogeneous invariant factors and column and row minimal indices of a matrix pencil equals its rank. For all details on matrix pencils, their invariants and canonical forms for the strict equivalence relation see [24, Chapter XII].

Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils, with $\operatorname{rank} B(\lambda) = n + \rho$, and $\operatorname{rank} C(\lambda) = n$, and $\rho \geq 0$. Let us denote the Kronecker invariants of $C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ as follows:

$$\gamma_1 \mid \cdots \mid \gamma_n \quad - \quad \text{homogeneous invariant factors,} \tag{2.2}$$

$$d_1 \geq \cdots \geq d_m \quad - \quad \text{column minimal indices,} \tag{2.3}$$

$$\bar{r}_1 \geq \cdots \geq \bar{r}_p \quad - \quad \text{row minimal indices.} \tag{2.4}$$

Let us denote the Kronecker invariants of $B(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ as follows:

$$\beta_1 \mid \cdots \mid \beta_{n+\rho} \quad - \quad \text{homogeneous invariant factors,} \tag{2.5}$$

$$g_1 \geq \cdots \geq g_{m-\rho} \quad - \quad \text{column minimal indices,} \tag{2.6}$$

$$\tilde{r}_1 \geq \cdots \geq \tilde{r}_{p-\rho} \quad - \quad \text{row minimal indices.} \tag{2.7}$$

We note that

$$\sum_{i=1}^{n} d(\gamma_i) + \sum_{i=1}^{m} d_i + \sum_{i=1}^{m} \bar{r}_i + \rho = \sum_{i=1}^{n+\rho} d(\beta_i) + \sum_{i=1}^{m-\rho} g_i + \sum_{i=1}^{p-\rho} \tilde{r}_i = n + \rho. \tag{2.8}$$

Definition 2.1. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be an arbitrary matrix pencil with* $\operatorname{rank} A(\lambda) = n$. *By $\bar{A}(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m)\times(n+2m+p)}$ we shall denote a pencil whose column and row minimal indices are equal to the column and row minimal indices of $A(\lambda)$ increased by 1, respectively, while nontrivial homogeneous invariant factors of $A(\lambda)$ and $\bar{A}(\lambda)$ coincide.*

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Let us see, for $C(\lambda)$ as above, with Kronecker invariants as in (2.2)–(2.4), what the Kronecker invariants for $\bar{C}(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m)\times(n+2m+p)}$ are

$$\begin{array}{rcl} \underbrace{1|\cdots|1}_{m+p}|\gamma_1 \mid \cdots \mid \gamma_n & - & \text{homogeneous invariant factors} \\ d_1+1 \geq \cdots \geq d_m+1 & - & \text{column minimal indices,} \\ \bar{r}_1+1 \geq \cdots \geq \bar{r}_p+1 & - & \text{row minimal indices.} \end{array}$$

Let us denote the homogeneous invariant factors $\underbrace{1|\cdots|1}_{m+p}|\gamma_1 \mid \cdots \mid \gamma_n$ by

$$\bar{\gamma}_1|\cdots|\bar{\gamma}_{n+m+p},$$

i.e.,

$$\bar{\gamma}_i = \gamma_{i-m-p}, \quad i = 1,\ldots,n+m+p.$$

Also, let $c \in \mathbb{F}$ be such that $\lambda - c\mu \not| \gamma_n$. Let x be a nonnegative integer.

Then, the Kronecker invariants of the pencil

$$\begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+2p+m+x)\times(n+2m+p+x)}$$

are

$$\tilde{\gamma}_1|\cdots|\tilde{\gamma}_{n+m+p+x} \quad - \quad \text{homogeneous invariant factors} \tag{2.9}$$

$$d_1+1 \geq \cdots \geq d_m+1 \quad - \quad \text{column minimal indices,} \tag{2.10}$$

$$\bar{r}_1+1 \geq \cdots \geq \bar{r}_p+1 \quad - \quad \text{row minimal indices.} \tag{2.11}$$

Here

$$\begin{array}{ll} \tilde{\gamma}_i := \gamma_{i-m-p-x}, & i = 1,\ldots,n+m+p, \\ \tilde{\gamma}_i := (\lambda-c\mu)\gamma_{i-m-p-x}, & i = n+m+p+1,\ldots,n+m+p+x. \end{array}$$

The following Remark and Lemma are straightforward and will be used in the proofs of the main results.

Remark 2.2. *Let $\alpha, \beta \in \mathbb{F}[\lambda,\mu]$ be homogeneous polynomials, and let $c \in \mathbb{F}$ be such that $(\lambda - c\mu) \not| \alpha$ and $(\lambda - c\mu) \not| \beta$. Then*

$$\operatorname{lcm}((\lambda-c\mu)\alpha, \beta) = (\lambda-c\mu)\operatorname{lcm}(\alpha,\beta),$$

and so

$$d(\operatorname{lcm}((\lambda-c\mu)\alpha,\beta)) = 1 + d(\operatorname{lcm}(\alpha,\beta)).$$

Also

$$\gcd((\lambda-c\mu)\alpha,\beta) = \gcd(\alpha,\beta).$$

By the above definitions of the homogeneous invariant factors of the pencils $\bar{B}(\lambda)$ and $\begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix}$, in the case $\rho = 0$, we obtain the following result:

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LEMMA 2.3. *Let $\beta_1|\cdots|\beta_n$ and $\gamma_1|\cdots|\gamma_n$ be two chains of homogeneous polynomials. Let $\bar\beta_1|\cdots|\bar\beta_{n+m+p}$ be*

$$\bar\beta_i := \beta_{i-m-p}, \quad i=1,\dots,n+m+p.$$

Let $\tilde\gamma_1|\cdots|\tilde\gamma_{n+m+p+x}$ be

$$\tilde\gamma_i := \gamma_{i-m-p-x}, \quad i=1,\dots,n+m+p,$$
$$\tilde\gamma_i := (\lambda-c\mu)\gamma_{i-m-p-x}, \quad i=n+m+p+1,\dots,n+m+p+x.$$

Here $c\in\mathbb{F}$ is such that $(\lambda-c\mu)\not|\beta_n$.

Then

$$\tilde\gamma_i|\bar\beta_i|\tilde\gamma_{i+2x}, \quad i=1,\dots,n+m+p,$$

is equivalent to

$$\beta_i|\gamma_{i+x},\ i=1,\dots,n-x,\ \text{and}\quad \gamma_i|\beta_{i+x},\ i=1,\dots,n-x.$$

Proof. Recall that by convention $\beta_i=\gamma_i=0$ for $i\ge n+1$, and $\beta_i=\gamma_i=1$, for $i\le 0$. Thus

$$\tilde\gamma_i|\bar\beta_i, \quad i=1,\dots,n+m+p,$$

is equivalent to

$$\gamma_{i-m-p-x}|\beta_{i-m-p}, \quad i=1,\dots,n+m+p,$$

i.e., to

$$\gamma_i|\beta_{i+x}, \quad i=1,\dots n-x.$$

Analogously, since $(\lambda-c\mu)\not|\beta_n$, we have that

$$\bar\beta_i|\tilde\gamma_{i+2x}, \quad i=1,\dots,n+m+p,$$

is equivalent to

$$\beta_{i-m-p}|\gamma_{i+x-m-p}, \quad i=1,\dots,n+m+p,$$

i.e., to

$$\beta_i|\gamma_{i+x}, \quad i=1,\dots,n-x,$$

as desired. ∎

Now, let us cite the main result from [15]. For all details see this reference.

THEOREM 2.4. *[15, Corollary 5] Let $B(\lambda),C(\lambda)\in\mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be matrix pencils. Let $\operatorname{rank} C(\lambda)=n$ and $\operatorname{rank} B(\lambda)=n+\rho$, with $\rho\ge 0$. Let $c\in\mathbb{F}$ be such that it is not an eigenvalue neither of $C(\lambda)$ nor of $B(\lambda)$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ such that $B'(\lambda)\sim B(\lambda)$ and $C'(\lambda)\sim C(\lambda)$ and

$$\operatorname{rank}(B'(\lambda)-C'(\lambda))=\rho, \tag{2.12}$$

if and only if

there exist matrix pencils $X(\lambda)\in\mathbb{F}[\lambda]^{(n+p)\times\rho}$, $Y(\lambda)\in\mathbb{F}[\lambda]^{\rho\times(n+m)}$, and $Z(\lambda)\in\mathbb{F}[\lambda]^{\rho\times\rho}$ such that

$$\left[\begin{array}{c|c} C(\lambda) & X(\lambda)\\ \hline Y(\lambda) & Z(\lambda)\end{array}\right]\sim\begin{bmatrix} B(\lambda) & 0\\ 0 & (\lambda-c)I_\rho\end{bmatrix}.$$

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Remark 2.5. *We note that since*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \geq \operatorname{rank} B'(\lambda) - \operatorname{rank} C'(\lambda) = \rho,$$

condition (2.12) is equivalent to

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq \rho.$$

3. Rank one perturbations. In [2,12], a solution to Problem 1 in the case $x = 1$ has been obtained. By analyzing the obtained necessary and sufficient conditions, we have realized that they are resembling the necessary and sufficient conditions for the Problem 2 in the case of one row and one column completion obtained in [20]. This led us to carefully study these two problems, and we present the obtained direct link between them in the following theorem. Further on we shall use the notation from Section 2. Before showing the results, we note that

$$x \geq \operatorname{rank}(B'(\lambda) - C'(\lambda)) \geq \operatorname{rank} B'(\lambda) - \operatorname{rank} C'(\lambda) = \rho.$$

So, if $x = 1$, then

$$0 \leq \rho \leq 1,$$

i.e., $\rho \in \{0, 1\}$.

3.1. Auxiliary results. If $\rho = 1$, then $\rho = x = 1$, i.e., we are in the minimal limit case already studied in [15]. Hence, by applying Theorem 2.4, we obtain the wanted connection.

So, we are left with the case $\rho = 0$. This is a novel case, which has never been studied in this new light, and we shall need additional notation in order to give its solution.

Before stating the main result, we cite the main result from [12] in the case $\rho = 0$. This is a joint version of Theorems 3.3, 3.4 and 3.5 from [12].

Let $w = \max\{i \in \{1, \ldots, m\} | g_i \neq d_i\}$, if any such i exists, and let $w = 0$ if $g_i = d_i$, for all $i = 1, \ldots, m$.

Let $z = \max\{i \in \{1, \ldots, p\} | \tilde{r}_i \neq \bar{r}_i\}$, if any such i exists, and let $z = 0$ if $\tilde{r}_i = \bar{r}_i$, for all $i = 1, \ldots, p$.

Also, let us define integers h and h' in the following way:

$$h := \begin{cases} \max\{i \in \{1, \ldots, w\} | g_i < d_{i-1}\}, & \text{if } w > 0 \text{ and } g_w > d_w, \\ \max\{i \in \{1, \ldots, w\} | d_i < g_{i-1}\}, & \text{if } w > 0 \text{ and } d_w > g_w, \\ 0 & \text{if } w = 0. \end{cases}$$

$$h' := \begin{cases} \max\{i \in \{1, \ldots, z\} | \tilde{r}_i < \bar{r}_{i-1}\}, & \text{if } z > 0 \text{ and } \tilde{r}_z > \bar{r}_z, \\ \max\{i \in \{1, \ldots, z\} | \bar{r}_i < \tilde{r}_{i-1}\}, & \text{if } z > 0 \text{ and } \bar{r}_z > \tilde{r}_z, \\ 0 & \text{if } z = 0. \end{cases}$$

Theorem 3.1. *[12] Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with Kronecker invariants (2.5)–(2.7) and (2.2)–(2.4), respectively, with $\rho = 0$. There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1,$$

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if and only if the following conditions hold:

$$\beta_i|\gamma_{i+1} \quad \text{and} \quad \gamma_i|\beta_{i+1}, \quad i=1,\dots,n-1, \tag{3.13}$$

and

(a) *if there exists $l \in \{1,\dots,m\}$ such that $g_l \neq d_l$, then*

$$(a.1) \quad \tilde{r}_i = \bar{r}_i, \quad i=1,\dots,p,$$

$$(a.2) \quad \sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) + \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) \geq$$

$$\geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - 1,$$

(b) *if there exists $l \in \{1,\dots,p\}$ such that $\tilde{r}_l \neq \bar{r}_l$, then*

$$(b.1) \quad d_i = g_i, \quad i=1,\dots,m,$$

$$(b.2) \quad \sum_{i=1}^{p} \min(\tilde{r}_i, \bar{r}_i) + \max(\tilde{r}_{h'}, \bar{r}_{h'}) + \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) \geq$$

$$\geq \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - 1.$$

Next, by using the notation from Section 2, we shall cite the main result from [20] applied to the pencils $\bar{B}(\lambda)$ and $\operatorname{diag}(\bar{C}(\lambda), \lambda - c)$.

Theorem 3.2. *[20, Theorem 3.1] Let $\bar{B}(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m)\times(n+2m+p)}$ be a matrix pencils having $\bar{\beta}_1|\cdots\cdots|\bar{\beta}_{n+m+p}$ as homogeneous invariant factors, $g_1+1 \geq \cdots \geq g_m+1$ as column minimal indices and $\tilde{r}_1+1 \geq \cdots \geq \tilde{r}_p+1$ as row minimal indices. Let $\operatorname{diag}(\bar{C}(\lambda), \lambda - c) \in \mathbb{F}[\lambda]^{(n+2p+m+1)\times(n+2m+p+1)}$ be a matrix pencil having $\tilde{\gamma}_1|\cdots|\tilde{\gamma}_{n+m+p+1}$ as homogeneous invariant factors, $d_1+1 \geq \cdots \geq d_m+1$ as column minimal indices and $\bar{r}_1+1 \geq \cdots \geq \bar{r}_p+1$ as row minimal indices.*

There exist pencils $x(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m)\times 1}$, $y(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+2m+p)}$, and $z(\lambda) \in \mathbb{F}[\lambda]^{1\times 1}$, such that

$$\left[\begin{array}{c|c} \bar{B}(\lambda) & x(\lambda) \\ \hline y(\lambda) & z(\lambda) \end{array}\right] \sim \begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & \lambda - c \end{bmatrix}, \tag{3.14}$$

if and only if the following conditions are valid

$$(i) \quad \tilde{\gamma}_i|\bar{\beta}_i|\tilde{\gamma}_{i+2}, \quad i=1,\dots,n+m+p,$$

and one of the following sets of conditions is satisfied

$$(ii.1) \qquad g_i = d_i, \quad i=1,\dots,m, \qquad \text{and}$$

$$\sum_{i=1}^{p} \min(\tilde{r}_i+1, \bar{r}_i+1) + \max(\tilde{r}_{h'}+1, \bar{r}_{h'}+1) - \sum_{i=1}^{p}(\bar{r}_i+1) \geq$$

$$\geq \sum_{i=1}^{n+m+p+1} d(\tilde{\gamma}_i) - \sum_{i=1}^{n+m+p} d(\gcd(\bar{\beta}_i, \tilde{\gamma}_{i+1})) - 1,$$

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or

$$(ii.2) \qquad \tilde{r}_i = \bar{r}_i, \quad i = 1, \ldots, p, \quad \textit{and}$$
$$\sum_{i=1}^{m} \min(g_i+1, d_i+1) + \max(g_h+1, d_h+1) - \sum_{i=1}^{m}(d_i+1) \geq$$
$$\sum_{i=1}^{n+m+p+1} d(\tilde{\gamma}_i) - \sum_{i=1}^{n+m+p} d(\gcd(\bar{\beta}_i, \tilde{\gamma}_{i+1})) - 1.$$

Remark 3.3. *Note that if $g_i = d_i$ for all $i = 1, \ldots, m$, and $\tilde{r}_i = \bar{r}_i$ for all $i = 1, \ldots, p$, conditions $(ii.1)$ and $(ii.2)$ are both trivially satisfied. Indeed, in that case we have $w = 0$ and $z = 0$, and hence $h = 0$ and $h' = 0$, and by convention we have $g_0 = d_0 = \bar{r}_0 = \tilde{r}_0 = +\infty$.*

Now we have all the tools for giving our main result.

3.2. The first main result. In this subsection, we give a direct link between Problems 1 and 2 when $x = 1$ and $\rho = 0$.

Theorem 3.4. *Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be matrix pencils. Let $\operatorname{rank} C(\lambda) = \operatorname{rank} B(\lambda) = n$. Let $c \in \mathbb{F}$ be such that it is not an eigenvalue neither of $C(\lambda)$ nor of $B(\lambda)$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ such that $B'(\lambda) \sim B(\lambda)$ and $C'(\lambda) \sim C(\lambda)$ and

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1,$$

if and only if

there exist matrix pencils $x(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m)\times 1}$, $y(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+2m+p)}$, and $z(\lambda) \in \mathbb{F}[\lambda]^{1\times 1}$ such that

$$\left[\begin{array}{c|c} \bar{B}(\lambda) & x(\lambda) \\ \hline y(\lambda) & z(\lambda) \end{array}\right] \sim \left[\begin{array}{cc} \bar{C}(\lambda) & 0 \\ 0 & \lambda - c \end{array}\right]. \tag{3.15}$$

Proof. Rank-one perturbations problem in the case when $\rho = 0$ has been solved in Theorem 3.1. Also, the general matrix pencil completion problem in the case of one row and one column completion has been solved in [20]. When we apply the main result from [20] to completion (3.14), we get Theorem 3.2.

Next, we shall compare the obtained necessary and sufficient conditions from Theorems 3.1 and 3.2.

By Lemma 2.3 condition (i) from Theorem 3.2 is equivalent to

$$\beta_i | \gamma_{i+1} \quad \text{and} \quad \gamma_i | \beta_{i+1}, \quad i = 1, \ldots, n-1.$$

In the case $g_i = d_i$, $i = 1, \ldots, m$, and $\tilde{r}_i = \bar{r}_i$, $i = 1, \ldots, p$, by Remark 3.3 both conditions $(ii.1)$ and $(ii.2)$ from Theorem 3.2 are trivially satisfied.

Otherwise, if there exists $i \in \{1, \ldots, m\}$ such that $g_i \neq d_i$, i.e., if case (a) of Theorem 3.1 holds, then

$$\tilde{r}_i = \bar{r}_i, \quad i = 1, \ldots, p.$$

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So, we are left with comparing condition $(a.2)$ from Theorem 3.1 with condition $(ii.2)$ from Theorem 3.2. Since $c \in \mathbb{F}$ satisfies

$$(\lambda - c\mu) \not| \beta_n \quad \text{and} \quad (\lambda - c\mu) \not| \gamma_n,$$

by the definition of the homogeneous polynomial chains $\bar{\beta}_1 | \cdots | \bar{\beta}_{n+m+p}$ and $\tilde{\gamma}_1 | \cdots | \tilde{\gamma}_{n+m+p+1}$ we have that

$$\sum_{i=1}^{n+m+p+1} d(\tilde{\gamma}_i) = \sum_{i=1}^{n} d(\gamma_i) + 1,$$

and

$$\sum_{i=1}^{n+m+p} d(\gcd(\bar{\beta}_i, \tilde{\gamma}_{i+1})) = \sum_{i=1}^{n+m+p} d(\gcd(\beta_{i-m-p}, \gamma_{i-m-p})) = \sum_{i=1}^{n} d(\gcd(\beta_i, \gamma_i)).$$

Next, we shall prove that $d(\gcd(\beta_1, \gamma_1)) = 0$. Let us suppose on the contrary that $d(\gcd(\beta_1, \gamma_1)) > 0$. Then since (2.8) holds, we must have $d(\gcd(\beta_1, \gamma_1)) = 1$, and in this case, we must also have $\beta_1 = \cdots = \beta_n = \gamma_1 = \cdots = \gamma_n$, while $d_1 = \cdots = d_m = 0$, $g_1 = \cdots = g_m = 0$, $\bar{r}_1 = \cdots = \bar{r}_p = 0$, and $\tilde{r}_1 = \cdots = \tilde{r}_p = 0$. Hence, we would have $g_i = d_i$ for all $i = 1, \ldots, m$, which contradicts our assumption that there exists $l \in \{1, \ldots, m\}$ such that $g_l \neq d_l$.

Therefore, we have $d(\gcd(\beta_1, \gamma_1)) = 0$, and so

$$\sum_{i=1}^{n+m+p} d(\gcd(\bar{\beta}_i, \tilde{\gamma}_{i+1})) = \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)).$$

So, condition

$$\sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) \geq$$
$$\geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n+m+p+1} d(\tilde{\gamma}_i) - \sum_{i=1}^{n+m+p} d(\gcd(\bar{\beta}_i, \tilde{\gamma}_{i+1})) - 2,$$

coincides with

$$\sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) + \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - 1,$$

which proves the equivalence between the case (a) from Theorem 3.1 and the case $(ii.2)$ from Theorem 3.2. Completely analogously, we get that the case (b) from Theorem 3.1 is equivalent to the case $(ii.1)$ from Theorem 3.2.

Altogether, the necessary and sufficient conditions from Theorems 3.1 and 3.2 coincide. Hence, the two problems are equivalent. This ends our proof. ◻

4. Perturbations of quasi-regular matrix pencils. In this section, we present the second main result of the paper, that is a direct link between Problems 1 and 2 when both pencils $B(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ are quasi-regular. Without loss of generality, we shall assume that both pencils have full row rank. In the notation (2.5)–(2.7) and (2.2)–(2.4), this implies $\rho = 0$ (i.e., $\operatorname{rank} B(\lambda) = \operatorname{rank} C(\lambda)$). Note that having full row rank means that there are no row minimal indices, i.e., $p = 0$. In order to give a main result, we shall need to recall previous results given in the following subsection.

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4.1. Generalized majorization and auxiliary results. The concept of *generalized majorization* has appeared in [3] when considering column completions of rectangular matrices, and later on in many other completion problems [11, 21]:

Definition 4.1. *Let* $\mathbf{f} = (f_1, \ldots, f_{m+x})$, $\mathbf{d} = (d_1, \ldots, d_m)$, *and* $\mathbf{a} = (a_1, \ldots, a_x)$ *be partitions. If*

$$d_i \geq f_{i+x}, \qquad i = 1, \ldots, m, \tag{4.16}$$

$$\sum_{i=1}^{h_j} f_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \qquad j = 1, \ldots, x, \tag{4.17}$$

$$\sum_{i=1}^{m+x} f_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{x} a_i, \tag{4.18}$$

where

$$h_j := \min\{i \in \{1, \ldots, m+x\} | d_{i-j+1} < f_i\}, \quad j = 1, \ldots, x,$$

then we say that $\mathbf{f}$ *is majorized by the pair* $(\mathbf{d}, \mathbf{a})$. *This type of majorization we call generalized majorization, and we write*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{a}).$$

Notice that, if (4.18) is satisfied, then (4.17) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+x} f_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{x} a_i, \quad j = 1, \ldots, x. \tag{4.19}$$

By using the notation from Definition 4.1, we shall also give a lighter form of the generalized majorization called the *weak generalized majorization*:

Definition 4.2. *If*

$$d_i \geq f_{i+x}, \qquad i = 1, \ldots, m, \tag{4.20}$$

$$\sum_{i=h_j+1}^{m+x} f_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{x} a_i, \quad j = 0, \ldots, x, \tag{4.21}$$

where $h_0 := 0$, *then we say that* $\mathbf{f}$ *is weakly majorized by the pair* $(\mathbf{d}, \mathbf{a})$. *This type of majorization we call weak generalized majorization, and we write*

$$\mathbf{f} \prec'' (\mathbf{d}, \mathbf{a}).$$

Definition 4.3. *Let* $\mathbf{a} = (a_1, \ldots, a_x)$ *be a partition. Then one can define*

$$\mathbf{a} + \mathbf{1} = (a_1 + 1, \ldots, a_x + 1),$$

and

$$\mathbf{a} - \mathbf{1} = (a_1 - 1, \ldots, a_x - 1).$$

By the above definitions, it is straightforward to get the following:

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Lemma 4.4. *Let us suppose that partitions* $\mathbf{f}, \mathbf{d}, \mathbf{a}$ *are such that*

$$\mathbf{f} \prec'' (\mathbf{d}, \mathbf{a}).$$

Then

$$\mathbf{f}+\mathbf{1} \prec'' (\mathbf{d}+\mathbf{1}, \mathbf{a}+\mathbf{1}),$$

and

$$\mathbf{f}-\mathbf{1} \prec'' (\mathbf{d}-\mathbf{1}, \mathbf{a}-\mathbf{1}).$$

The so-called double majorization problem has been extensively studied (for more details see, e.g. [13]). It consists of the following: let $\mathbf{d} = (d_1, \ldots, d_m)$, $\mathbf{g} = (g_1, \ldots, g_n)$, $\mathbf{a} = (a_1, \ldots, a_x)$, and $\mathbf{b} = (b_1, \ldots, b_y)$, be partitions such that $m + x = n + y$. Determine whether there exists a partition $\mathbf{f} = (f_1, \ldots, f_{m+x})$, such that

$$\mathbf{f} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{f} \prec'' (\mathbf{g}, \mathbf{b}). \tag{4.22}$$

In [13, Theorem 11], the explicit necessary and sufficient conditions for the existence of such partition $\mathbf{f}$ have been obtained in the form of inequalities involving partitions $\mathbf{g}$, $\mathbf{d}$, $\mathbf{a}$, and $\mathbf{b}$. Such explicit form of the conditions has been denoted by

$$\overline{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{a}, \mathbf{b}). \tag{4.23}$$

The exact form of $\overline{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ is not used in this paper, for an interested reader, we suggest [13] for more details.

Now, we can cite the main result from [14].

Theorem 4.5. *[14, Theorem 4.1] Let* $B(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ *be a quasi-regular matrix pencil, having* $\beta_1|\cdots|\beta_n$ *as homogeneous invariant factors and* $g_1 \geq \cdots \geq g_m$ *as column minimal indices. Let* $C(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ *be a quasi-regular matrix pencil, having* $\gamma_1|\cdots|\gamma_n$ *as homogeneous invariant factors and* $d_1 \geq \cdots \geq d_m$ *as column minimal indices. Let* x *be a nonnegative integer, such that* $x \leq n$. *There exist matrix pencils* $B'(\lambda) \sim B(\lambda)$, *and* $C'(\lambda) \sim C(\lambda)$, *such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq x, \tag{4.24}$$

if and only if

$$\begin{aligned} &(i) \quad \gamma_i|\beta_{i+x}, \quad i = 1, \ldots, n-x, \quad \beta_i|\gamma_{i+x}, \quad i = 1, \ldots, n-x, \\ &(ii) \quad \overline{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{a}, \bar{\mathbf{a}}). \end{aligned}$$

Here, $\mathbf{a} = (a_1, \ldots, a_x)$ is a partition defined by

$$\sum_{i=j+1}^{x} a_i = \sum_{i=1}^{n-j} d(\gamma_i) - \sum_{i=1}^{n-j} d(\gcd(\beta_{i+j}, \gamma_i)) - (x-j), \quad j = 0, \ldots, x-1. \tag{4.25}$$

Also, $\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_x)$ is a partition defined by

$$\sum_{i=j+1}^{x} \bar{a}_i = \sum_{i=1}^{n-j} d(\beta_i) - \sum_{i=1}^{n-j} d(\gcd(\beta_i, \gamma_{i+j})) - (x-j), \quad j = 0, \ldots, x-1. \tag{4.26}$$

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Finally, we shall need the main result from [9] applied to the case when both of the involved matrix pencils are quasi-regular. By using the notation from Section 2, we cite [9, Theorem 3.1] for a completion of $\bar{B}(\lambda)$ up to $\begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix}$(for an interested reader we also suggest [9,23] for more general results on this topic).

THEOREM 4.6. *[9, Theorem 3.1] Let $B(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ be a quasi-regular matrix pencil, having $\beta_1|\cdots|\beta_n$ as homogeneous invariant factors and $g_1 \geq \cdots \geq g_m$ as column minimal indices. Let $C(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ be a quasi-regular matrix pencil, having $\gamma_1|\cdots|\gamma_n$ as homogeneous invariant factors and $d_1 \geq \cdots \geq d_m$ as column minimal indices. Let $x \geq 0$.*

There exist pencils $X(\lambda) \in \mathbb{F}^{(n+m)\times x}$, $Y(\lambda) \in \mathbb{F}^{x\times(n+2m)}$ and $Z(\lambda) \in \mathbb{F}^{x\times x}$ such that

$$\begin{bmatrix} \bar{B}(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \sim \begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix}, \tag{4.27}$$

if and only if

$$\begin{aligned} &(i) \quad \bar{\gamma}_i \mid \tilde{\beta}_i \mid \bar{\gamma}_{i+2x}, \quad i=1,\ldots,n+m, \\ &(ii) \quad \overline{\Omega}(\mathbf{g}+\mathbf{1}, \mathbf{d}+\mathbf{1}, \mathbf{w}, \bar{\mathbf{w}}). \end{aligned}$$

Here, $\bar{\beta}_1|\cdots|\bar{\beta}_{n+m}$ are the homogeneous invariant factors of $\bar{B}(\lambda)$ (see Definition 2.1), while $\tilde{\gamma}_1|\cdots|\tilde{\gamma}_{n+m+x}$ are the homogeneous invariant factors of the matrix pencil $\operatorname{diag}(\bar{C}(\lambda), (\lambda-c)I_x)$. Also

$$\mathbf{g}+\mathbf{1} = (g_1+1,\ldots,g_m+1), \quad \mathbf{d}+\mathbf{1} = (d_1+1,\ldots,d_m+1),$$

while $\mathbf{w} = (w_1,\ldots,w_x)$ is a partition given by

$$\sum_{i=j+1}^{x} w_i = \sum_{i=1}^{n+m+x-j} d(\operatorname{lcm}(\bar{\beta}_{i-x+j}, \tilde{\gamma}_i)) - \sum_{i=1}^{n+m} d(\bar{\beta}_i) - x + j, \quad j=0,\ldots,x-1,$$

and $\bar{\mathbf{w}} = (\bar{w}_1,\ldots,\bar{w}_x)$ is a partition given by

$$\sum_{i=j+1}^{x} \bar{w}_i = \sum_{i=1}^{n+m+x} d(\operatorname{lcm}(\bar{\beta}_{i-x-j}, \tilde{\gamma}_i)) - \sum_{i=1}^{n+m+x} d(\tilde{\gamma}_i), \quad j=0,\ldots,x-1.$$

4.2. The second main result. The second main result of the paper is finding a completion problem that is equivalent to Problem 1 in the case of full row rank matrix pencils. It is given in the following theorem:

THEOREM 4.7. *Let $B(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+m)}$ be two quasi-regular matrix pencils. Let $x \geq 0$. Let $c \in \mathbb{F}$ be such that it is not an eigenvalue neither of $C(\lambda)$ nor of $B(\lambda)$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq x, \tag{4.28}$$

if and only if there exist matrix pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{x\times(n+2m)}$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{x\times x}$ such that

$$\left[\begin{array}{c|c} \bar{B}(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim \begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix}. \tag{4.29}$$

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Proof. As in the proof of Theorem 3.4, we shall use the fact that both of the problems appearing in Theorem 4.7 have already been solved in Theorems 4.5 and 4.6, so we can compare the necessary and sufficient conditions from these two theorems. To that end, let us first compare the partitions $\mathbf{a}$ and $\mathbf{w}$ from Theorems 4.5 and 4.6, respectively. By definition of the homogeneous invariant factors of the pencils $\bar{B}(\lambda)$ and $\begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda - c)I_x \end{bmatrix}$, and since $c \in \mathbb{F}$ is such that $(\lambda - c\mu) \nmid \beta_n$ and $(\lambda - c\mu) \nmid \gamma_n$, we have that for every $j = 0, \ldots, x-1$:

$$\sum_{i=j+1}^{x} w_i = \sum_{i=1}^{n+m+x-j} d(\operatorname{lcm}(\bar{\beta}_{i-x+j}, \tilde{\gamma}_i)) - \sum_{i=1}^{n+m} d(\bar{\beta}_i) - x + j =$$

$$= \sum_{i=1}^{n+m+x-j} d(\tilde{\gamma}_i) - \sum_{i=1}^{n+m+x-j} d(\gcd(\bar{\beta}_{i-x+j}, \tilde{\gamma}_i)) - x + j =$$

$$= \sum_{i=1}^{n-j} d(\gamma_i) - \sum_{i=1}^{n+m+x-j} d(\gcd(\beta_{i-m-x+j}, \gamma_{i-m-x})) =$$

$$= \sum_{i=1}^{n-j} d(\gamma_i) - \sum_{i=1}^{n-j} d(\gcd(\beta_{i+j}, \gamma_i)) = \sum_{i=j+1}^{x} a_i + x - j = \sum_{i=j+1}^{x} (a_i + 1).$$

Hence, we conclude

$$\mathbf{w} = \mathbf{a} + \mathbf{1}. \tag{4.30}$$

Next, let us compare the partitions $\bar{\mathbf{a}}$ and $\bar{\mathbf{w}}$ from Theorems 4.5 and 4.6, respectively. As before, we have that for every $j = 0, \ldots, x-1$,

$$\sum_{i=j+1}^{x} \bar{w}_i = \sum_{i=1}^{n+m+x} d(\operatorname{lcm}(\bar{\beta}_{i-x-j}, \tilde{\gamma}_i)) - \sum_{i=1}^{n+m+x} d(\tilde{\gamma}_i) =$$

$$= \sum_{i=1}^{n+m-j} d(\bar{\beta}_i) - \sum_{i=1}^{n+m+x} d(\gcd(\bar{\beta}_{i-x-j}, \tilde{\gamma}_i)) =$$

$$= \sum_{i=1}^{n-j} d(\beta_i) - \sum_{i=1}^{n} d(\gcd(\beta_{i-j}, \gamma_i)) = \sum_{i=1}^{n-j} d(\beta_i) - \sum_{i=1}^{n-j} d(\gcd(\beta_i, \gamma_{i+j})) = \sum_{i=j+1}^{x} (\bar{a}_i + 1).$$

Hence, we conclude

$$\bar{\mathbf{w}} = \bar{\mathbf{a}} + \mathbf{1}. \tag{4.31}$$

Thus, condition (ii) in Theorem 4.6 becomes

$$\overline{\Omega}(\mathbf{g} + \mathbf{1}, \mathbf{d} + \mathbf{1}, \mathbf{a} + \mathbf{1}, \bar{\mathbf{a}} + \mathbf{1}). \tag{4.32}$$

Now, we can prove the equivalence. Let us start by assuming that there exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq x.$$

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By Theorem 4.5 this implies

$$\gamma_i|\beta_{i+x}, \quad i=1,\dots,n, \quad \beta_i|\gamma_{i+x}, \quad i=1,\dots,n-x, \tag{4.33}$$

$$\overline{\Omega}(\mathbf{g},\mathbf{d},\mathbf{a},\bar{\mathbf{a}}). \tag{4.34}$$

Condition (4.33) is equivalent to (i) from Theorem 4.6, by Lemma 2.3. Also, by notation we introduced in (4.22)–(4.23), condition (4.34) implies the existence of a partition $\mathbf{f}=(f_1,\dots,f_{m+x})$ such that

$$\mathbf{f} \prec'' (\mathbf{d},\mathbf{a}), \quad \text{and} \quad \mathbf{f} \prec'' (\mathbf{g},\bar{\mathbf{a}}).$$

Thus, by Lemma 4.4, we get

$$\mathbf{f}+\mathbf{1} \prec'' (\mathbf{d}+\mathbf{1},\mathbf{a}+\mathbf{1}), \quad \text{and} \quad \mathbf{f}+\mathbf{1} \prec'' (\mathbf{g}+\mathbf{1},\bar{\mathbf{a}}+\mathbf{1}),$$

and therefore we have (4.32).

Altogether, by applying Theorem 4.6, we get the existence of the wanted completion.

On the other hand, let us suppose that there exists a completion by x rows and x columns of a pencil $\bar{B}(\lambda)$ up to a pencil strictly equivalent to

$$\begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-c)I_x \end{bmatrix}.$$

By Theorem 4.6, and by (4.30) and (4.31), this implies condition (4.32) and

$$\tilde{\gamma}_i \mid \bar{\beta}_i \mid \tilde{\gamma}_{i+2x}, \quad i=1,\dots,n+m. \tag{4.35}$$

Again, by Lemma 2.3 condition (4.35) gives (4.33).

By condition (4.32), we get that there exists a partition $\mathbf{u}=(u_1,\dots,u_{m+x})$ such that

$$\mathbf{u} \prec'' (\mathbf{d}+\mathbf{1},\mathbf{a}+\mathbf{1}), \quad \text{and} \quad \mathbf{u} \prec'' (\mathbf{g}+\mathbf{1},\bar{\mathbf{a}}+\mathbf{1}). \tag{4.36}$$

Hence, by Lemma 4.4, we have

$$\mathbf{u}-\mathbf{1} \prec'' (\mathbf{d},\mathbf{a}), \quad \text{and} \quad \mathbf{u}-\mathbf{1} \prec'' (\mathbf{g},\bar{\mathbf{a}}), \tag{4.37}$$

i.e., we obtain

$$\overline{\Omega}(\mathbf{g},\mathbf{d},\mathbf{a},\bar{\mathbf{a}}).$$

Finally, by applying Theorem 4.5, we get the existence of matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda)-C'(\lambda)) \le x,$$

as wanted.

This proves the equivalence of the problems given in the theorem. $\square$

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5. A general conjecture on expressing Problem 1 as a case of Problem 2. As a result of work on Problems 1 and 2, and also by thorough analysis of the conditions in the cases solved till now, we bravely state the following conjecture. It is still far from being proved in the full generality. However, there are strong indications for it, as we manage to prove it for three unrelated cases: when $x=1$; when both pencils $B(\lambda)$ and $C(\lambda)$ are quasi-regular; and when $x = |\operatorname{rank} B(\lambda) - \operatorname{rank} C(\lambda)|$.

Conjecture 1. *Let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with $\operatorname{rank} B(\lambda) = n+\rho$ and $\operatorname{rank} C(\lambda) = n$, with $\rho \geq 0$. Let $x \geq 0$. Let $c, d \in \mathbb{F}$, with $c \neq d$, be such that they are not eigenvalues neither of $C(\lambda)$ nor of $B(\lambda)$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq x, \tag{5.38}$$

if and only if

there exist matrix pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m-\rho)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{x\times(n+2m+p-\rho)}$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{x\times x}$ such that

$$\left[\begin{array}{cc|c} \bar{B}(\lambda) & 0 & \multirow{2}{*}{$X(\lambda)$} \\ 0 & (\lambda-c)I_\rho & \\ \hline \multicolumn{2}{c|}{Y(\lambda)} & Z(\lambda) \end{array}\right] \sim \begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-d)I_{x-\rho} \end{bmatrix}. \tag{5.39}$$

5.1. Evidence for Conjecture 1. Below we shall give some evidence for Conjecture 1. In the following subsection, we shall give a proof of Conjecture 1 in three different cases covered by Theorems 2.4, 3.4 and 4.7. Before proceeding, we give some preliminary observations concerning Conjecture 1, and the completion (5.39).

First, we note that the sizes of the pencils in (5.39) match. Indeed, since $B(\lambda), C(\lambda) \in F[\lambda]^{(n+p)\times(n+m)}$, with $\operatorname{rank} B(\lambda) = n+\rho$ and $\operatorname{rank} C(\lambda) = n$, we have $\bar{B}(\lambda) \in F[\lambda]^{(n+2p+m-2\rho)\times(n+2m+p-2\rho)}$, with $\operatorname{rank} \bar{B}(\lambda) = n+m+p-\rho$, and $\bar{C}(\lambda) \in F[\lambda]^{(n+2p+m)\times(n+2m+p)}$, with $\operatorname{rank} \bar{C}(\lambda) = n+m+p$. Therefore, the pencils on the left-hand side and on the right-hand side of (5.39) have the same size $(n+2p+m+x-\rho)\times(n+2m+p+x-\rho)$.

As we have explained in the Introduction, the bounded rank perturbation (5.38) can always be expressed as two completion problems with a common pencil (1.1). From that we can obtain some necessary conditions that need to be satisfied. One of them is that the homogeneous invariant factors of the pencils $B(\lambda)$ and $C(\lambda)$ need to satisfy the following

$$\beta_i | \gamma_{i+x}, \quad i = 1, \ldots, n-x; \quad \gamma_i | \beta_{i+x}, \quad i = 1, \ldots, n+\rho-x. \tag{5.40}$$

On the other hand, the matrix pencil completion (5.39) implies divisibility relation between the homogeneous invariant factors of the pencils

$$\begin{bmatrix} \bar{B}(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{C}(\lambda) & 0 \\ 0 & (\lambda-d)I_{x-\rho} \end{bmatrix}. \tag{5.41}$$

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If we denote the homogeneous invariant factors of the two pencils in (5.41) by $\tilde\beta_1|\cdots|\tilde\beta_{n+m+p}$ and $\tilde\gamma_1|\cdots|\tilde\gamma_{n+m+p+x-\rho}$, respectively, then by Sá-Thompson theorem [34, 35], the completion (5.39) gives the following interlacing divisibility relations

$$\tilde\gamma_i|\tilde\beta_i,\quad i=1,\dots,n+m+p,\quad \tilde\beta_i|\tilde\gamma_{i+2x},\quad i=1,\dots,n+m+p-\rho-x. \tag{5.42}$$

Since

$$\begin{aligned}\tilde\gamma_i &= \gamma_{i-m-p-x+\rho}, && i=1,\dots,n+m+p,\\ \tilde\gamma_i &= (\lambda-d\mu)\gamma_{i-m-p-x+\rho}, && i=n+m+p+1,\dots,n+m+p+x-\rho,\end{aligned}$$

and

$$\begin{aligned}\tilde\beta_i &= \beta_{i-m-p+\rho}, && i=1,\dots,n+m+p-\rho,\\ \tilde\beta_i &= (\lambda-c\mu)\beta_{i-m-p+\rho}, && i=n+m+p-\rho+1,\dots,n+m+p,\end{aligned}$$

and since c and d are not eigenvalues of $B(\lambda)$ and $C(\lambda)$, i.e., since $(\lambda-c\mu)\nmid\gamma_n$ and $(\lambda-d\mu)\nmid\beta_{n+\rho}$, we have that (5.42) is equivalent to (5.40). Therefore, the interlacing divisibilities between chains of homogeneous invariant factors of $B(\lambda)$ and $C(\lambda)$ implied by the bounded rank perturbation (5.38) are the same as the ones implied by the completion problem (5.39).

5.1.1. A proof of Conjecture 1 in the quasi-regular case. If both $B(\lambda)$ and $C(\lambda)$ are quasi-regular, we have that $\rho=0$, and so the completion (5.39) becomes

$$\left[\begin{array}{c|c}\bar B(\lambda) & *\\ \hline * & *\end{array}\right]\sim\left[\begin{array}{cc}\bar C(\lambda) & 0\\ 0 & (\lambda-d)I_x\end{array}\right].$$

Hence, Theorem 4.7 proves Conjecture 1 in this case. □

5.1.2. A proof of Conjecture 1 in the limit case $x=\rho$. In this subsection, we shall prove Conjecture 1 in the case when $x=\rho$. In this case, the completion (5.39) from Conjecture 1 becomes

$$\left[\begin{array}{cc|c}\bar B(\lambda) & 0 & \\ 0 & (\lambda-c)I_\rho & *\\ \hline \multicolumn{2}{c|}{*} & *\end{array}\right]\sim\bar C(\lambda). \tag{5.43}$$

On the other hand, by Theorem 2.4, the perturbations problem (5.38) in the case $x=\rho$ is equivalent to the following completion problem

$$\left[\begin{array}{c|c}C(\lambda) & *\\ \hline * & *\end{array}\right]\sim\left[\begin{array}{cc}B(\lambda) & 0\\ 0 & (\lambda-c)I_\rho\end{array}\right]. \tag{5.44}$$

However, the completion problems (5.43) and (5.44) are equivalent, as we prove below in Lemma 5.1. This shows that Conjecture 1 holds if $x=\rho$. □

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LEMMA 5.1. *Let* $B(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ *and* $C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ *be two matrix pencils with* $\operatorname{rank} B(\lambda) = n+\rho$ *and* $\operatorname{rank} C(\lambda) = n$, *with* $\rho \geq 0$. *Let* $c \in \mathbb{F}$ *be such that it is not an eigenvalue neither of* $C(\lambda)$ *nor of* $B(\lambda)$.

There exist matrix pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times\rho}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{\rho\times(n+m)}$, *and* $Z(\lambda) \in \mathbb{F}[\lambda]^{\rho\times\rho}$ *such that*

$$\left[\begin{array}{c|c} C(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim \begin{bmatrix} B(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix}, \tag{5.45}$$

if and only if

there exist matrix pencils $X'(\lambda) \in \mathbb{F}[\lambda]^{(n+2p+m-\rho)\times\rho}$, $Y'(\lambda) \in \mathbb{F}[\lambda]^{\rho\times(n+2m+p-\rho)}$, *and* $Z'(\lambda) \in \mathbb{F}[\lambda]^{\rho\times\rho}$ *such that*

$$\left[\begin{array}{cc|c} \bar{B}(\lambda) & 0 & \\ 0 & (\lambda-c)I_\rho & X'(\lambda) \\ \hline \multicolumn{2}{c|}{Y'(\lambda)} & Z'(\lambda) \end{array}\right] \sim \bar{C}(\lambda). \tag{5.46}$$

Proof. Let us denote the homogeneous invariant factors of $\begin{bmatrix} B(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix}$ by $\hat{\beta}_1|\cdots|\hat{\beta}_{n+2\rho}$.

Let us denote the homogeneous invariant factors of $\begin{bmatrix} \bar{B}(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix}$ by $\tilde{\beta}_1|\cdots|\tilde{\beta}_{n+m+p}$.

And, let us denote by $\bar{\gamma}_1|\cdots|\bar{\gamma}_{n+m+p}$ the homogeneous invariant factors of $\bar{C}(\lambda)$.

In the rest of the proof, we shall use notation and results from Section 2. In addition, let $h_0 = 0$, $h_j = \min\{i | g_{i-j+1} < d_i\}$, $j = 1, \ldots, \rho$, $v_0 = 0$, $v_k = \min\{i | \tilde{r}_{i-k+1} < \bar{r}_i\}$, $k = 1, \ldots, \rho$.

Since $\operatorname{rank} C(\lambda) = n$, $\operatorname{rank} \begin{bmatrix} B(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix} = n+2\rho$, and the completion (5.45) is done by adding ρ rows and ρ columns, it is an example of the so-called minimal completion of type I. These kind of completions are well studied and covered by [18, Theorem 4.1]. Hence, by applying [18, Theorem 4.1], we get the following necessary and sufficient conditions for the existence of completion (5.45):

$$\hat{\beta}_i | \gamma_i | \hat{\beta}_{i+2\rho}, \quad i = 1, \ldots, n, \tag{5.47}$$

$$\tilde{r}_i \geq \bar{r}_{i+\rho}, \quad i = 1, \ldots, p-\rho, \tag{5.48}$$

$$g_i \geq d_{i+\rho}, \quad i = 1, \ldots, m-\rho, \tag{5.49}$$

$$\begin{aligned} &\sum_{i=1}^{h_j}(d_i+1) - \sum_{i=1}^{h_j-j}(g_i+1) + \sum_{i=1}^{v_k}(\bar{r}_i+1) - \sum_{i=1}^{v_k-k}(\tilde{r}_i+1) \leq \\ &\leq \sum_{i=1}^{n+2\rho} d(\hat{\beta}_i) - \sum_{i=1}^{n+2\rho-k-j} d(\operatorname{lcm}(\gamma_{i-2\rho+j+k}, \hat{\beta}_i)), \quad j,k = 0, \ldots, \rho. \end{aligned} \tag{5.50}$$

Also, since $\operatorname{rank} \begin{bmatrix} \bar{B}(\lambda) & 0 \\ 0 & (\lambda-c)I_\rho \end{bmatrix} = n+p+m$ and $\operatorname{rank} \bar{C}(\lambda) = n+m+p$, the completion (5.46) is an example of the so-called minimal completion of type II. These kind of completions are well studied and

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covered by [18, Theorem 4.2]. Hence, by applying [18, Theorem 4.2] (see also Remarks 2 and 3 in [15]), we get the necessary and sufficient conditions for the existence of completion (5.46):

$$\bar{\gamma}_i|\tilde{\beta}_i|\bar{\gamma}_{i+2\rho}, \quad i=1,\dots,n+m+p, \tag{5.51}$$

$$\sum_{i=1}^{h_j}(d_i+1)-\sum_{i=1}^{h_j-j}(g_i+1)+\sum_{i=1}^{v_k}(\bar{r}_i+1)-\sum_{i=1}^{v_k-k}(\tilde{r}_i+1)\leq \tag{5.52}$$
$$\leq \sum_{i=1}^{n+m+p} d(\tilde{\beta}_i)-\sum_{i=1-2\rho}^{n+m+p-k-j} d(\operatorname{lcm}(\bar{\gamma}_{i+j+k},\tilde{\beta}_i)), \quad j,k=0,\dots,\rho,$$

together with (5.48) and (5.49).

By the definitions of the homogeneous polynomials $\hat{\beta}_i$, for $i=1,\dots,n+2\rho$, $\bar{\gamma}_i$, for $i=1,\dots,n+m+p$, and $\tilde{\beta}_i$, for $i=1,\dots,n+m+p$ (see Section 2), we get the following

$$\sum_{i=1}^{n+m+p} d(\bar{\gamma}_i)=\sum_{i=1}^{n} d(\gamma_i),$$
$$\sum_{i=1}^{n+2\rho} d(\hat{\beta}_i)=\sum_{i=1}^{n+m+p} d(\tilde{\beta}_i)=\rho+\sum_{i=1}^{n+\rho} d(\beta_i)$$
$$\sum_{i=1}^{n+2\rho-k-j} d(\operatorname{lcm}(\gamma_{i-2\rho+j+k},\hat{\beta}_i))=\sum_{i=1}^{n+2\rho-k-j} d(\operatorname{lcm}(\gamma_{i-2\rho+j+k},\beta_{i-\rho}))+\max(0,\rho-j-k),$$

for all $j,k=0,\dots,\rho$, as well as

$$\sum_{i=1-2\rho}^{n+m+p-k-j} d(\operatorname{lcm}(\bar{\gamma}_{i+j+k},\tilde{\beta}_i))=\sum_{i=1}^{n+2\rho-k-j} d(\operatorname{lcm}(\gamma_{i-2\rho+j+k},\beta_{i-\rho}))+\max(0,\rho-j-k),$$

for all $j,k=0,\dots,\rho$.

So, we have that (5.50) coincide with (5.52).

Finally, analogously as in Lemma 2.3 and as in Section 5.1, by the definition of the involved polynomial chains, we conclude that both conditions (5.47) and (5.51) are equivalent to

$$\gamma_i|\beta_{i+\rho}, \quad i=1,\dots,n, \quad \text{and} \quad \beta_i|\gamma_{i+\rho}, \quad i=1,\dots,n-\rho.$$

So, in this way, we have proved the equivalence between the completion problems (5.45) and (5.46), as desired. ∎

5.1.3. A proof of Conjecture 1 in the case of rank one perturbations. As we have seen in Section 3 if $x=1$, the value of $\rho\in\{0,1\}$.

If $\rho=0$, we have that the completion problems (3.15) and (5.39) coincide. Hence, by applying Theorem 3.4, we prove Conjecture 1 in this case.

If $\rho=1$, we are in situation already discussed in Section 5.1.2. Hence, by applying Theorem 2.4, we prove Conjecture 1 in this case. ∎

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6. Conclusion. In this paper, we directly relate two fundamental linear algebra problems: the bounded rank perturbations problem for matrix pencils and the general matrix pencil completion problem. In Conjecture 1 we give, to our opinion, a perfect candidate for the completion problem which is to be equivalent to the bounded rank perturbations problem.

Also, we show evidences for such a claim. In particular, in Theorem 3.4, we give a completion problem which is equivalent to Problem 1 in the case $x = 1$. Next, in Theorem 4.7, we give a completion problem which is equivalent to Problem 1 in the case when both pencils $B(\lambda)$ and $C(\lambda)$ are of full row rank.

By using these results, together with Theorem 2.4, in Section 5, we prove Conjecture 1 in three separate cases: if $x = 1$; if both pencils $B(\lambda)$ and $C(\lambda)$ are quasi-regular; and if $x = |\operatorname{rank} B(\lambda) - \operatorname{rank} C(\lambda)|$.

This paper opens various question for the future work. The completion problem (5.39) appearing in the conjecture becomes very important, and it presents one of the open problems for the future study.

We expect the results from this paper to be milestones toward proving Conjecture 1, and consequently solving Problem 1.

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