Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

ELA

THE POWER EQUATION*

MOSHE GOLDBERG† AND ELIAHU LEVY†

Abstract. Let $\mathcal{A}$ be a power-associative algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. The purpose of this paper is to study real-valued solutions of the *power equation* on $\mathcal{A}$, i.e., functions $f : \mathcal{A} \to \mathbb{R}$ which satisfy $f(a^k) = f(a)^k$ for all $a \in \mathcal{A}$ and $k = 1, 2, 3, \ldots$

Key words. Power-associative algebras, The power equation.

AMS subject classifications. 16B99, 17A05, 17A35, 17D05, 39B22.

1. The power equation and the radius. Let $\mathcal{A}$ be a nontrivial algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. Throughout this paper we shall assume that $\mathcal{A}$ is *power-associative*; that is, the subalgebra generated by any one element of $\mathcal{A}$ is associative, thus ensuring that powers of each element in $\mathcal{A}$ are unambiguously defined.

Let

$$f : \mathcal{A} \to \mathbb{R}$$

be a real-valued function on $\mathcal{A}$. We say that f is a *solution of the power equation on* $\mathcal{A}$ if

$$f(a^k) = f(a)^k \text{ for all } a \in \mathcal{A} \text{ and } k = 1, 2, 3, \ldots \tag{1.1}$$

Obviously, the only constant solutions of (1.1) are $f = 0$ and $f = 1$. Less trivial examples are not hard to come by:

Example 1.1. Viewing the complex numbers

$$\mathbb{C} = \{z = \alpha + i\beta : \ \alpha, \beta \in \mathbb{R}\}$$

as a 2-dimensional algebra over the reals, we see that the absolute-value function

$$|z| = \sqrt{\alpha^2 + \beta^2}$$

satisfies the power equation (1.1).

Similarly, addressing the real 4-dimensional associative algebra of the *quaternions*,

$$\mathbb{H} = \{q = \alpha + i\beta + j\gamma + k\delta : \ \alpha, \beta, \gamma, \delta \in \mathbb{R}\}, \quad i^2 = j^2 = k^2 = ijk = -1,$$

we note that the absolute-value function

$$|q| = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$$

*Received by the editors on July 13, 2011. Accepted for publication on August 14, 2011. Handling Editor: Robert Guralnick.

†Department of Mathematics, Technion–Israel Institute of Technology, Haifa 32000, Israel (goldberg@math.technion.ac.il, eliahu@math.technion.ac.il).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

is again a solution of (1.1).

In the same way, we reflect upon the real 8-dimensional alternative[1] (but not associative) algebra of the *octonions*,

$$\mathbb{O} = \{c = \alpha_1 + i_2\alpha_2 + \cdots + i_8\alpha_8 :\ \alpha_j \in \mathbb{R}\},$$

with its intricate multiplication rule (e.g., [Ba, CS]). We observe that analogously to the two previous cases, the absolute-value function

$$|c| = \sqrt{\alpha_1^2 + \cdots + \alpha_8^2}$$

is a solution of (1.1), a fact that stems from the Eight Square Theorem, [D], which implies that $|cd| = |c||d|$ for all $c, d \in \mathbb{O}$.

Example 1.2. Let $\mathbb{F}^{n\times n}$ be the algebra of $n \times n$ matrices over $\mathbb{F}$. Then clearly, the classical spectral radius

$$\rho(A) = \max\{|\lambda| :\ \lambda \in \mathbb{C} \text{ an eigenvalue of } A\}, \quad A \in \mathbb{F}^{n\times n},$$

is a solution of (1.1).

We notice that the solutions of the power equation presented in Examples 1.1 and 1.2 are all continuous and have the following three properties which will occupy us during the course of this paper:

(i) For all $a \in \mathcal{A}$,

$$f(a) \geq 0.$$

(ii) For all $a \in \mathcal{A}$ and $\alpha \in \mathbb{F}$,

$$f(\alpha a) = |\alpha| f(a).$$

(iii) f vanishes only on nilpotent elements of $\mathcal{A}$.

As we shall see in Theorem 1.2, on every finite-dimensional power-associative algebra over $\mathbb{F}$ there exists a continuous solution of (1.1) which has the above three properties. To this end we shall need the following preliminaries.

As usual, by a *minimal polynomial* of an element a in a power-associative algebra $\mathcal{A}$ over $\mathbb{F}$, we mean a monic polynomial of lowest positive degree with coefficients in $\mathbb{F}$ that annihilates a.

With this familiar definition, and taking into account that $\mathcal{A}$ may or may not have a unit element, we record:

[1] An algebra $\mathcal{A}$ is called *alternative* if the subalgebra generated by any two elements in $\mathcal{A}$ is associative; hence an alternative algebra is power-associative.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

Theorem 1.1 ([Go1, Theorem 1.1(a)]). *Let $\mathcal{A}$ be a finite-dimensional power-associative algebra over $\mathbb{F}$. Then every element in $\mathcal{A}$ possesses a unique minimal polynomial.*

Letting $\mathcal{A}$ be finite-dimensional, and denoting the minimal polynomial of an element $a \in \mathcal{A}$ by p_a, we proceed by defining the *radius* of a as

$$r(a) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ a root of } p_a\}.$$

Collecting and appropriately rephrasing three of the results in [Go1], we may now post:

Theorem 1.2 ([Go1, Theorems 2.1, 2.4, and 3.3]). *Let $\mathcal{A}$ be a finite-dimensional power-associative algebra over $\mathbb{F}$. Then:*

(a) *The radius r is a continuous solution of the power equation on $\mathcal{A}$ which has properties* (i)–(iii).

(b) *If $\mathcal{A}$ is void of nonzero nilpotent elements, then r is the only continuous solution of the power equation on $\mathcal{A}$ which has properties* (i)–(iii).

We point out that p_a, the minimal polynomial of an element a, may depend not only on a, but also on the underlying algebra. For example, [Go3, Sec. 4], fix an idempotent matrix $M \in \mathbb{F}^{n\times n}$, $M \neq I$, and consider the matrix algebra

$$\mathcal{A}_M = \{MAM : A \in \mathbb{F}^{n\times n}\} \tag{1.2}$$

with the usual matrix operations. Apparently, M is the unit element in $\mathcal{A}_M$; hence the minimal polynomial of M in $\mathcal{A}_M$ is $t - 1$. On the other hand, M is an element of the larger algebra $\mathbb{F}^{n\times n}$ whose unit is I; so it is easy to verify that the minimal polynomial of M in $\mathbb{F}^{n\times n}$ is $t^2 - t$.

This example, as well as others which can be found in [Go1], are special cases of the following, more general observation.

Theorem 1.3 ([Go1, Theorem 1.2]). *Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional power-associative algebras over $\mathbb{F}$, such that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Let a be an element of $\mathcal{A}$, and let p_a and q_a denote the minimal polynomials of a in $\mathcal{A}$ and in $\mathcal{B}$, respectively. Then either $p_a = q_a$ or $q_a(t) = tp_a(t)$.*

Unlike the minimal polynomial, the radius of an element is independent of the underlying algebra in the following sense.

Theorem 1.4 ([Go1, p. 4060]). *Let $\mathcal{A}$ and $\mathcal{B}$ be a finite-dimensional power-associative algebras over $\mathbb{F}$, such that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Let a be an element of $\mathcal{A}$. Then the radii of a in $\mathcal{A}$ and in $\mathcal{B}$ coincide.*

In light of the important role that the radius r plays in Theorem 1.2, it seems useful to determine r on various algebras.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

For example, [Go3, p. 371], since $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are void of nonzero nilpotents, Theorem 1.2(b) tells us that on these algebras the radius r is the only continuous solution of the power equation which has properties (i)–(iii). Hence, consulting Example 1.1, we conclude that on these algebras, the radius coincides with the corresponding absolute-value function.

Another example, [Go1, equation (2.2)], is obtained by considering $\mathbb{F}^{n\times n}$, and recalling that the roots of the minimal polynomial of a matrix $A \in \mathbb{F}^{n\times n}$ are its eigenvalues. Thus, on $\mathbb{F}^{n\times n}$ we have $r = \rho$ where ρ is the spectral radius. In fact, by Theorem 1.4, we get a little more; namely,

$$r = \rho$$

on any subalgebra of $\mathbb{F}^{n\times n}$ (with or without a unit element).

We realize, of course, that if $\mathcal{A}$ is an infinite-dimensional algebra, then the minimal polynomial of an element may fail to exist, in which case the definition of the radius is rendered meaningless and Theorems 1.2 and 1.4 become irrelevant.

In passing, we mention that the radius r is intimately connected with the notion of stability of continuous subnorms on finite-dimensional algebras [Go1, Sec. 3; Go3, Sec. 4], and with the well-known Spectral Radius Formula [R, Theorem 18.9; Go2, Sec. 2; Go3, Sec. 4].

2. More on continuous solutions of the power equation. We begin this section by exhibiting a simple 2-dimensional algebra on which the radius is not the only nontrivial continuous solution of (1.1) which has properties (i)–(iii).

Theorem 2.1. *Consider the 2-dimensional associative algebra*

$$\mathcal{A} = \{\alpha 1_{\mathcal{A}} + \beta b :\ \alpha, \beta \in \mathbb{F}\}, \tag{2.1}$$

where $1_{\mathcal{A}}$ is the unit in $\mathcal{A}$ and b is a fixed nonzero nilpotent element with $b^2 = 0$. Then, for each constant $\kappa \geq 0$, the function

$$f_\kappa(\alpha 1_{\mathcal{A}} + \beta b) = \begin{cases} |\alpha| e^{-\kappa\left|\frac{\beta}{\alpha}\right|}, & \alpha \neq 0, \\ 0, & \alpha = 0, \end{cases} \tag{2.2}$$

is a continuous solution of the power equation on $\mathcal{A}$ which has properties (i)–(iii).

Proof. Since $b^2 = 0$, for every element $a = \alpha 1_{\mathcal{A}} + \beta b$ in $\mathcal{A}$ we have

$$a^k = (\alpha 1_{\mathcal{A}} + \beta b)^k = \alpha^k 1_{\mathcal{A}} + k\alpha^{k-1}\beta b, \quad k = 1, 2, 3\ldots$$

So for $\alpha = 0$,

$$f_\kappa(a^k) = 0, \quad k = 1, 2, 3\ldots,$$

and for $\alpha \neq 0$,

$$f_\kappa(a^k) = f_\kappa(\alpha^k 1_{\mathcal{A}} + k\alpha^{k-1}\beta b) = |\alpha|^k e^{-\kappa k\left|\frac{\alpha^{k-1}\beta}{\alpha^k}\right|} = |\alpha|^k e^{-\kappa k\left|\frac{\beta}{\alpha}\right|}, \quad k = 1, 2, 3\ldots$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

Hence f_κ is a solution of (1.1) on $\mathcal{A}$.

As for the other properties of f_κ, we immediately see that (i) and (ii) hold. Further, we realize that $f_\kappa(\alpha 1_\mathcal{A} + \beta b) = 0$ precisely when $\alpha = 0$; i.e., f vanishes only on nilpotent elements, so (iii) holds as well. Finally, we use the fact that $\kappa \geq 0$ to obtain that for each $\beta_0 \in \mathbb{F}$,

$$|\alpha| e^{-\kappa \left|\frac{\beta}{\alpha}\right|} \to 0 \text{ as } (\alpha, \beta) \to (0, \beta_0);$$

thus f is continuous on $\mathcal{A}$, and the proof is complete. ∎

We remark that for $\kappa = 0$, the function f_κ coincides with the radius r on the algebra $\mathcal{A}$ in (2.1). Indeed, it is not hard to see that the minimal polynomial of an element $a = \alpha 1_\mathcal{A} + \beta b$ in $\mathcal{A}$ is

$$p_a(t) = \begin{cases} (t-\alpha)^2, & \beta \neq 0, \\ t - \alpha, & \beta = 0. \end{cases}$$

Consequently, α is the only root of p_a, so

$$f_0(a) = |\alpha| = r(a).$$

Although our essential concern in this paper is with *real-valued* solutions of the power equation, we observe that if κ is chosen to be a complex number with $\operatorname{Re}\kappa \geq 0$ and $\operatorname{Im}\kappa \neq 0$, then f_κ in (2.2) is a *complex-valued* continuous solution of the power equation on the algebra in (2.1) which satisfies (ii) and (iii).

Aided by Theorem 2.1, we next state:

Corollary 2.1. *Let $\mathcal{A}$ be a power-associative algebra over $\mathbb{F}$ with $\dim \mathcal{A} \geq 2$. Suppose $\mathcal{A}$ has a unit as well as a nonzero nilpotent element. Then there exists a 2-dimensional associative subalgebra of $\mathcal{A}$ on which the power equation has infinitely many continuous solutions that have properties* (i)–(iii).

Proof. Select a nilpotent element b in $\mathcal{A}$, and let m be the smallest positive integer for which $b^m = 0$. We may assume that $m = 2$; for if not, put $n = \left\lfloor \frac{m+1}{2} \right\rfloor$ and replace b by $c = b^n$. Then, $c \neq 0$ because $n < m$, and $c^2 = 0$ because $2n \geq m$.

Now, let $1_\mathcal{A}$ be the unit element in $\mathcal{A}$, and consider the 2-dimensional subalgebra

$$\mathcal{B} = \{\alpha 1_\mathcal{A} + \beta b : \ \alpha, \beta \in \mathbb{F}\}.$$

Thus, by Theorem 2.1, for every constant $\kappa \geq 0$, the function f_κ in (2.2) is a continuous solution of (1.1) on $\mathcal{B}$ with the desired properties. ∎

3. Positivity of solutions of the power equation. We commence this section by addressing the behavior of the solutions of the power equation on nilpotent elements.

Theorem 3.1. *Let $\mathcal{A}$ be a power-associative algebra over $\mathbb{F}$, and let f be a solution of the power equation. Then either f vanishes on every nilpotent element in*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

$\mathcal{A}$, or f takes the value 1 on each such element.

Proof. Let a be a nilpotent element of $\mathcal{A}$. Then $a^m = 0$ for some positive integer m, thus

$$f(a)^m = f(a^m) = f(0). \tag{3.1}$$

Further,

$$f(0) = f(0^2) = f(0)^2,$$

hence $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 0$ then by (3.1),

$$f(a)^m = 0.$$

so $f(a) = 0$. If $f(0) = 1$, then again by (3.1),

$$f(a)^m = 1;$$

thus,

$$f(a) = f(a)^m f(a) = f(a)^{m+1} = f(a^{m+1}) = f(0) = 1,$$

and the assertion is in the bag. ∎

Theorem 3.1 has two elementary consequences:

Corollary 3.1. *Let $\mathcal{A}$ be a power-associative algebra over $\mathbb{F}$, and let f be a solution of the power equation.*

(a) *If $\mathcal{A}$ consists only of nilpotent elements, then either $f = 0$ on $\mathcal{A}$ or $f = 1$ on $\mathcal{A}$.*

(b) *If f is homogeneous in the sense of property* (ii)*, then f vanishes on every nilpotent element in $\mathcal{A}$.*

Proof. Part (a) is a trivial outcome of Theorem 3.1. Part (b) follows from the same theorem and the fact that if f satisfies (ii) then $f(0) = 0$. ∎

We proceed by providing a simple condition under which all real-valued solutions of the power equation are nonnegative.

Proposition 3.1. *If $\mathcal{A}$ is a power-associative algebra over $\mathbb{F}$, and if every element in $\mathcal{A}$ is a square, then the real-valued solutions of the power equation are nonnegative.*

Proof. Let f be a real-valued solution of (1.1), and let a be an element of $\mathcal{A}$. Then $a = b^2$ for some $b \in \mathcal{A}$. So $f(a) = f(b^2) = f(b)^2 \geq 0$, and we are done. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

ELA

Example 3.1. Since every element in $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ is a square, it follows from Proposition 3.1 that all real-valued solutions of (1.1) on these algebras are nonnegative.

A more subtle variant of Proposition 3.1 is the following:

Proposition 3.2. *Let $\mathcal{A}$ be a power-associative algebra over $\mathbb{F}$. Assume that the set*

$$\mathcal{S} = \{a \in \mathcal{A} : \ a = b^2 \ for\ some\ b\ in\ \mathcal{A}\},$$

of all squares in $\mathcal{A}$, is dense in $\mathcal{A}$ with respect to some topology. Then every real-valued solution of the power equation which is continuous with respect to the above topology is nonnegative.

Proof. Let f be a continuous solution of (1.1). If $a \in \mathcal{S}$, then we can find $b \in \mathcal{A}$ such that $a = b^2$; thus, as in the previous proof, $f(a) = f(b^2) = f(b)^2 \geq 0$. If $a \in \mathcal{A} \smallsetminus \mathcal{S}$, then there exists a sequence $\{a_j\}_{j=1}^{\infty} \subset \mathcal{S}$ with $\lim a_j = a$. So by the continuity of f,

$$f(a) = \lim_{j\to\infty} f(a_j) \geq 0,$$

and the proposition follows. $\square$

Example 3.2. To illustrate Proposition 3.2, consider $\mathbb{C}^{n\times n}$, the algebra of $n \times n$ matrices over $\mathbb{C}$, and recall (e.g., [HJ, Theorem 6.4.12], [Ga, Ch. VIII, Sec. 6]) that every nonsingular matrix in $\mathbb{C}^{n\times n}$ is a square. Since the nonsingular matrices form a dense set in $\mathbb{C}^{n\times n}$, we infer that every real-valued continuous solution of (1.1) on $\mathbb{C}^{n\times n}$ is nonnegative.

We remark that the only solution of the power equation satisfying $f(a) \leq 0$ for all $a \in \mathcal{A}$ is $f = 0$. This is so because if f is a solution such that $f(a) < 0$ for some a in $\mathcal{A}$, then $f(a^2) = f(a)^2 > 0$.

4. Continuous solutions that vanish on more than nilpotent elements. Our purpose in this section is to construct a nontrivial continuous solution of the power equation which has properties (i) and (ii) but, in contrast to (iii), may vanish on more than just nilpotents.

This can be done, for example, by letting $\mathcal{A}$ be a finite-dimensional power-associative algebra over $\mathbb{F}$, and defining

$$s(a) = \min\{|\lambda| : \ \lambda \in \mathbb{C} \text{ a root of } p_a\}, \quad a \in \mathcal{A}, \tag{4.1}$$

where p_a is the minimal polynomial of a in $\mathcal{A}$. As we shall see, the function s has the desired properties.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

ELA

We note that, unlike the radius r, the function s may depend on the underlying algebra. For instance, appealing to the algebra $\mathcal{A}_M$ in (1.2), we recall that on this algebra the minimal polynomial of the idempotent matrix M is $t-1$, so $s(M)=1$. On the other hand, the minimal polynomial of M as a member of $\mathbb{F}^{n\times n}$ is t^2-t, hence in this case $s(M)=0$.

This last example is a special case of a more general phenomenon which readily follows from Theorem 1.3.

THEOREM 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional power-associative algebras over $\mathbb{F}$, such that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Let a be an element of $\mathcal{A}$, and let $s_{\mathcal{A}}(a)$ and $s_{\mathcal{B}}(a)$ denote the values of $s(a)$ in $\mathcal{A}$ and in $\mathcal{B}$, respectively. Then either $s_{\mathcal{A}}(a)=s_{\mathcal{B}}(a)$ or $s_{\mathcal{A}}(a)>s_{\mathcal{B}}(a)=0$.*

We are now ready to prove:

THEOREM 4.2. *Let $\mathcal{A}$ be a finite-dimensional power-associative algebra over $\mathbb{F}$. Then the function s in* (4.1) *is a continuous solution of the power equation on $\mathcal{A}$ which has properties* (i) *and* (ii).

Proof. We begin by showing that s is a solution of (1.1). Indeed, if $\mathcal{A}$ does not have a unit element, our job is easy. For if p is a nonzero member of $\mathbb{F}[t]$, the ring of polynomials over $\mathbb{F}$, then an element a in $\mathcal{A}$ can be substituted in p if and only if p does not have a constant term. So in this simple case, the minimal polynomial of every a in $\mathcal{A}$ is void of a constant term. Hence s vanishes everywhere in $\mathcal{A}$ and the power equation holds.

Assume therefore that $\mathcal{A}$ has a unit $1_{\mathcal{A}}$. Select $a\in\mathcal{A}$, and let $\mathcal{A}_a$ denote the subalgebra of $\mathcal{A}$ generated by a and $1_{\mathcal{A}}$. Since $\mathcal{A}$ and $\mathcal{A}_a$ have the same unit, it is not hard to confirm, [Go1, Theorem 1.2(a)], that the minimal polynomial of every element b in $\mathcal{A}_a$ coincides with the minimal polynomial of b in $\mathcal{A}$. Thus,

$$s_{\mathcal{A}}(b)=s_{\mathcal{A}_a}(b),$$

where $s_{\mathcal{A}}(b)$ and $s_{\mathcal{A}_a}(b)$ denote the values of $s(b)$ in $\mathcal{A}$ and in $\mathcal{A}_a$, respectively.

Since $\mathcal{A}_a$ is associative, we may employ the algebraic isomorphism φ in Example 1.5 of [Go1], which shows that $\mathcal{A}_a$ is algebraically isomorphic to a matrix algebra $\mathcal{A}_\varphi$, such that the roots of the minimal polynomial of each element b in $\mathcal{A}_a$ are the eigenvalues of the corresponding matrix $A_b\equiv\varphi(b)$ in $\mathcal{A}_\varphi$. Therefore,

$$s_{\mathcal{A}_a}(b)=\sigma(A_b)\quad\text{for all } b\in\mathcal{A}_a,$$

where

$$\sigma(A_b)=\min\{|\lambda|:\ \lambda\in\mathbb{C}\text{ an eigenvalue of } A_b\}.$$

Now, as φ is an algebraic isomorphism, we have, of course,

$$A_{b^k}=A_b^k,\quad k=1,2,3,\ldots$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

So recalling that

$$\sigma(A_b^k) = \sigma(A_b)^k,$$

we get,

$$s_{\mathcal{A}}(a^k) = s_{\mathcal{A}_a}(a^k) = \sigma(A_{a^k}) = \sigma(A_a^k) = \sigma(A_a)^k = s_{\mathcal{A}_a}(a)^k = s_{\mathcal{A}}(a)^k,$$

hence s satisfies (1.1) on $\mathcal{A}$.

To prove that s is continuous, select $a \in \mathcal{A}$, and let N be a norm on $\mathcal{A}$. By Lemma 2.1 in [Go1], for each $\varepsilon > 0$ there exists $\delta > 0$, such that if x is an element of $\mathcal{A}$ with $N(x-a) < \delta$, then every root λ_x of the minimal polynomial of x lies in the ε-neighborhood of some root λ_a of the minimal polynomial of a, and symmetrically, every root λ_a of the minimal polynomial of a lies in the ε-neighborhood of some root λ_x of the minimal polynomial of x. It follows that

$$s(x) < s(a) + \varepsilon \quad \text{and} \quad s(a) < s(x) + \varepsilon.$$

Consequently, $|s(x) - s(a)| < \varepsilon$, which forces the continuity of s.

Addressing property (ii), we note that if $\alpha = 0$, then there is nothing to prove; so suppose that $\alpha \neq 0$. Let p_a and $p_{\alpha a}$ be the minimal polynomials of a and αa in $\mathcal{A}$, respectively. Then obviously,

$$p_{\alpha a}(t) = p_a(\alpha^{-1}t);$$

so the roots of $p_{\alpha a}$ are those of p_a multiplied by α, and (ii) follows. □

With Theorem 4.2 at hand, it remains to discuss the set of elements on which s vanishes, and show, as promised, that this set may be larger than just the nilpotents.

We first note that since s satisfies (ii), Corollary 3.1(b) implies that s vanishes on all the nilpotent elements in $\mathcal{A}$.

Now, observe that if a is an element of $\mathcal{A}$ with minimal polynomial p_a, then $s(a) = 0$ if and only if $p_a(0) = 0$. Thus, for example, choosing $\mathcal{A} = \mathbb{F}^{n\times n}$ $(n \geq 2)$, we see that s vanishes not only on nilpotent matrices, but also on the larger set of all matrices whose spectrum contains zero.

5. Discontinuous solutions of the power equation. In this section we concern ourselves with discontinuous solutions of the power equation. We begin by offering an example of a family of such solutions which have properties (i)–(iii).

Example 5.1 (compare [GGL, p. 218]). Take $\mathbb{F}^{n\times n}$ $(n \geq 2)$. Fix a real constant $\kappa \neq 0$, and for every $A \in \mathbb{F}^{n\times n}$ define

$$g_\kappa(A) = \begin{cases} \rho(A)^{\kappa+1}\sigma(A)^{-\kappa}, & \sigma(A) > 0, \\ \rho(A), & \sigma(A) = 0, \end{cases} \tag{5.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

ELA

where $\rho(A)$ is the spectral radius of A and

$$\sigma(A) = \min\{|\lambda| : \ \lambda \in \mathbb{C} \text{ an eigenvalue of } A\}.$$

It can be easily verified that g_κ is a solution of (1.1) which satisfies (i)–(iii). Moreover, g_κ is discontinuous at every matrix A with $\rho(A) > 0$ and $\sigma(A) = 0$. This is so since if $\{A_j\}_{j=1}^{\infty}$ is a sequence of matrices whose limit is A, such that $\sigma(A_j) > 0$, then by the continuity of ρ and σ, we have

$$\lim_{j\to\infty} g_\kappa(A_j) = \begin{cases} \infty, & \kappa > 0, \\ 0, & \kappa < 0, \end{cases}$$

whereas

$$g_\kappa(A) = \rho(A).$$

We remark that g_κ is continuous at every matrix A with $\sigma(A) > 0$.

In the remainder of this section we shall fall back on $\mathbb{C}$, the algebra of the complex numbers over the reals, and construct a solution of the power equation which is discontinuous *everywhere*. For this purpose, let us consider the Cauchy functional equation

$$\psi(x + y) = \psi(x) + \psi(y), \quad x, y \in \mathbb{R}, \tag{5.2}$$

whose (real) solutions have been widely discussed in the literature (e.g., [H], [HLP, Sec. 3.20], [HR], [Bo, Sec. 20], [GL, Sec. 2]). We recall that equation (5.2) has discontinuous solutions, and that all such solutions are discontinuous everywhere and unbounded (both from below and above) on any interval in $\mathbb{R}$. we also recall that given a positive number c, one may select a discontinuous solution of (5.2) which is c-periodic. With this in mind, we register the following result whose proof is contained in that of Theorem 2.1 in [GL] when appropriate changes in terminology are made.

Theorem 5.1. *Let $\mathbb{C}$ be the 2-dimensional algebra of complex numbers over the reals. Let f be a solution of the power equation on $\mathbb{C}$, and let ψ be a discontinuous π-periodic solution of* (5.2)*. Then:*

(a) *The function*

$$g(z) = f(z)e^{\psi(\arg z)}, \quad z \in \mathbb{C},$$

(*where* $\arg z$ *denotes the principal argument of z, i.e.,* $0 \le \arg z < 2\pi$ *with* $\arg 0 = 0$) *is a solution of the power equation.*

(b) *If f possesses any of the properties* (i)–(iii)*, so does g.*

(c) *If f is continuous, then g is discontinuous everywhere in $\mathbb{C}$.*

To illustrate this theorem, note, for example, that if ψ be a discontinuous π-periodic solution of (5.2), then

$$g(z) = |z|e^{\psi(\arg z)}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

is a solution of the power equation on $\mathbb{C}$ which satisfies (i)–(iii) and has no shred of continuity.

6. Products and powers of solutions. Clearly, products and positive powers of solutions of the power equation are also solutions of this equation.

For example, let $\mathcal{A}$ be a finite-dimensional power-associative algebra over $\mathbb{F}$, and consider the radius r and the function s defined in (4.1). Then $r^\kappa s^\tau$ is a continuous nonnegative solution of the power equation on $\mathcal{A}$ for any pair of positive constants κ and τ.

Barring the trivial case where s vanishes everywhere in $\mathcal{A}$, we observe that $r^\kappa s^\tau$ possesses property (ii) if and only if $\kappa + \tau = 1$. Furthermore, $r^\kappa s^\tau$ satisfies (iii) precisely when s vanishes only on nilpotent elements. This implies, for instance, that if $\mathcal{A} = \mathbb{F}^{n\times n}$ and $\kappa + \tau \neq 1$, then $r^\kappa s^\tau$ is a continuous solution of (1.1) which satisfies (i) but neither (ii) nor (iii).

We conclude this short section by mentioning that negative powers of solutions of the power equation may also be used to construct new solutions. A suitable example is the function g_k in (5.1) which provides a discontinuous solution of (1.1) on $\mathbb{F}^{n\times n}$.

To obtain a continuous solution of a similar nature, take a finite-dimensional power-associative algebra $\mathcal{A}$, fix $\kappa \geq 0$, and for all $a \in \mathcal{A}$ define

$$h_\kappa(a) = \begin{cases} s(a)^{\kappa+1} r(a)^{-\kappa}, & s(a) > 0, \\ 0, & s(a) = 0. \end{cases}$$

Evidently, the function h_κ is a continuous solution of the power equation which satisfies (i) and (ii). As for property (iii), it holds, of course, if and only if s vanishes only on nilpotent elements.

REFERENCES

[Ba] J.C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39:145–205, 2002.

[Bo] R.P. Boas Jr. *A Primer of Real Functions*. Amer. Math. Soc., Providence, Rhode Island, 1960.

[CS] J.H. Conway and D.A. Smith. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*. A.K. Peters, Wellesley, Massachusetts, 2003.

[D] L.E. Dickson. On quaternions and their generalization and the history of the Eight Square Theorem. *Ann. of Math.*, 20:155–171, 1918/1919.

[Ga] F.R. Gantmacher. *The Theory of Matrices*, vol. 1. Chelsea Publishing Co., New York, 1960.

[Go1] M. Goldberg. Minimal polynomials and radii of elements in finite-dimensional power-associative algebras. *Trans. Amer. Math. Soc.*, 359:4055–4072, 2007.

[Go2] M. Goldberg. Radii and subnorms on finite-dimensional power-associative algebras. *Linear Multilinear Algebra*, 55:405–415, 2007.

[Go3] M. Goldberg. Stable subnorms on finite-dimensional power-associative algebras. *Electron. J. Linear Algebra*, 17:359–375, 2008.

[GGL] M. Goldberg, R. Guralnick, and W.A.J. Luxemburg. Stable subnorms II. *Linear Multilinear Algebra*, 51:209–219, 2003.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 810-821, August 2011

[GL] M. Goldberg and W.A.J. Luxemburg. Discontinuous subnorms. *Linear Multilinear Algebra*, 49:1–24, 2001.

[H] G. Hamel. Eine Basis aller Zahlen und die unstetigen Lösungen der Functionalgleichung $f(x+y) = f(x) + f(y)$. *Math. Ann.*, 60:450–462, 1905.

[HJ] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.

[HLP] G.H. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities*, 2nd edition. Cambridge University Press, Cambridge, 1952.

[HR] H. Hahn and A. Rosenthal. *Set Functions*. University of New Mexico Press, Albuquerque, 1948.

[R] W. Rudin. *Real and Complex Analysis*, 3rd edition. McGraw-Hill, New York, 1987.