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THE MINIMUM ALGEBRAIC CONNECTIVITY OF CATERPILLAR UNICYCLIC GRAPHS*

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Abstract. A caterpillar unicyclic graph is a unicyclic graph in which the removal of all pendant vertices makes it a cycle. In this paper, the unique caterpillar unicyclic graph with minimum algebraic connectivity among all caterpillar unicyclic graphs is determined.

Key words. Algebraic connectivity, Caterpillar unicyclic graph, Characteristic polynomial.

AMS subject classifications. 05C50.

1. Introduction. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let $d(v_i)$ be the degree of the vertex $v_i \in V(G)$ $(i = 1, 2, \ldots, n)$, and $D = D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* $L(G) = D(G) - A(G)$ is the difference between $D(G)$ and the adjacency matrix $A(G)$. It is easy to see that $L(G)$ is a positive semidefinite symmetric matrix with the smallest eigenvalue 0 and the corresponding eigenvector is the all ones column vector, which is denoted by e. Denote its eigenvalues by

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. Fiedler [4] showed that the second smallest eigenvalue of $L(G)$ is 0 if and only if G is disconnected. Thus, the second smallest eigenvalue of $L(G)$ is popularly known as the *algebraic connectivity* of G and is usually denoted by $\alpha(G)$. Let P_n and C_n be the path and the cycle on n vertices, respectively. It is a known fact that

$$\alpha(P_n) = 4\sin^2 \tfrac{\pi}{2n} \text{ and } \alpha(C_n) = 4\sin^2 \tfrac{\pi}{n}.$$

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Let $Y \in \mathbb{R}^n$ be a column vector. It will be convenient to assume that the entry y_i of Y is corresponding to the vertex v_i of G. Such a Y is sometimes called a *valuation* of the graph G (see, [11]). In the following, y_i will be written as $Y(v_i)$ for convenient. If X is a unit eigenvector of G corresponding to $\alpha(G)$, we commonly call it a *Fiedler vector* of G. It is obvious that $X^T e = 0$ and

$$\alpha(G) = X^T L(G) X = \sum_{v_i v_j \in E} (X(v_i) - X(v_j))^2 = \min_{\substack{Y \in \mathbb{R}^n \setminus \{0\} \\ Y^T e = 0}} \frac{Y^T L(G) Y}{Y^T Y}.$$

Furthermore, from $L(G)X = \alpha(G)X$, we also have the set of equations below, known as eigenvalue equations of G:

$$(d(v) - \alpha(G))X(v) = \sum_{u \in N(v)} X(u) \quad \text{for } v \in V(G),$$

where $N_G(v)$ (or $N(v)$ for short) denotes the set of vertices which are adjacent to v in G.

A *caterpillar unicyclic graph* is a unicyclic graph in which the removal of all pendant vertices makes it a cycle. Let $C_g = v_1 v_2 \cdots v_g v_1$ be a cycle with length g, where $v_i v_{i+1} \in E(C_g)$ for $i = 1, 2, \ldots, g-1$ and $v_g v_1 \in E(C_g)$, and let $C^{n_1, n_2, \ldots, n_k}_{g; i_1, i_2, \ldots, i_k}$ be the caterpillar unicyclic graph obtained from C_g by attaching n_j pendant edges at v_{i_j} $(1 \le i_1 < \cdots < i_k \le g)$, respectively. By symmetry, we may always assume that $i_1 = 1$. For example, $C^{n_1, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1}$ and $C^{n_1, n_2, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1, \lfloor \frac{g}{2} \rfloor + 2}$ (see Fig. 1.1) are two caterpillar unicyclic graphs which will be used in the next section. If $C^{n_1, n_2, \ldots, n_k}_{g; i_1, i_2, \ldots, i_k}$ has n vertices, then it is easy to see that $n_1 + n_2 + \cdots + n_k = n - g$.

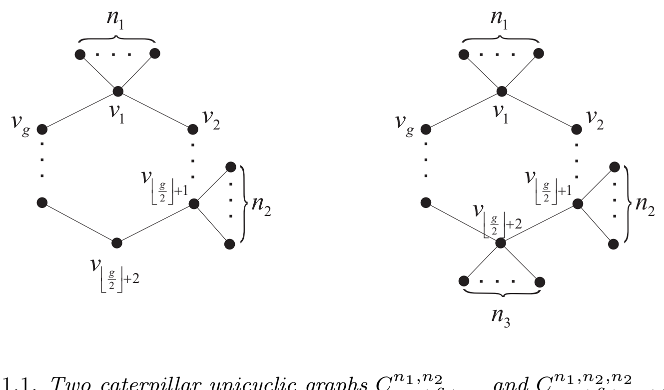

FIG. 1.1. *Two caterpillar unicyclic graphs* $C^{n_1, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1}$ *and* $C^{n_1, n_2, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1, \lfloor \frac{g}{2} \rfloor + 2}$.

In [1], Fallat and Kirkland proved that for some choice of the parameters n_1, n_2, $\ldots$, n_g, the graph $C^{n_1, n_2, \ldots, n_g}_{g; 1, 2, \ldots, g}$ maximizes the algebraic connectivity over the

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class of unicyclic graphs with girth g. In particular, they proved that among all unicyclic graphs on n vertices with girth 3, the graph $C_{3;1}^{n-3}$ has the maximum algebraic connectivity. In [2], Fallat, Kirkland and Pati proved that the graph $C_{4;1}^{n-4}$ has the maximum algebraic connectivity among all unicyclic graphs on n vertices with girth 4. Furthermore, they proved that there is an N such that for each $n > N$, the graph $C_{g;1}^{n-g}$ has the maximum algebraic connectivity among all unicyclic graphs on n vertices with girth g. On the other hand, when g is large relative to n, they showed that this graph does not maximize the algebraic connectivity. For the minimum algebraic connectivity, Guo [7] proved that the graph $C_{n,g}$ has the minimum algebraic connectivity among all connected graphs with girth g, where $C_{n,g}$ is called the lollipop graph, which is obtained by appending a g-cycle C_g to a pendant vertex of a path on $n-g$ vertices. This confirmes the conjecture proposed by Fallat and Kirkland (see [1], [3]).

In this paper, we prove that the graph $C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}$ has the minimum algebraic connectivity among all caterpillar unicyclic graphs on n vertices with girth g.

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B;x) = \det(xI - B)$ the characteristic polynomial of the square matrix B. In particular, if $B = L(G)$, we write $\Phi(L(G))$ by $\Phi(G;x)$ or simply by $\Phi(G)$ and call $\Phi(G)$ the *Laplacian characteristic polynomial* of G.

2. Lemmas and results. Let G be a graph and let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G. The following lemma follows from Courant-Weyl inequalities (see [9]).

Lemma 2.1. *The Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \mu_2(G) \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0.$$

By Lemma 2.1, we immediately have the following:

Corollary 2.2. *Let G be a connected graph and v be a pendant vertex of G. Then $\alpha(G) \leq \alpha(G - v)$.*

The following inequalities are known as Cauchy's inequalities and the whole theorem is also known as interlacing theorem [9].

Lemma 2.3. *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B be a principal sub-matrix of A. Let B has eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ $(m \leq n)$. Then the inequalities $\lambda_{n-m+i} \leq \rho_i \leq \lambda_i$ hold for $i = 1, 2, \ldots, m$.*

Lemma 2.4. [5] *Let $G_1 = (V, E_1)$ be a graph on n vertices and $G_2 = (V, E_2)$ be*

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a graph obtained from G_1 by removing an edge and adding a new edge that was not there before. Then

$$\mu_i(G_1) \geq \mu_{i+1}(G_2) \text{ and } \mu_i(G_2) \geq \mu_{i+1}(G_1) \text{ for } 1 \leq i \leq n-1.$$

LEMMA 2.5. [8] *Suppose that $g \geq 4$. Then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$.*

For $U \subseteq V(G)$, let $L_U(G)$ be the principal sub-matrix of $L(G)$ formed by deleting the rows and columns corresponding to all vertices in U. If $U = \{v\}$, then we simple write $L_U(G)$ as $L_v(G)$. Let H_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to two end vertices of P_{n+2}.

LEMMA 2.6. [7] *Set $\Phi(P_0) = 0, \Phi(H_0) = 1$. Then we have*

(1) $\Phi(P_n) = x\Phi(H_{n-1})$, $(n \geq 1)$;
(2) $\Phi(P_m)\Phi(P_n) - \Phi(P_{m-1})\Phi(P_{n+1}) = \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-2})\Phi(P_n)$, $(m \geq 2, n \geq 1)$.

COROLLARY 2.7. *For $m \geq k+1,\ n \geq 1$,*

$$\Phi(P_m)\Phi(P_n) - \Phi(P_{m-k})\Phi(P_{n+k}) = \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1}).$$

Proof. From Lemma 2.6, we have

$$\begin{aligned}
&\Phi(P_m)\Phi(P_n) - \Phi(P_{m-k})\Phi(P_{n+k})\\
&= \Phi(P_m)\Phi(P_n) - \Phi(P_{m-1})\Phi(P_{n+1}) + \Phi(P_{m-1})\Phi(P_{n+1})\\
&\quad - \Phi(P_{m-2})\Phi(P_{n+2}) + \Phi(P_{m-2})\Phi(P_{n+2}) - \cdots - \Phi(P_{m-k+1})\Phi(P_{n+k-1})\\
&\quad + \Phi(P_{m-k+1})\Phi(P_{n+k-1}) - \Phi(P_{m-k})\Phi(P_{n+k})\\
&= \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-2})\Phi(P_n) + \Phi(P_{m-2})\Phi(P_n) - \Phi(P_{m-3})\Phi(P_{n+1})\\
&\quad + \Phi(P_{m-3})\Phi(P_{n+1}) - \cdots - \Phi(P_{m-k})\Phi(P_{n+k-2})\\
&\quad + \Phi(P_{m-k})\Phi(P_{n+k-2}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1})\\
&= \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1}). \quad \square
\end{aligned}$$

Suppose G_1 and G_2 are two disjoint graphs. $u \in V(G_1)$ and $v \in V(G_2)$. Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 with an edge.

LEMMA 2.8. [6] *Let G_1 and G_2 be two disjoint graphs. Then*

$$\Phi(G_1u : vG_2) = \Phi(G_1)\Phi(G_2) - \Phi(G_1)\Phi(L_v(G_2)) - \Phi(L_u(G_1))\Phi(G_2).$$

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By a similar argument as the proof of Lemma 2.8, which was shown in [6], we also have the following result.

COROLLARY 2.9. *Let G_1 and G_2 be two disjoint graphs. Suppose w is a vertex of G_1 which is different from u. Then*

$$\begin{aligned}\Phi(L_w(G_1u : vG_2)) = \Phi(L_w(G_1))\Phi(G_2) - \Phi(L_w(G_1))\Phi(L_v(G_2))\\ - \Phi(L_{\{w,u\}}(G_1))\Phi(G_2).\end{aligned}$$

Let G and H be two disjoint graphs with $|V(G)| = s \geq 2, |V(H)| = t \geq 2$. Let $u \in V(G)$ and $r \in V(H)$. Let $Gu \cdot rH$ be the graph obtained from G and H by identifying the two vertices u and r (suppose that the new vertex is still u). It is easy to see that $Gu \cdot rH$ has $n = s + t - 1$ vertices.

LEMMA 2.10. *Suppose u and v are two distinct vertices of G. Suppose X is a Fiedler vector of $Gu \cdot rH$. If $(X(v) - X(u)) \sum_{\substack{w\in V(H)\\ w\neq r}} X(w) \geq 0$, then $\alpha(Gu \cdot rH) \geq \alpha(Gv \cdot rH)$. Moreover, the inequality is strict if $X(u) \neq X(v)$.*

Proof. Let Y be a valuation of $Gv \cdot rH$ defined by

$$Y(w) = \begin{cases} X(w) - \frac{(t-1)(X(v)-X(u))}{n}, & w \in V(G);\\ X(w) + \frac{s(X(v)-X(u))}{n}, & w \in V(H), w \neq r.\end{cases}$$

It is easy to see that $Y^T e = 0$, $Y^T L(Gv \cdot rH)Y = X^T L(Gu \cdot rH)X = \alpha(Gu \cdot rH)$ and

$$\begin{aligned} Y^TY &= X^TX - 2\sum_{w\in V(G)} \frac{(t-1)(X(v)-X(u))}{n}X(w) + \frac{s(t-1)^2(X(v)-X(u))^2}{n^2}\\ &\quad + 2\sum_{\substack{w\in V(H)\\ w\neq r}} \frac{s(X(v)-X(u))}{n}X(w) + \frac{(t-1)s^2(X(v)-X(u))^2}{n^2}\\ &= 1 + 2(X(v)-X(u))\sum_{\substack{w\in V(H)\\ w\neq r}} X(w) + \frac{s(t-1)(X(v)-X(u))^2}{n}\\ &\geq 1.\end{aligned}$$

Clearly, the inequality is strict if $X(v) \neq X(u)$.

Thus, we have

$$\alpha(Gu \cdot rH) = X^T L(Gu \cdot rH)X \geq \frac{Y^T L(Gv \cdot rH)Y}{Y^TY} \geq \alpha(Gv \cdot rH),$$

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and the inequality is strict if $X(v) \neq X(u)$. $\square$

From Lemma 2.10, we immediately have the following.

Corollary 2.11. *Let u, v be two vertices of a connected graph G and there exist s pendant edges $uu_1, uu_2, \ldots, uu_s$ at u. Suppose X is a Fiedler vector of G. Let $G' = G - uu_1 - uu_2 - \cdots - uu_s + vu_1 + vu_2 + \cdots + vu_s$. If $X(v) \geq X(u) \geq 0$, then $\alpha(G) \geq \alpha(G')$. Moreover, the inequality is strict if $X(v) \neq X(u)$.*

Lemma 2.12. *Let $C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}$ be the caterpillar unicyclic graph defined in Section 1. Then for $n \geq g+1$,*

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) < \alpha(P_{g-\lfloor \frac{g}{2} \rfloor}).$$

Proof. From Corollary 2.2 and Lemma 2.5, we have

$$\begin{aligned}\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) \leq \alpha(C_{g+1,g}) < \alpha(C_{g+1}) &= 4\sin^2 \frac{\pi}{g+1} \\ &\leq 4\sin^2 \frac{\pi}{2(g-\lfloor \frac{g}{2} \rfloor)} = \alpha(P_{g-\lfloor \frac{g}{2} \rfloor}). \quad \square\end{aligned}$$

Remark 1. Since $\alpha(P_n)$ is a decreasing function on n, $\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) < \alpha(P_j)$ for $j \leq g - \lfloor \frac{g}{2} \rfloor$.

Lemma 2.13. [10] *Let G be a connected graph with a cut vertex v. Then $\alpha(G) \leq 1$, the equality holds if and only if v is adjacent to every vertex of G.*

Lemma 2.14. *Let $C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}$ and $C^{n_1,n_2}_{g;1,i}$ $(2 \leq i \leq \lfloor \frac{g}{2} \rfloor)$ be the caterpillar unicyclic graphs defined in Section 1. Then for $n_1, n_2 \geq 1$,*

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) \leq \mu_{n-2}(C^{n_1,n_2}_{g;1,i}).$$

Proof. If $\mu_{n-2}(C^{n_1,n_2}_{g;1,i}) \geq 1$, then the result follows from Lemma 2.13. Thus, in the following, we assume that $\mu_{n-2}(C^{n_1,n_2}_{g;1,i}) < 1$. From Corollary 2.2 and Lemma 2.3, we have

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) \leq \alpha(C^{n_1,0}_{g;1,\lfloor \frac{g}{2} \rfloor+1}) = \alpha(C^{n_1}_{g;1}) \leq \lambda_{g+n_1-2}(L_{v_i}(C^{n_1}_{g;1})) \leq \mu_{n-2}(C^{n_1,n_2}_{g;1,i}),$$

where $\lambda_{g+n_1-2}(L_{v_i}(C^{n_1}_{g;1}))$ denotes the second smallest eigenvalue of $L_{v_i}(C^{n_1}_{g;1})$. $\square$

Lemma 2.15. *For $2 \leq i \leq \lfloor \frac{g}{2} \rfloor$ and $n_1, n_2 \geq 1$, we have*

$$\alpha(C^{n_1,n_2}_{g;1,i}) > \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2} \rfloor+1}).$$

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Proof. Applying Lemma 2.8 and Corollary 2.9 for n_1 times, we have

$$\begin{aligned}
&\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{n_1,n_2})\\
&= (x-1)\Phi(C_{g;1,i}^{n_1-1,n_2}) - x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,i}^{0,n_2})) - (x-1)\Phi(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{n_1-1,n_2})\\
&\quad + x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2}))\\
&\vdots \qquad \vdots\\
&= (x-1)^{n_1}\Phi(C_{g;1,i}^{0,n_2}) - n_1x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,i}^{0,n_2})) - (x-1)^{n_1}\Phi(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2})\\
&\quad + n_1x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2})).
\end{aligned}$$

Note that $C_{g;1,i}^{0,n_2} = C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2} = C_{g;1}^{n_2}$. Then from the above equation, we have

$$\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{n_1,n_2}) = n_1x(x-1)^{n_1-1}[\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))] \tag{2.1}$$

Applying Corollary 2.9 again, we have

$$\begin{aligned}
&\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))\\
&= (x-1)\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2-1})) - x(x-1)^{n_2-1}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})\\
&\quad - (x-1)\Phi(L_{v_1}(C_{g;1,i}^{0,n_2-1})) + x(x-1)^{n_2-1}\Phi(H_{i-2})\Phi(H_{g-i})\\
&\qquad\qquad \vdots\\
&= (x-1)^{n_2}\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,0})) - n_2x(x-1)^{n_2-1}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})\\
&\quad - (x-1)^{n_2}\Phi(L_{v_1}(C_{g;1,i}^{0,0})) + n_2x(x-1)^{n_2-1}\Phi(H_{i-2})\Phi(H_{g-i})
\end{aligned}$$

Note that $L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,0}) = L_{v_1}(C_{g;1,i}^{0,0}) = L_{v_1}(C_g) = \Phi(H_{g-1})$. Thus, from the above equation, we have

$$\begin{aligned}
&\Phi(L_{v_1}(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))\\
&= n_2x(x-1)^{n_2-1}[\Phi(H_{i-2})\Phi(H_{g-i}) - \Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})].
\end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), and from Lemma 2.6 and Corollary 2.7, we have

$$\begin{aligned}
&\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{n_1,n_2})\\
&= n_1n_2x^2(x-1)^{n_1+n_2-2}[\Phi(H_{i-2})\Phi(H_{g-i}) - \Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})]\\
&= n_1n_2(x-1)^{n_1+n_2-2}[\Phi(P_{i-1})\Phi(P_{g-i+1}) - \Phi(P_{\lfloor\frac{g}{2}\rfloor})\Phi(P_{g-\lfloor\frac{g}{2}\rfloor})]\\
&= -n_1n_2(x-1)^{n_1+n_2-2}\Phi(P_{\lfloor\frac{g}{2}\rfloor-i+1})\Phi(P_{g-\lfloor\frac{g}{2}\rfloor-i+1}).
\end{aligned}$$

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Let $\alpha = \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. From Lemma 2.12, we have

$$\begin{aligned}&(-1)^{n_1+n_2+g-1}[\Phi(C^{n_1,n_2}_{g;1,i},\alpha) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1},\alpha)]\\ &= (-1)^{n_1+n_2+g}[n_1n_2(\alpha-1)^{n_1+n_2-2}\Phi(P_{\lfloor \frac{g}{2}\rfloor-i+1},\alpha)\Phi(P_{g-\lfloor \frac{g}{2}\rfloor-i+1},\alpha)].\end{aligned}$$

By Remark 1 and the fact $0 < \alpha < 1$, the above expression is positive. Note that $n_1+n_2+g = n$ is the order of the graph $C^{n_1,n_2}_{g;1,i}$. So $(-1)^{n-1}\Phi(C^{n_1,n_2}_{g;1,i},\alpha) > 0$. Thus, from Lemma 2.14, we have $\alpha(C^{n_1,n_2}_{g;1,i}) > \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. ∎

LEMMA 2.16. *For $n_1 \geq n_2+2$, $\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) > \alpha(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$.*

Proof. We separate the proof into two cases.

Case 1. $n_2 \geq 1$.

Applying Lemma 2.8 and Corollary 2.9 for several times, we have

$$\begin{aligned}&\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})\\ &= (x-1)\Phi(C^{n_1-1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - x(x-1)^{n_2}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &\quad - (x-1)\Phi(C^{n_1-1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) + x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &= x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x(x-1)^{n_2}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &= x(x-1)^{n_1}\Phi(L_{v_1}(C^{0,n_2-1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\ &\quad - x(x-1)^{n_2+1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-2,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &\quad + x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\ &= x(x-1)^{n_1}\Phi(L_{v_1}(C^{0,n_2-1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x(x-1)^{n_2+1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-2,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &= x(x-1)^{n_1+n_2-1}\Phi(L_{v_1}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &\quad - n_2x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\ &\quad - x(x-1)^{n_1+n_2-1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\ &\quad + (n_1-1)x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1}).\end{aligned}$$

Note that $L_{v_1}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) = L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) = L_{v_1}(C_g)$. Then from Lemma 2.6 and the above equation we have

$$\begin{aligned}&\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})\\ &= x^2(x-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\ &= (x-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(P_{\lfloor \frac{g}{2}\rfloor})\Phi(P_{g-\lfloor \frac{g}{2}\rfloor}).\end{aligned}$$

Let $\alpha = \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. From Lemma 2.4, we have $\alpha \leq \mu_{n-2}(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$.

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Thus, similar to the proof of Lemma 2.15, we have

$$\begin{aligned}&(-1)^{n_1+n_2+g-1}[\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor\frac{g}{2}\rfloor+1},\alpha)-\Phi(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1},\alpha)]\\&=(-1)^{n_1+n_2+g-1}(\alpha-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(P_{\lfloor\frac{g}{2}\rfloor},\alpha)\Phi(P_{g-\lfloor\frac{g}{2}\rfloor},\alpha)<0.\end{aligned}$$

Then, we have $\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1})>\alpha(C^{n_1-1,n_2+1}_{g;1,\lfloor\frac{g}{2}\rfloor+1})$.

Case 2. $n_2=0$.

From Lemma 2.8 and Corollary 2.9, we have

$$\begin{aligned}&\Phi(C^{n_1-1,1}_{g;1,\lfloor\frac{g}{2}\rfloor+1})-\Phi(C^{n_1}_{g;1})\\&=(x-1)\Phi(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1})-x\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\&\quad-(x-1)\Phi(C^{n_1-1}_{g;1})+x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0}_{g;1}))\\&=x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0}_{g;1}))-x\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\&=x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0}_{g;1}))-x(x-1)^{n_1-1}\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\&\quad+(n_1-1)x^2(x-1)^{n_1-2}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})\\&=(n_1-1)x^2(x-1)^{n_1-2}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1}).\end{aligned}$$

By a similar argument as that of Case 1, the result follows. □

Now we give the main result of this paper.

THEOREM 2.17. *Let G be a caterpillar unicyclic graph on n vertices with girth g. Then*

$$\alpha(G)\geq\alpha(C^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}_{g;1,\lfloor\frac{g}{2}\rfloor+1}),$$

and the equality holds if and only if $G=C^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}_{g;1,\lfloor\frac{g}{2}\rfloor+1}$.

Proof. Since G is a caterpillar unicyclic graph on n vertices with girth g, we may assume that

$$G=C^{n_1,n_2,\ldots,n_k}_{g;\,i_1,i_2,\ldots,i_k},\quad n_j\geq1 \text{ for } 1\leq j\leq k\leq g;1\leq i_1<i_2<\cdots<i_k\leq g.$$

For $k=1$, the result follows from Case 2 of the proof of Lemma 2.16. For $k=2$, the result follows from Lemmas 2.15 and 2.16.

For $k=3$, let X be a Fiedler vector of $G=C^{n_1,n_2,n_3}_{g;\,i_1,i_2,i_3}$. Since $-X$ is also a Fiedler vector of G, without loss of generality, we may assume that there are at least two of $X(v_{i_1})$, $X(v_{i_2})$ and $X(v_{i_3})$ being nonnegative. By renumbering, we may assume that $G\cong C^{n_1,n_2,n_3}_{g;\,1,i,j}$ with $X(v_j)\geq X(v_i)\geq X(v_1)$ and $X(v_i)\geq0$. Suppose $X(v_j)>X(v_i)$. Then by Corollary 2.11, $\alpha(G)>\alpha(C^{n_1,n_2+n_3}_{g;\,1,j})$. By renumbering of the vertices, we

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may assume that $j \leq \lfloor \frac{g}{2} \rfloor + 1$. By Lemmas 2.15 and 2.16 if necessary, we obtain the result.

So now we assume that $X(v_j) = X(v_i)$. If the distance between v_1 and v_i or the distance between v_1 and v_j less than $\lfloor \frac{g}{2} \rfloor$, then (by renumbering the vertices if necessary) we may assume that $i \leq \lfloor \frac{g}{2} \rfloor$. Then by Corollary 2.11, Lemmas 2.15 and 2.16, we obtain that $\alpha(G) \geq \alpha(C^{n_1,n_2+n_3}_{g;\ 1,i}) > \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1}) \geq \alpha(C^{\lfloor \frac{n-g}{2} \rfloor, \lceil \frac{n-g}{2} \rceil}_{g;1,\lfloor \frac{g}{2} \rfloor+1})$.

Thus, we have to deal with the case that the distance between v_1 to both v_i and v_j are $\lfloor \frac{g}{2} \rfloor$. Note that the necessary condition for the occurrence of this case is g being odd. So now $G \cong C^{n_1,n_2,n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1,\lfloor \frac{g}{2} \rfloor+2}$ with $X(v_{\lfloor \frac{g}{2} \rfloor+1}) = X(v_{\lfloor \frac{g}{2} \rfloor+2})$. Note that

$$\alpha(G) = X^T L(C^{n_1,n_2,n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1,\lfloor \frac{g}{2} \rfloor+2})X = X^T L(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2})X \geq \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}).$$

Suppose that $\alpha(G) = \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2})$. From the above equation we can see that X is also a Fiedler vector of $C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}$. From the eigenvalue equations of G, we have

$$\begin{aligned} &\Big(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - \alpha(G)\Big) X(v_{\lfloor \frac{g}{2} \rfloor+1}) \\ &= \sum_{w \in N(v_{\lfloor \frac{g}{2} \rfloor+1})} X(w) = X(v_{\lfloor \frac{g}{2} \rfloor+1}) + X(v_{\lfloor \frac{g}{2} \rfloor}) + \sum_{\substack{w \in N(v_{\lfloor \frac{g}{2} \rfloor+1}) \\ w \neq v_{\lfloor \frac{g}{2} \rfloor}, v_{\lfloor \frac{g}{2} \rfloor+2}}} X(w). \end{aligned}$$

Note that for $w \in N(v_{\lfloor \frac{g}{2} \rfloor+1}) \setminus \{v_{\lfloor \frac{g}{2} \rfloor}, v_{\lfloor \frac{g}{2} \rfloor+2}\}$, $(1 - \alpha(G))X(w) = X(v_{\lfloor \frac{g}{2} \rfloor+1})$. Thus, the above equation becomes

$$X(v_{\lfloor \frac{g}{2} \rfloor}) = \left(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 1 - \alpha(G) - \frac{d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2}{1 - \alpha(G)}\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}). \tag{2.3}$$

Similarly, from the eigenvalue equations of $C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}$, we have

$$\Big(2 - \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2})\Big) X(v_{\lfloor \frac{g}{2} \rfloor+1}) = X(v_{\lfloor \frac{g}{2} \rfloor+2}) + X(v_{\lfloor \frac{g}{2} \rfloor}).$$

Then

$$X(v_{\lfloor \frac{g}{2} \rfloor}) = \Big(1 - \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2})\Big) X(v_{\lfloor \frac{g}{2} \rfloor+1}). \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\left(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2 - \frac{d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2}{1 - \alpha(G)}\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}) = 0.$$

Since $n_2 \geq 1$, $d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) > 2$. Thus, we have $X(v_{\lfloor \frac{g}{2} \rfloor+1}) = 0$. From the eigenvalue equations of G, it is easy to see that $X = \mathbf{0}$. It yields a contradiction. So $\alpha(G) > \alpha(C^{n_1,n_2+n_3}_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2})$.

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For $k \geq 4$, from Corollary 2.11, there exists some caterpillar unicyclic graph, say $C^{n_1,n_2,n_3}_{g;\,1,i,j}$ for $n_1, n_2, n_3 \geq 1$ and $1 < i < j$, such that $\alpha(G) \geq \alpha(C^{n_1,n_2,n_3}_{g;\,1,i,j})$. This case is referred to the case when $k = 3$.

Hence, the proof is completed. □

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