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SPECTRAL CHARACTERIZATION OF †-SHAPE TREES*

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Abstract. The †-shape tree is the coalescence of the star $K_{1,4}$ and the path P_{n-4} with respect to two pendent vertices. In this paper, it is showed that the †-shape tree is determined by its adjacency spectrum if and only if $n \neq 2k+9$ $(k = 0, 1, \ldots)$. Furthermore, all the cospectral mates of the †-shape tree are found when $n = 2k + 9$.

Key words. †-shape tree, Adjacency spectrum, Spectral characterization, Cospectral graphs.

AMS subject classifications. 05C50.

1. Introduction. In this paper, we are concerned only with simple undirected graphs (loops and multiple edges are not allowed). Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Let $A(G)$ be the $(0,1)$-adjacency matrix of G, the polynomial $P_G(\lambda) = \det(\lambda I - A(G))$, where I is the identity matrix, is the *characteristic polynomial* of G with respect to $A(G)$. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers, which will be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and be called the adjacency eigenvalues of G. The eigenvalues of G together with their multiplicities is called the *adjacency spectrum* of G. Two graphs G and H are said to be *cospectral* if they share the same spectrum (i.e., equal characteristic polynomial). A graph G is said to be determined by its adjacency spectrum (DAS for short) if for any graph H, $P_G(\lambda) = P_H(\lambda)$ implies that H is isomorphic to G. Up to now, numerous examples of cospectral but non-isomorphic graphs have been found. On the other hand, only few graphs with very special structures have been proved to be determined by their spectra, see [4, 6, 7, 8, 11, 13, 15] for some examples.

Determining which graphs are determined by their spectrum is a difficult problem, far from resolved, in the theory of graph spectra. In [4], van Dam and Haemers proposed the following more modest problem: which trees are determined by their spectrum? It is well-know that the path P_n is DAS (see [4]). It is proved by Shen et al. [13] that the graph Z_n is DAS and by Wang and Xu [15] that all T-shape trees are DAS except for a few well-defined cases. In 1973, Schwenk [12] proved the most

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striking result that almost all trees are cospectral. For a recent survey of the subject, one can consult [5].

There have been some attempts to characterize graphs having spectral radius at most a given number, for example, the class of all graphs G whose largest eigenvalue λ_1 is bounded by 2 has been completely determined by Smith [14]. Subsequently, Cvetković et al. [2] gave a nearly complete description of all graphs G with $2 < \lambda_1 \leq \sqrt{2+\sqrt{5}}$ and their description was completed by Brouwer and Neumaier [1]. Later Woo and Neumaier [16] examine the structure of graphs G with $\sqrt{2+\sqrt{5}} < \lambda_1 \leq \frac{3}{2}\sqrt{2}$ and their result is presented in Section 2.

Most of the connected graphs with spectral radius at most 2 are known to be DAS [4, 13, 15]. Ghareghani [7] showed that all connected graphs except for three well-defined trees with spectral radius in the interval $(2, \sqrt{2+\sqrt{5}}]$ are DAS. They also posed a problem to determine all DAS ones among the graphs characterized in [16]. Now we cite some terminologies used in the paper [16].

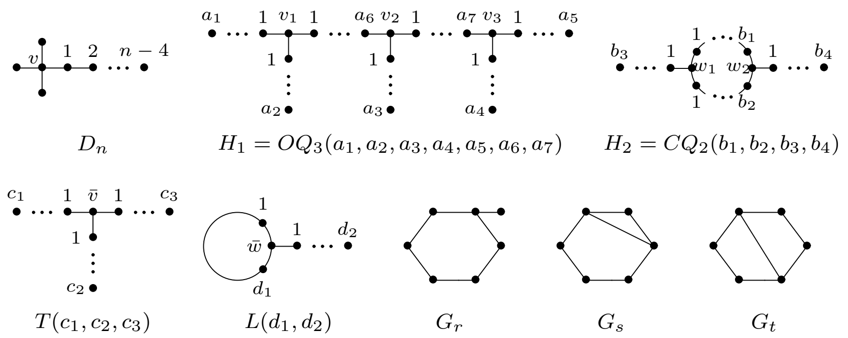

FIG. 1.1. *Graphs* $D_n, H_1, H_2, T(c_1, c_2, c_3), L(d_1, d_2), G_r, G_s, G_t$.

An *open quipu* is a tree G of maximum degree 3 such that all vertices of degree 3 lie on a path. A *closed quipu* is a connected graph G of maximum degree 3 such that all vertices of degree 3 lie on a cycle, and no other cycle exists. Let OQ_n (CQ_n) denote the set of open quipus (closed quipus) with exactly n vertices of degree 3, e.g., $OQ_3(a_1, \ldots, a_7)$ and $CQ_2(b_1, \ldots, b_4)$ is shown in Fig. 1.1. A †-*shape* tree D_n ($n \geq 7$) is the coalescence of the star $K_{1,4}$ and the path P_{n-4} with respect to two pendent vertices (see Fig. 1.1). A T-shape tree is a tree with exactly one of its vertices having maximum degree 3. Denote by $T(c_1, c_2, c_3)$ the T-shape tree such that deleting the unique vertex of degree 3 leaves three disjoint paths $P_{c_1} \cup P_{c_2} \cup P_{c_3}$(see Fig. 1.1). The *lollipop graph* $L(d_1, d_2)$ is obtained from the cycle C_{d_1+1} by appending a path P_{d_2} (also see Fig. 1.1). Without loss of generality, in what follows we always assume that $a_1 \geq a_2 > 0$, $a_3 > 0$, $a_4 \geq a_5 > 0$, $a_6, a_7 \geq 0$; $b_1 \geq b_2 \geq 0$, $b_3 \geq b_4 > 0$; $c_1 \geq c_2 \geq c_3 > 0$; $d_1 \geq 2$, $d_2 > 0$ where a_i, b_i, c_i, d_i correspond to the lengths of paths that are combined

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to form the graphs in Fig. 1.1. Throughout this paper $G - v$ and $G - uv$ denote the graph obtained from G by deleting a vertex v and an edge uv, respectively. The degree sequence of a graph G is written as $\pi(G) = (0^{x_0}, 1^{x_1}, ..., k^{x_k}, ..., \Delta^{x_\Delta})$ where k^{x_k} means that G has x_k vertices of degree k and $x_0 + x_1 + \cdots + x_\Delta = n$. The notion and symbols not defined here are standard, see [3] for any undefined terms.

In this paper, we complete the spectral characterization of one of the three graphs determined by Woo and Neumaier. Our main result is the following.

THEOREM 1.1. *The †-shape tree D_n is determined by its adjacency spectrum if and only if $n \neq 2k+9$ $(k = 0, 1, 2, \ldots)$. If $n = 2k+9$ then when k is odd the unique cospectral mate of D_n is $OQ_3(k+1, 1, 1, 1, 1, 1, k)$ while when $k = 2\ell$ is even the cospectral mates of D_n are $OQ_3(k+1, 1, 1, 1, 1, 1, k)$ and $OQ_3(1, 1, \ell+1, 1, 1, 2\ell+1, \ell)$.*

The paper is organized as follows. In Section 2, some useful lemmas are cited. In Section 3, the degree sequences of graphs which are cospectral with †-shape tree D_n are determined. In Section 4, the spectral characterization of the †-shape tree D_n and graphs with degree sequence $(1^5, 2^{n-8}, 3^3)$ is finished. In Section 5, our main result is obtained.

2. Preliminaries. First, we give some lemmas that will be frequently used in the next section. Most of them are basic tools in studying the graph DAS problem, moreover, it is worth to mention that Lemma 2.8 [16] plays an important role in our research.

LEMMA 2.1. [3] *Let uv be an edge of a graph G, $\mathscr{C}(u)$ and $\mathscr{C}(uv)$ be the sets of all cycles Z containing u or uv, respectively. Then*

(*i*) $P_G(\lambda) = \lambda P_{G-u}(\lambda) - \sum_{uv \in E(G)} P_{G-u-v}(\lambda) - 2\sum_{Z \in \mathscr{C}(u)} P_{G-V(Z)}(\lambda)$.
(*ii*) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-u-v}(\lambda) - 2\sum_{Z \in \mathscr{C}(uv)} P_{G-V(Z)}(\lambda)$.

LEMMA 2.2 (Interlacing). [3] *Suppose that A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ of a principal submatrix of A of size m satisfy $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, ..., m$.*

LEMMA 2.3. [3] *Let G be a connected graph and H be a proper subgraph of G. Then $\lambda_1(G) > \lambda_1(H)$.*

LEMMA 2.4. [4] *Let G be a graph. For the adjacency matrix the following can be deduced from the spectrum:*

(1) *The number of vertices.*
(2) *The number of edges.*
(3) *The number of closed walks of any fixed length.*

Let $N_G(H)$ be the number of subgraphs of a graph G which are isomorphic to

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H and let $N_G(i)$ be the number of closed walks of length i in G. Let $N'_H(i)$ be the number of closed walks of H of length i which contain all the edges of H and $S_i(G)$ be the set of all the connected subgraphs H of G such that $N'_H(i) \neq 0$. Then

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i). \tag{2.1}$$

The following two lemmas can be obtained from (2.1) and it provides some formulae for calculating the number of closed walks of length $2, 3, 4$ for any graphs and $6, 8$ for graphs without cycles C_i $(i = 3, 4, 5, 7)$.

Lemma 2.5. [10] *The number of closed walks of length* $2, 3, 4$ *of a graph* G *are giving in the following where* m *is number of edges of* G.

(*i*) $N_G(2) = 2m$, $N_G(3) = 6N_G(C_3)$.
(*ii*) $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$.

Lemma 2.6. [9] *Let* G *be a connected graph without cycles* C_i $(i = 3, 4, 5, 7)$. *Then*

(*i*) $N_G(6) = 2m + 12N_G(P_3) + 6N_G(P_4) + 12N_G(K_{1,3}) + 12N_G(C_6)$.
(*ii*) $N_G(8) = 2m + 28N_G(P_3) + 32N_G(P_4) + 72N_G(K_{1,3}) + 8N_G(P_5)$
$+ 16N_G(T(2,1,1)) + 48N_G(K_{1,4}) + 96N_G(C_6) + 16N_G(G_r) + 48N_G(G_s)$
$+ 48N_G(G_t) + 16N_G(C_8)$ *(see Fig. 1.1).*

Let $P_{P_r}(\lambda)$ be the characteristic polynomial of the path P_r. For the sake of simplicity, we denote $P_{P_r}(\lambda)$ by P_r if there is no confusion. By convention, let $P_0 = 1$, $P_{-1} = 0$ and $P_{-2} = -1$.

Lemma 2.7. [11] $P_r = \frac{x^{2r+2}-1}{x^{r+2}-x^r}$ *and* $P_{P_r}(2) = r+1$, *where* x *satisfies* $x^2 - \lambda x + 1 = 0$.

Lemma 2.8. [16] *A connected graph* G *whose largest eigenvalue* λ_1 *satisfies* $2 < \lambda_1 \leq \frac{3}{2}\sqrt{2}$ *is either an open quipu, a closed quipu, or a dagger.*

Lemma 2.9. *Let* H *be a graph cospectral with the* †*-shape tree* D_n, *then* H *is either an open quipu or the union of a closed quipu and an induced subgraph of a* T*-shape tree.*

Proof. Clearly, removing the only vertex of degree 4 of D_n leaves three isolated vertices and a path. By Lemma 2.2 $\lambda_1(D_n) > 2 > \lambda_1(D_n - v) \geq \lambda_2(D_n)$. Since H and D_n are cospectral, $\lambda_1(H) = \lambda_1(D_n)$, $\lambda_2(H) = \lambda_2(D_n) < 2$. It follows that H has at most one component which contains cycles. Moreover, Lemma 2.8 implies that the component of H corresponding to the maximal eigenvalue is the induced subgraph of the open quipu or the closed quipu. Note that D_n is a tree, we get if H contains no cycle, then H is an open quipu. Otherwise H is the union of a closed quipu and an

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induced graph of an open quipu. Observe that $\lambda_2(H) < 2$, it is easy to see that the open quipu cannot have two vertices of degree 3, thus is a T-shape tree. $\square$

3. Degree sequences of graphs cospectral with a †-shape tree. In this section, we determine the degree sequences of graphs which are cospectral with the †-shape tree D_n. Let H be a graph cospectral with the †-shape tree D_n, by Lemma 2.4, D_n and H have the same number of vertices, edges and closed walks of any given length. Denote by x_i and y_i the number of vertices of degree i in D_n and H, respectively. By counting the number of vertices, edges and closed walks of length 4 in D_n and H, we have the following three equations:

$$\sum_{i=0}^{\Delta} x_i = n = \sum_{i=0}^{\Delta'} y_i,$$
$$\sum_{i=0}^{\Delta} ix_i = 2(n-1) = \sum_{i=0}^{\Delta'} iy_i,$$
$$\sum_{i=0}^{\Delta} ix_i + 4\sum_{i=0}^{\Delta}\binom{i}{2}x_i + 8n_4 = 6n+2 = \sum_{i=0}^{\Delta'} iy_i + 4\sum_{i=0}^{\Delta'}\binom{i}{2}y_i + 8n_4',$$

where $n_4 = N_{D_n}(C_4) = 0$ and $n_4' = N_H(C_4)$. From the structure of D_n we see that $\Delta = 4$, $x_0 = 0$, $x_1 = 4$, $x_2 = n-5$, $x_3 = 0$ and $x_4 = 1$. By adding up these three equations with coefficients 4, -5 and 1, respectively, we have

$$6 = \sum_{i=0}^{\Delta'} (i^2 - 3i + 2)y_i + 4n_4'. \tag{3.1}$$

The following lemma gives the degree sequence of H according to $n_4' = 1$ or $n_4' = 0$.

LEMMA 3.1. *Let H be a graph cospectral with the †-shape tree D_n. Let n_4' be the number of C_4 in H. Then $\pi(H)$ is $(0^1, 2^{n-1})$ or $(1^3, 2^{n-4}, 3^1)$ if $n_4' = 1$; $\pi(H)$ is either $(1^4, 2^{n-5}, 4^1)$, $(1^5, 2^{n-8}, 3^3)$ or $(0^1, 1^2, 3^{n-5}, 4^2)$ if $n_4' = 0$.*

Proof. If $n_4' = 1$, Eq. (3.1) yields $\sum_{i=0}^{\Delta'}(i^2 - 3i + 2)y_i = 2$, this implies that $y_0 + y_3 = 1$ and $y_i = 0$ for $i \geq 4$. Thus, $\pi(H)$ is $(0^1, 2^{n-1})$ or $(1^3, 2^{n-4}, 3^1)$.

If $n_4' = 0$, Eq. (3.1) gives $\sum_{i=0}^{\Delta'}(i^2 - 3i + 2)y_i = 6$, this implies that $y_0 + y_3 + 3y_4 = 3$ and $y_i = 0$ for $i \geq 5$. So $(y_0, y_1, y_2, y_3, y_4) \in \{(0, 4, n-5, 0, 1), (0, 5, n-8, 3, 0), (1, 2, n-5, 2, 0), (2, -1, n-2, 1, 0), (3, -4, n+1, 0, 0)\}$, i.e., $\pi(H)$ is one of $(1^4, 2^{n-5}, 4^1)$, $(1^5, 2^{n-8}, 3^3)$ and $(0^1, 1^2, 2^{n-5}, 3^2)$. $\square$

Moreover, we can say further that if H is cospectral with the †-shape tree D_n, then H has no cycle C_4, thus the degree sequences $(0^1, 2^{n-1})$ and $(1^3, 2^{n-4}, 3^1)$ do not need to be considered.

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LEMMA 3.2. *Let H be a graph cospectral with the †-shape tree D_n, then H has no cycle C_4.*

Proof. Suppose that H contains C_4, then Lemma 3.1 gives the two possible degree sequences of H. First suppose that $\pi(H) = (0^1, 2^{n-1})$. Lemma 2.9 implies that H has only one cycle, thus $H = C_4 \cup K_1$, however $5 = |V(H)| < 6 < |V(D_n)|$, contradicting H and D_n are cospectral. Next suppose that $\pi(H) = (1^3, 2^{n-4}, 3^1)$, also by Lemma 2.9 we get $H = C_4 \cup T(c_1, c_2, c_3)$ or $L(3, d_2) \cup P_r$. If $H = C_4 \cup T(c_1, c_2, c_3)$, then 2 is an eigenvalue of H, but it is easy to see that $\lambda_1(D_n) > 2 > \lambda_2(D_n)$, therefore they are not cospectral and this case is impossible. If $H = L(3, d_2) \cup P_r$, then $\lambda_1(H) = \lambda_1(L(3, d_2))$, however by Lemma 2.3, $\lambda_1(L(3, d_2)) \geq \lambda_1(L(3, 1)) = 2.1358 \cdots > \frac{3}{2}\sqrt{2} \geq \lambda_1(D_n)$, we also get a contradiction. ∎

We complete this section by a remark.

REMARK 3.3. Let H be a graph cospectral with the †-shape tree D_n, if $\pi(H) = (1^4, 2^{n-5}, 4^1)$, we claim that H is connected, then Lemma 2.8 implies that $H = D_n$. Suppose by the contradiction that H is not connected, then $\pi(H)$ and Lemma 2.8 give that H is the union of a †-shape tree $D_{n'}$ $(n \geq n' + 3)$ and some cycles. Clearly, $D_{n'}$ is a proper subgraph of D_n, by Lemma 2.3, $\lambda_1(H) = \lambda_1(D_{n'}) < \lambda_1(D_n)$, a contradiction. In the next two sections we investigate H with degree sequences $(1^5, 2^{n-8}, 3^3)$ and $(0^1, 1^2, 2^{n-5}, 3^2)$, respectively.

4. †-shape tree and graphs with degree sequence $(1^5, 2^{n-8}, 3^3)$**.** Let H be a graph with degree sequence $(1^5, 2^{n-8}, 3^3)$, from Lemma 2.9 one can easily get the following result.

LEMMA 4.1. *Let H be a graph cospectral with the †-shape tree D_n $(n \geq 7)$ and $\pi(H) = (1^5, 2^{n-8}, 3^3)$, then H may be one of the following graphs: $H_a = OQ_3(a_1, \ldots, a_7)$; $H_b = CQ_3 \cup P_r$; $H_c = CQ_2(b_1, \ldots, b_4) \cup T(c_1, c_2, c_3)$.*

In addition, we have more properties of the †-shape tree D_n below.

LEMMA 4.2. *Let D_n $(n \geq 7)$ be the †-shape tree and $P_{D_n}(\lambda)$ be its characteristic polynomial. Let x satisfy $x^2 - \lambda x + 1 = 0$, then*

(*i*) $P_{D_n}(2) = -4n + 20$.
(*ii*) $P_{D_n}(\lambda) = \frac{(x^2+1)^2(x^{n+2} - 2x^n + 2x^{6-n} - x^{4-n})}{x^4(x^2-1)}$.
(*iii*) $N_{D_n}(6) = 20n + 40$.
(*iv*) $N_{D_n}(8) = 70n + 282$.

Proof. Let v be the vertex of degree 4 of D_n. Use Lemma 2.1 (*i*) at v to get

$$P_{D_n}(\lambda) = \lambda^4 P_{P_{n-4}}(\lambda) - 3\lambda^2 P_{P_{n-4}}(\lambda) - \lambda^3 P_{P_{n-5}}(\lambda). \tag{4.1}$$

By Lemma 2.7, $P_{P_r}(2) = r + 1$, substituting this into (4.1) gives $P_{D_n}(2) = -4n + 20$.

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Also by Lemma 2.7 $P_{P_r}(\lambda) = \frac{x^{2r+2}-1}{x^{r+2}-x^r}$ where x satisfy $x^2 - \lambda x + 1 = 0$. Putting this into (4.1) and using Maple we obtain $P_{D_n}(\lambda) = \frac{(x^2+1)^2(x^{n+2}-2x^n+2x^{6-n}-x^{4-n})}{x^4(x^2-1)}$.

For an acyclic graph Γ, Lemma 2.6 (i) implies

$$N_\Gamma(6) = 2|E(\Gamma)| + 12N_\Gamma(P_3) + 6N_\Gamma(P_4) + 12N_\Gamma(K_{1,3}). \tag{4.2}$$

For any triangle free graph Γ, let us define

$$d(uv) = (d(u)-1)(d(v)-1), \text{ where } uv \in E(\Gamma).$$

It is easy to verify $N_\Gamma(P_3) = \sum_{v\in V(\Gamma)} \binom{d(v)}{2}$, $N_\Gamma(P_4) = \sum_{uv\in E(\Gamma)} d(uv)$ and $N_\Gamma(K_{1,3}) = \sum_{v\in V(\Gamma)} \binom{d(v)}{3}$. Set $\Gamma = D_n$, we have $|E(D_n)| = n-1$, $N_{D_n}(P_3) = 1\times\binom{4}{2} + (n-5)\times\binom{2}{2} = n+1$, $N_{D_n}(K_{1,3}) = 4$. $N_{D_n}(P_4)$, related to the structure of H, is more complicated. It is easy to see that for an edge $uv \in E(D_n)$, $d(uv) \in \{0,1,3\}$. uv is said to be the edge of i*-type* if $d(uv) = i$ $(i = 0,1,3)$. It is clear that uv is 0-type if and only if uv is a pendent edge; uv is 1-type if and only if $d(u) = d(v) = 2$; uv is 3-type if and only if $\{d(u), d(v)\} = \{2,4\}$. Denote by $m_i(D_n)$ the number of i-type edges in D_n. Then

$$m_0(D_n) + m_1(D_n) + m_2(D_n) + m_4(D_n) = |E(D_n)| = n-1. \tag{4.3}$$

Since D_n has four pendent edges, $m_0(D_n) = 4$. Moreover, $m_3(D_n) = 1$ and so $m_1(D_n) = n-6$ and

$$N_{D_n}(P_4) = \sum_{uv\in E(D_n)} d(uv) = m_1 + 3m_3 = n-3.$$

By (4.2), $N_{D_n}(6) = 20n + 40$.

Since D_n has no cycle, Lemma 2.6 (ii) implies

$$\begin{aligned} N_{D_n}(8) =& 2|E(D_n)| + 28N_{D_n}(P_3) + 32N_{D_n}(P_4) + 72N_{D_n}(K_{1,3}) + 8N_{D_n}(P_5) \\ &+ 16N_{D_n}(T(2,1,1)) + 48N_{D_n}(K_{1,4}). \end{aligned} \tag{4.4}$$

It is easy to verify that $N_{D_n}(P_5) = n-4$, $N_{D_n}(T(2,1,1)) = 3$ and $N_{D_n}(K_{1,4}) = 1$. Thus, by (4.4), $N_{D_n}(8) = 70n + 282$. $\square$

4.1. †-shape tree and open quipus. In this subsection, all cospectral graphs of the †-shape tree D_n that are open quipus are determined.

LEMMA 4.3. *Let* $H_a = OQ_3(a_1, \dots, a_7)$ *be an open quipu with three vertices of degree* 3 *and be cospectral with the †-shape tree* D_n $(n \geq 7)$, *then* $N_{H_a}(P_4) = n-1$ *and* H_a *is one of the following graphs:* $H_a^1 = OQ_3(1,1,1,1,1,n-8,0)$, $H_a^2 =$

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$OQ_3(a_1,1,1,1,1,a_6,a_7)$ $(a_1,a_6,a_7>0)$, $H_a^3 = OQ_3(1,1,a_3,1,1,a_6,a_7)$ $(a_3,a_6,a_7>0)$.

Proof. Since H_a and D_n are cospectral, they have the same number of closed walks of length 6. From the structure of H_a we see that $|E(H_a)| = n-1$, $N_{H_a}(P_3) = n+1$ and $N_{H_a}(K_{1,3}) = 3$. By Lemma 4.2 (iii) and (4.2),

$$2|E(H_a)| + 12N_{H_a}(P_3) + 6N_{H_a}(P_4) + 12N_{H_a}(K_{1,3}) = N_{H_a}(6) = N_{D_n}(6) = 20n + 40.$$

Solving the above equation gives $N_{H_a}(P_4) = n-1$.

Analogously to D_n, for an edge $uv \in E(H_a)$, $d(uv) \in \{0,1,2,4\}$. It is clear that uv is 0-type if and only if uv is a pendent edge; uv is 1-type if and only if $d(u)=d(v)=2$; uv is 2-type if and only if $\{d(u),d(v)\} = \{2,3\}$ and uv is 4-type if and only if $d(u)=d(v)=3$. Denote by $m_i(H_a)$ the number of i-type edges in H_a, respectively. Hence,

$$N_{H_a}(P_4) = \sum_{uv\in E(H_a)} d(uv) = m_1(H_a) + 2m_2(H_a) + 4m_4(H_a). \tag{4.5}$$

Since H_a has five pendent edges, $m_0(H_a)=5$. We claim that $m_4(H_a)\le 1$. Otherwise, from the structure of H_a we have $m_4(H_a)=2$, then $m_1(H_a)+m_2(H_a) = n-8$. Thus, by (4.5), $N_{H_a}(P_4) \ge m_1(H_a)+m_2(H_a)+4m_4(H_a) = n$, a contradiction.

Now if $m_4(H_a)=1$, we obtain

$$\begin{cases} m_1(H_a)+m_2(H_a) = n-7, \\ m_1(H_a)+2m_2(H_a) = n-5, \end{cases}$$

so, $m_1(H_a) = n-9$ and $m_2(H_a)=2$. It is easy to check that $H_a = OQ_3(1,1,1,1,1,n-8,0)$.

Finally, if $m_4(H_a)=0$, then we have

$$\begin{cases} m_1(H_a)+m_2(H_a) = n-6, \\ m_1(H_a)+2m_2(H_a) = n-1, \end{cases}$$

thus, $m_1(H_a) = n-11$ and $m_2(H_a)=5$. One can easily obtain that $H_a = OQ_3(a_1,1,1,1,1,a_6,a_7)$ $(a_1,a_6,a_7>0)$, or $OQ_3(1,1,a_3,1,1,a_6,a_7)$ $(a_3,a_6,a_7>0)$. ∎

Lemma 4.4. *Let $P_{T(c_1,c_2,c_3)}(\lambda)$ $(c_1\ge c_2\ge c_3>0)$ be the characteristic polynomial of the T-shape tree $T(c_1,c_2,c_3)$, then $P_{T(c_1,c_2,c_3)}(2) = c_1+c_2+c_3+2-c_1c_2c_3$.*

Proof. Let v be the vertex of degree 3 of $T(c_1,c_2,c_3)$. Apply Lemma 2.1 (i) at v to get

$$\begin{aligned} P_{T(c_1,c_2,c_3)}(\lambda) &= \lambda P_{P_{c_1}}(\lambda)P_{P_{c_2}}(\lambda)P_{P_{c_3}}(\lambda) - P_{P_{c_1-1}}(\lambda)P_{P_{c_2}}(\lambda)P_{P_{c_3}}(\lambda) \\ &\quad - P_{P_{c_1}}(\lambda)P_{P_{c_2-1}}(\lambda)P_{P_{c_3}}(\lambda) - P_{P_{c_1}}(\lambda)P_{P_{c_2}}(\lambda)P_{P_{c_3-1}}(\lambda). \end{aligned} \tag{4.6}$$

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By Lemma 2.7 $P_{P_r}(2) = r + 1$, substituting this into (4.6) gives $P_{T(c_1,c_2,c_3)}(2) = c_1 + c_2 + c_3 + 2 - c_1c_2c_3$. ☐

THEOREM 4.5. *Let both the open quipu $H_a^1 = OQ_3(1, 1, 1, 1, 1, n-8, 0)$ and the †-shape tree D_n $(n \geq 7)$ be of order n. Then H_a^1 is cospectral with D_n if and only if $n = 9$.*

Proof. Suppose that H_a^1 and D_n are cospectral. Let v_2 be the middle vertex of degree 3 of H_a^1. Applying Lemma 2.1 (i) at v_2 to obtain

$$\begin{aligned} P_{H_a^1}(\lambda) =& \lambda^2 P_{T(n-8,1,1)}(\lambda)P_{P_3}(\lambda) - \lambda P_{T(n-9,1,1)}(\lambda)P_{P_3}(\lambda) \\ &- P_{T(n-8,1,1)}(\lambda)P_{P_3}(\lambda) - \lambda^3 P_{T(n-8,1,1)}(\lambda). \end{aligned} \tag{4.7}$$

By Lemma 2.7 and Lemma 4.4, $P_{P_3}(2) = 4$, $P_{T(n-8,1,1)}(2) = P_{T(n-9,1,1)}(2) = 4$. Putting them into (4.7), we get $P_{H_a^1}(2) = -16$. Lemma 4.2 (i) gives $P_{D_n}(2) = -4n + 20$, thus, $-16 = -4n + 20$, i.e., $n = 9$.

Conversely, it is easy to check that $OQ_3(1, 1, 1, 1, 1, 1, 0)$ and D_9 are cospectral. ☐

THEOREM 4.6. *Let both the open quipu $H_a^2 = OQ_3(a_1, 1, 1, 1, 1, a_6, a_7)$ $(a_1, a_6, a_7 > 0)$ and the †-shape tree D_n $(n \geq 7)$ be of order n where $n = a_1 + a_6 + a_7 + 7$. Then H_a^2 is cospectral with D_n if and only if $a_1 = a_7 + 1, a_6 = 1$ and $n = 2a_7 + 9$.*

Proof. Suppose that H_a^2 and D_n are cospectral. Let v_2 be the middle vertex of degree 3 of H_a^2. Again using Lemma 2.1 (i) at v_2 we get

$$\begin{aligned} P_{H_a^2}(\lambda) =& \lambda^2 P_{T(a_1,a_6,1)}(\lambda)P_{T(a_7,1,1)}(\lambda) - P_{T(a_1,a_6,1)}(\lambda)P_{T(a_7,1,1)}(\lambda) \\ &- \lambda P_{T(a_1,a_6-1,1)}(\lambda)P_{T(a_7,1,1)}(\lambda) - \lambda P_{T(a_1,a_6,1)}(\lambda)P_{T(a_7-1,1,1)}(\lambda). \end{aligned} \tag{4.8}$$

By Lemma 4.4, $P_{T(a_1,a_6,1)}(2) = a_1 + a_6 + 3 - a_1a_6$, $P_{T(a_7,1,1)}(2) = P_{T(a_7-1,1,1)}(2) = 4$, $P_{T(a_1,a_6-1,1)}(2) = 2a_1 + a_6 + 2 - a_1a_6$, $P_{T(a_1,a_6,1)}(2) = a_1 + a_6 + 3 - a_1a_6$, substituting them into (4.8) implies $P_{H_a^2}(2) = 4a_1a_6 - 12a_1 - 4a_6 - 4$. By Lemma 4.2, $P_{D_n}(2) = -4n + 20$, thus

$$\begin{cases} -4n + 20 = 4a_1a_6 - 12a_1 - 4a_6 - 4 \\ n = a_1 + a_6 + a_7 + 7, \end{cases}$$

so, $2a_1 = a_1a_6 + a_7 + 1$. Since a_1, a_6, a_7 are positive integers, this implies $a_6 = 1, a_1 = a_7 + 1$ and $n = 2a_7 + 9$.

Conversely, one can easily verify that $OQ_3(a_7 + 1, 1, 1, 1, 1, 1, a_7)$ and D_{2a_7+9} are cospectral. ☐

THEOREM 4.7. *Let both the open quipu $H_a^3 = OQ_3(1, 1, a_3, 1, 1, a_6, a_7)$ $(a_3 > 0, a_6 \geq a_7 > 0)$ and the †-shape tree D_n $(n \geq 7)$ be of order n where $n = a_3+a_6+a_7+7$. Then H_a^3 is cospectral with D_n if and only if $a_3 = a_7+1, a_6 = 2a_7+1$ and $n = 4a_7+9$.*

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Proof. Suppose that H_a^3 and D_n are cospectral. Let v_2 be the middle vertex of degree 3 of H_a^3. Applying Lemma 2.1 (i) at v_2 to obtain

(4.9)
$$\begin{aligned}P_{H_a^3}(\lambda) =&\lambda P_{T(a_6,1,1)}(\lambda)P_{T(a_7,1,1)}(\lambda)P_{P_{a_3}}(\lambda) - P_{T(a_6-1,1,1)}(\lambda)P_{T(a_7,1,1)}(\lambda)P_{P_{a_3}}(\lambda)-\\ &P_{T(a_6,1,1)}(\lambda)P_{T(a_7-1,1,1)}(\lambda)P_{P_{a_3}}(\lambda) - P_{T(a_6,1,1)}(\lambda)P_{T(a_7,1,1)}(\lambda)P_{P_{a_3-1}}(\lambda).\end{aligned}$$

By Lemma 2.7 and Lemma 4.4, $P_{P_{a_3}}(2) = a_3 + 1$, $P_{P_{a_3-1}}(2) = a_3$, $P_{T(a_6,1,1)}(2) = P_{T(a_6-1,1,1)}(2) = P_{T(a_7,1,1)}(2) = P_{T(a_7-1,1,1)}(2) = 4$. Putting them into (4.9) we get $P_{H_a^3}(2) = -16a_3$. By Lemma 4.2, $P_{D_n}(2) = -4n + 20$, thus

$$\begin{cases}-4n + 20 = -16a_3\\ n = a_3 + a_6 + a_7 + 7,\end{cases}$$

so, $n = 4a_3 + 5$, $a_7 = 3a_3 - a_6 - 2$.

By putting $n = 4a_3 + 5$ into Lemma 4.2 (ii) we have

$$P_{D_{4a_3+5}}(\lambda) = x^{-5}(x^2 - 1)^{-1}(x^2 + 1)^2(x^{4a_3+8} - 2x^{4a_3+6} + 2x^{-4a_3+2} - x^{-4a_3}),$$

we denote by $N_1(x) = x^{4a_3+8} - 2x^{4a_3+6} + 2x^{-4a_3+2} - x^{-4a_3}$.

Similarly, from Lemma 2.7, (4.6), (4.9), by substituting $a_7 = 3a_3 - a_6 - 2$ into (4.9) one can use Maple to verify that

$$\begin{aligned}P_{H_a^3}(\lambda) =&x^{-5}(x^2 - 1)^{-1}(x^2 + 1)^2(x^{4a_3+8} - 2x^{4a_3+6} + x^{2a_3+4} - x^{4-2a_3} + 2x^{2-4a_3}\\ &- x^{-4a_3} - x^{2a_6-2a_3+6} + x^{2a_6-4a_3+6} - x^{4a_3-2a_6+2} + x^{2a_3-2a_6+2}).\end{aligned}$$

Denote by $N_2(x) = x^{4a_3+8} - 2x^{4a_3+6} + x^{2a_3+4} - x^{4-2a_3} + 2x^{2-4a_3} - x^{-4a_3} - x^{2a_6-2a_3+6} + x^{2a_6-4a_3+6} - x^{4a_3-2a_6+2} + x^{2a_3-2a_6+2}$. Since H_a^3 and D_{4a_3+5} are cospectral, $H_a^3(\lambda) = D_{4a_3+5}(\lambda)$, then

$$\begin{aligned}N_2(x) - N_1(x) =&x^{2a_3+4} - x^{4-2a_3} - x^{2a_6-2a_3+6} + x^{2a_6-4a_3+6}\\ &- x^{4a_3-2a_6+2} + x^{2a_3-2a_6+2} = 0.\end{aligned}$$

Thus, $x^{2a_3+4} - x^{2a_6-2a_3+6} = 0$ or $x^{2a_3+4} - x^{4a_3-2a_6+2} = 0$. This implies

$$\text{(a)}\begin{cases}2a_3 + 4 = 2a_6 - 2a_3 + 6\\ a_7 = 3a_3 - a_6 - 2,\end{cases} \quad \text{or} \quad \text{(b)}\begin{cases}2a_3 + 4 = 4a_3 - 2a_6 + 2\\ a_7 = 3a_3 - a_6 - 2.\end{cases}$$

Option (a) leads to $a_3 = a_7 + 1$ and $a_6 = 2a_7 + 1$. While option (b) gives $a_6 = a_3 - 1$ and $a_7 = 2a_3 - 1$, this contradicts $a_6 \geq a_7$. Thus, $a_3 = a_7 + 1$, $a_6 = 2a_7 + 1$ and $n = 4a_7 + 9$.

Conversely, it is easy to verify that $OQ_3(1, 1, a_7 + 1, 1, 1, 2a_7 + 1, a_7)$ and D_{4a_7+9} are cospectral. □

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4.2. †-shape tree and closed quipus. In this subsection, it is shown that no closed quipus union a tree are cospectral with the †-shape tree D_n.

LEMMA 4.8. *Let both $H_b = CQ_3 \cup P_r$ and the †-shape tree D_n $(n \geq 7)$ be of order n, then H_b is not cospectral with the †-shape tree D_n.*

Proof. Similar to $H_a = OQ_3$, denote by $m_i(H_b)$ $(i = 0, 1, 2, 4)$ the number of i-type edges in H_b, respectively. Since H_b has no odd cycle, from the structure of H_b we have $|E(H_b)| = n - 1$, $N_{H_b}(P_3) = n + 1$, $N_{H_b}(K_{1,3}) = 3$, $4 \leq m_0(H_b) \leq 5$ and $m_4(H_b) \leq 2$. We also have $2 \leq m_2(H_b) \leq 5$ if $m_4(H_b) = 2$; $4 \leq m_2(H_b) \leq 7$ if $m_4(H_b) = 1$; or $6 \leq m_2(H_b) \leq 9$ if $m_4(H_b) = 0$. Since

$$\begin{aligned} N_{H_b}(P_4) &= m_1(H_b) + 2m_2(H_b) + 4m_4(H_b) \\ &= (n - 1 - m_0(H_b) - m_2(H_b) - m_4(H_4)) + 2m_2(H_b) + 4m_4(H_b) \\ &= n - 1 - m_0(H_b) + m_2(H_b) + 3m_4(H_b), \end{aligned} \tag{4.10}$$

we obtain $N_{H_b}(P_4) \geq n + 2$ if $m_4(H_b) = 2$; $N_{H_b}(P_4) \geq n + 1$ if $m_4(H_b) = 1$; $N_{H_b}(P_4) \geq n$ if $m_4(H_b) = 0$. Thus $N_{H_b}(P_4) > n - 1$, by Lemma 2.6 (i),

$$\begin{aligned} N_{H_b}(6) &= 2|E(H_b)| + 12N_{H_b}(P_3) + 6N_{H_b}(P_4) + 12N_{H_b}(K_{1,3}) + 12N_{H_b}(C_6) \\ &> 2(n - 1) + 12(n + 1) + 6(n - 1) + 12 \times 3 = 20n + 40. \end{aligned}$$

Recall that $N_{D_n}(6) = 20n + 40$ and cospectral graphs must have the same number of closed walks of any given length, thus H_b is not cospectral with the †-shape tree D_n. ☐

LEMMA 4.9. *Let both $H_c = CQ_2(b_1, \ldots, b_4) \cup T(c_1, c_2, c_3)$ $(b_1 \geq b_2 \geq 0, b_3 \geq b_4 > 0, c_1 \geq c_2 \geq c_3 > 0)$ and the †-shape tree D_n $(n \geq 7)$ be of order n, then H_c is not cospectral with the †-shape tree D_n.*

Proof. Suppose that H_c and the †-shape tree D_n are cospectral. Similar to $H_a = OQ_3$, denote by $m_i(H_c)$ $(i = 0, 1, 2, 4)$ the number of i-type edges in H_2, respectively. From the structure of H_c we have $|E(H_c)| = n - 1$, $N_{H_c}(P_3) = n + 1$, $N_{H_c}(K_{1,3}) = 3$, $m_0(H_c) = 5$ and $m_4(H_2) \leq 1$. We also have $2 \leq m_2(H_c) \leq 7$ if $m_4(H_c) = 1$; or $4 \leq m_2(H_c) \leq 9$ if $m_4(H_c) = 0$.

First if $m_4(H_c) = 1$, analogous to (4.10),

$$N_{H_c}(P_4) = n - 1 - m_0(H_c) + m_2(H_c) + 3m_4(H_c) \geq n - 1 - 5 + 2 + 3 = n - 1,$$

and equality holds if and only if $m_2(H_c) = 2$, i.e., $H_c = CQ_2(n - 8, 0, 1, 1) \cup K_{1,3}$. By Lemma 2.6 (i),

$$\begin{aligned} N_{H_c}(6) &= 2|E(H_c)| + 12N_{H_c}(P_3) + 6N_{H_c}(P_4) + 12N_{H_c}(K_{1,3}) + 12N_{H_c}(C_6) \\ &= 2(n - 1) + 12(n + 1) + 6N_{H_c}(P_4) + 36 + 12N_{H_c}(C_6), \end{aligned} \tag{4.11}$$

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thus,

$$N_{H_c}(6) \geq 14n + 10 + 6(n-1) + 36 + 12N_{H_c}(C_6) = 20n + 40 + 12N_{H_c}(C_6),$$

hence, $N_{H_c}(6) = 20n + 40$ if and only if $N_{H_c}(C_6) = 0$, i.e., $H_c = CQ_2(n-8,0,1,1) \cup K_{1,3}$ $(n > 12)$. Now let w_1w_2 be the unique 4-type edge of $CQ_2(n-8,0,1,1)$, using Lemma 2.1 (ii) at w_1w_2 we get

$$P_{CQ_2(n-8,0,1,1)}(\lambda) = P_{P_{n-4}}(\lambda) - \lambda^2 P_{P_{n-8}}(\lambda) - 2\lambda^2.$$

It is easy to see that $P_{K_{1,3}}(\lambda) = \lambda^4 - 3\lambda^2$, so,

$$P_{H_c}(\lambda) = P_{CQ_2(n-8,0,1,1)}(\lambda) P_{K_{1,3}}(\lambda) = (P_{P_{n-4}}(\lambda) - \lambda^2 P_{P_{n-8}}(\lambda) - 2\lambda^2)(\lambda^4 - 3\lambda^2). \tag{4.12}$$

Substituting $P_{P_r}(2) = r + 1$ into (4.12) leads to $P_{H_c}(2) = -12n + 68$. By Lemma 4.2 (i), $P_{D_n}(2) = -4n + 20$, note that H_c and D_n are cospectral, thus $P_{H_c}(2) = P_{D_n}(2)$, i.e., $n = 6$, this contradicts $n > 12$.

Second if $m_4(H_c) = 0$, by (4.11) we see that $N_{H_c}(6) = 20n + 40$ if and only if $N_{H_c}(P_4) = (n-1)$ and $N_{H_c}(C_6) = 0$ or $N_{H_c}(P_4) = (n-3)$ and $N_{H_c}(C_6) = 1$. However, similar to (4.10),

$$N_{H_c}(P_4) = n - 1 - m_0(H_c) + m_2(H_c) \geq n - 1 - 5 + 4 = n - 2, \tag{4.13}$$

thus the cases $N_{H_c}(P_4) = (n-3)$ and $N_{H_c}(C_6) = 1$ are eliminated. Also by (4.13), we have $N_{H_c}(P_4) = n-1$ if and only if $m_2(H_c) = 5$, i.e., $H_c = CQ_2(b_1, b_2, 1, 1) \cup T(c_1, 1, 1)$ $(b_1 + b_2 > 4, c_1 \geq 2)$ or $H_c = CQ_2(b_1, b_2, b_3, 1) \cup K_{1,3}$ $(b_1 + b_2 > 4, b_3 \geq 2)$, denote by them H_c^1 and H_c^2, respectively. We continue to count the number of closed walks of length 8.

Let $H_c^1 = CQ_2(b_1, b_2, 1, 1) \cup T(c_1, 1, 1)$. We will split this into three general cases (a): $b_1 \geq b_2 = 1$ and $c_1 \geq 3$; (b): $b_1 \geq b_2 \geq 2$ and $c_1 \geq 3$ or $b_1 \geq b_2 = 1$ and $c_1 = 2$; (c): $b_1 \geq b_2 \geq 2$ and $c_1 = 2$. Then

$$N_{H_c^1}(P_5) = \begin{cases} b_1 + b_2 + 7 + c_1 - 3 + 2 = n - 1 & \text{if in case (a);} \\ b_1 + b_2 + 6 + c_1 - 3 + 2 = n - 2 & \text{if in case (b);} \\ b_1 + b_2 + 6 = n - 3 & \text{if in case (c);} \end{cases}$$

$N_{H_c^1}(K_{1,4}) = 0$, $N_{H_c^1}(T(2,1,1)) = 5$, by Lemma 2.6 (ii),

$$\begin{aligned} N_{H_c^1}(8) &= 2|E(H_c^1)| + 28N_{H_c^1}(P_3) + 32N_{H_c^1}(P_4) + 72N_{H_c^1}(K_{1,3}) + 8N_{H_c^1}(P_5) \\ &\quad + 16N_{H_c^1}(T(2,1,1)) + 16N_{H_c^1}(C_8) \\ &= \begin{cases} 70n + 282 + 16N_{H_c^1}(C_8) & \text{if in case (a);} \\ 70n + 274 + 16N_{H_c^1}(C_8) & \text{if in case (b);} \\ 70n + 266 + 16N_{H_c^1}(C_8) & \text{if in case (c).} \end{cases} \end{aligned}$$

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Note that $N_{D_n}(8) = 70n + 282$, thus, $N_{H_c^1}(8) = 70n + 282$ if and only if $b_1 \geq b_2 = 1$, $c_1 \geq 3$ and $N_{H_c^1}(8) = 0$ or $b_1 \geq b_2 \geq 2 = c_1$ and $N_{H_c^1}(8) = 1$, that is $H_c^1 \in \{CQ_2(b_1,1,1,1)\cup T(c_1,1,1)(b_1 > 5, c_1 \geq 3), CQ_2(4,2,1,1)\cup T(2,1,1), CQ_2(3,3,1,1)\cup T(2,1,1)\}$. However, direct computation shows that $CQ_2(4,2,1,1) \cup T(2,1,1)$ and $CQ_2(3,3,1,1) \cup T(2,1,1)$ are not cospectral with D_{15}. Denote $\bar{H}_c^1 = CQ_2(b_1,1,1,1)$ $\cup T(c_1,1,1)$ $(b_1 > 5, c_1 \geq 3)$, we show that $\bar{H}_c^1$ is also not cospectral with the †-shape tree D_n.

Let w_1b_2 be the 2-type edge of $CQ_2(b_1,1,1,1)$ (see Fig. 1.1 H_2, where $b_2 = b_3 = b_4 = 1$). Applying Lemma 2.1 (ii) at w_1b_2 we get

$$P_{CQ_2(b_1,1,1,1)}(\lambda) = P_{T(b_1+2,1,1)}(\lambda) - \lambda P_{b_1+2}(\lambda) - 2\lambda^2,$$

thus,

$$P_{\bar{H}_c^1}(\lambda) = (P_{T(b_1+2,1,1)}(\lambda) - \lambda P_{b_1+2}(\lambda) - 2\lambda^2)P_{T(c_1,1,1)}(\lambda). \tag{4.14}$$

Denote by $N_3(x) = -2x^{c_1+7} + 2x^{c_1+3} + 2x^{c_1-b_1+2} - x^{c_1-b_1} + x^{b_1-c_1+6} - 2x^{b_1-c_1+4} - 2x^{3-c_1} + 2x^{-1-c_1}$, from Lemma 2.7, (4.6), (4.14), $n = b_1 + c_1 + 8$ and using Maple we get

$$P_{\bar{H}_c^1}(\lambda) = x^{-4}(x^2-1)(x^2+1)^2(x^{n+2} - 2x^n + 2x^{6-n} - x^{4-n} + N_3(x)).$$

Since the leading term of $N_3(x)$ is $-2x^{c_1+7}$ or $x^{b_1-c_1+6}$ or their sum, clearly, it is not equal to zero. We obtain $N_3(x) \neq 0$. Recall that

$$P_{D_n}(\lambda) = x^{-4}(x^2-1)(x^2+1)^2(x^{n+2} - 2x^n + 2x^{6-n} - x^{4-n}),$$

thus, $P_{\bar{H}_c^1}(\lambda) \neq P_{D_n}(\lambda)$, i.e., $\bar{H}_c^1$ is not cospectral with D_n.

Let $H_c^2 = CQ_2(b_1,b_2,b_3,1) \cup T(1,1,1)$, we will also split this into three general cases (d): $b_1 \geq b_2 = 1$ and $b_3 \geq 3$; (e): $b_1 \geq b_2 = 1$ and $b_3 = 2$ or $b_1 \geq b_2 \geq 2$ and $b_3 \geq 3$; (f): $b_1 \geq b_2 \geq 2$ and $b_3 = 2$. Then

$$N_{H_c^2}(P_5) = \begin{cases} b_1 + b_3 + 9 = n + 1 & \text{if in case (d);} \\ b_1 + b_2 + b_3 + 7 = n & \text{if in case (e);} \\ b_1 + b_2 + 8 = n - 1 & \text{if in case (f);} \end{cases}$$

$N_{H_c^2}(K_{1,4}) = 0$, $N_{H_c^2}(T(2,1,1)) = 5$, by Lemma 2.6 (ii),

$$\begin{aligned} N_{H_c^2}(8) &= 2|E(H_c^2)| + 28N_{H_c^2}(P_3) + 32N_{H_c^2}(P_4) + 72N_{H_c^2}(K_{1,3}) + 8N_{H_c^2}(P_5) \\ &\quad + 16N_{H_c^2}(T(2,1,1)) + 16N_{H_c^2}(C_8) \\ &= \begin{cases} 70n + 298 + 16N_{H_c^2}(C_8) & \text{if in case (d);} \\ 70n + 290 + 16N_{H_c^2}(C_8) & \text{if in case (e);} \\ 70n + 282 + 16N_{H_c^2}(C_8) & \text{if in case (f);} \end{cases} \end{aligned}$$

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hence, $N_{H_c^2}(8) = 70n + 282$ if and only if $b_1 \geq b_2 \geq 2 = b_3$ and $N_{H_c^2}(C_8) = 0$, that is $H_c^2 = CQ_2(b_1, b_2, 2, 1) \cup K_{1,3}$ $(b_1 + b_2 > 6, b_1 \geq b_2 \geq 2)$.

Let w_2 be the right vertex of degree 3 of $CQ_2(b_1, b_2, 2, 1)$ $(b_1+b_2 > 6, b_1 \geq b_2 \geq 2)$. Applying Lemma 2.1 (i) at w_2 we get

$$\begin{aligned} P_{CQ_2(b_1,b_2,2,1)}(\lambda) =& \lambda^2 P_{T(b_1,b_2,2)}(\lambda) - P_{T(b_1,b_2,2)}(\lambda) - \lambda P_{T(b_1-1,b_2,2)}(\lambda) \\ & - \lambda P_{T(b_1,b_2-1,2)}(\lambda) - 2\lambda(\lambda^2 - 1). \end{aligned} \tag{4.15}$$

By Lemma 4.4, $P_{T(b_1,b_2,2)}(2) = b_1+b_2+4-2b_1b_2$, $P_{T(b_1-1,b_2,2)}(2) = b_1+3b_2+3-2b_1b_2$, $P_{T(b_1,b_2-1,2)}(2) = 3b_1 + b_2 + 3 - 2b_1b_2$, $P_{K_{1,3}}(2) = 4$, substituting them into (4.15) we have $P_{CQ_2(b_1,b_2,2,1)}(2) = 2b_1b_2 - 5b_1 - 5b_2 - 12$. Thus

$$P_{H_c^2}(2) = P_{CQ_2(b_1,b_2,2,1)}(2)P_{K_{1,3}}(2) = 8b_1b_2 - 20b_1 - 20b_2 - 48.$$

Recall that $P_{D_n}(2) = -4n + 20$ and $n = b_1 + b_2 + 9$, we obtain

$$8b_1b_2 - 20b_1 - 20b_2 - 48 = -4(b_1 + b_2 + 9) + 20, \tag{4.16}$$

which is equivalent to $b_1 = 2 + \frac{8}{b_2-2}$. Note that $b_1 \geq b_2$, thus (4.16) has integral solution $(b_1, b_2) = (10, 3)$ or $(6, 4)$. But it is easy to verify that $CQ_2(10, 3, 2, 1) \cup K_{1,3}$ and $CQ_2(6, 4, 2, 1) \cup K_{1,3}$ are not cospectral with D_{22}, D_{19}, respectively. □

5. †-shape tree and graphs with degree sequence $(0^1, 1^2, 2^{n-5}, 3^2)$**.** Let H_d be a graph with degree sequence $(0^1, 1^2, 2^{n-5}, 3^2)$. From Lemma 2.9 we obtain if H_d is cospectral with the †-shape tree D_n, then H_d may be $CQ_2(b_1, b_2, b_3, b_4) \cup K_1$ $(b_1 \geq b_2 \geq 0, b_3 \geq b_4 > 0)$. Furthermore, we can prove the following result.

LEMMA 5.1. *Let both $H_d = CQ_2(b_1, b_2, b_3, b_4) \cup K_1$ $(b_1 \geq b_2 \geq 0, b_3 \geq b_4 > 0)$ and the †-shape tree D_n $(n \geq 7)$ be of order n, then H_d is not cospectral with D_n.*

Proof. Similar to H_a, denote by $m_i(H_d)$ $(i = 0, 1, 2, 4)$ the number of i-type edges in H_d, respectively. From the structure of H_d we get $|E(H_d)| = n-1$, $N_{H_d}(P_3) = n+1$, $N_{H_d}(K_{1,3}) = 2$, $m_0(H_d) = 2$ and $m_4(H_d) \leq 1$. We also have $2 \leq m_2(H_d) \leq 4$ if $m_4(H_d) = 1$ or $4 \leq m_2(H_d) \leq 6$ if $m_4(H_d) = 0$. First if $m_4(H_d) = 1$, similar to (4.10),

$$N_{H_d}(P_4) = n - 1 - m_0(H_d) + m_2(H_d) + 3m_4(H_d) \geq n - 1 - 2 + 2 + 3 = n + 2.$$

Next if $m_4(H_d) = 0$, also analogous to (4.10),

$$N_{H_d}(P_4) = n - 1 - m_0(H_d) + m_2(H_d) \geq n - 1 - 2 + 4 = n + 1,$$

and equality holds if and only if $m_2(H_d) = 4$, i.e., $H_d = CQ_2(b_1, b_2, 1, 1) \cup K_1$. By Lemma 2.6 (i),

$$\begin{aligned} N_{H_d}(6) &= 2|E(H_d)| + 12N_{H_d}(P_3) + 6N_{H_d}(P_4) + 12N_{H_d}(K_{1,3}) + 12N_{H_d}(C_6) \\ &\geq 2(n-1) + 12(n+1) + 6(n+1) + 24 + 12N_{H_d}(C_6) \\ &= 20n + 40 + 12N_{H_d}(C_6). \end{aligned}$$

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Hence, $N_{H_d}(6) = 20n + 40$ if and only if $N_{H_d}(P_4) = n + 1$ and $N_{H_d}(C_6) = 0$, that is $H_d = CQ_2(b_1, b_2, 1, 1) \cup K_1$ $(b_1 + b_2 > 4)$.

Let w_2 be the right vertex of degree 3 of $CQ_2(b_1, b_2, 1, 1)$. Applying Lemma 2.1 (i) at w_2 we get

$$\begin{aligned} P_{CQ_2(b_1,b_2,1,1)}(\lambda) =& \lambda^2 P_{T(b_1,b_2,1)}(\lambda) - P_{T(b_1,b_2,1)}(\lambda) - \lambda P_{T(b_1-1,b_2,1)}(\lambda) \\ & - \lambda P_{T(b_1,b_2-1,1)}(\lambda) - 2\lambda^2. \end{aligned} \tag{5.1}$$

By Lemma 4.4, $P_{T(b_1,b_2,1)}(2) = b_1 + b_2 + 3 - b_1 b_2$, $P_{T(b_1-1,b_2,1)}(2) = b_1 + 2b_2 + 2 - b_1 b_2$ and $P_{T(b_1,b_2-1,1)}(2) = 2b_1 + b_2 + 2 - b_1 b_2$. Substituting them into (5.1) we have $P_{CQ_2(b_1,b_2,1,1)}(2) = b_1 b_2 - 3b_1 - 3b_2 - 7$. Thus,

$$P_{H_d}(2) = 2P_{CQ_2(b_1,b_2,1,1)}(2) = 2b_1 b_2 - 6b_1 - 6b_2 - 14.$$

Note that $P_{D_n}(2) = -4n + 20$ and $n = b_1 + b_2 + 5$, we obtain

$$2b_1 b_2 - 6b_1 - 6b_2 - 14 = -4(b_1 + b_2 + 5) + 20, \tag{5.2}$$

which is equivalent to $b_1 = 1 + \frac{8}{b_2 - 1}$. Recall that $b_1 \geq b_2$, thus (5.2) has integral solution $(b_1, b_2) = (9, 2)$ or $(5, 3)$. But it is easy to verify that $CQ_2(9, 2, 1, 1) \cup K_1$ and $CQ_2(5, 3, 1, 1) \cup K_1$ are not cospectral with D_{16}, D_{13}, respectively. $\square$

Now our main result Theorem 1.1 follows from Lemmas 3.1, 4.1, 4.3, Theorems 4.5, 4.6, 4.7, and Lemmas 4.8, 4.9, 5.1.

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