ELA

THE A-LIKE MATRICES FOR A HYPERCUBE[*]

ŠTEFKO MIKLAVIČ[†] AND PAUL TERWILLIGER[‡]

Abstract. Let D denote a positive integer and let Q_D denote the graph of the D-dimensional hypercube. Let X denote the vertex set of Q_D and let $A \in \mathrm{Mat}_X(\mathbb{R})$ denote the adjacency matrix of Q_D. A matrix $B \in \mathrm{Mat}_X(\mathbb{R})$ is called A-*like* whenever both (i) $BA = AB$; (ii) for all $x, y \in X$ that are not equal or adjacent, the (x, y)-entry of B is zero. Let $\mathcal{L}$ denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ consisting of the A-like elements. The subspace $\mathcal{L}$ is decomposed into the direct sum of its symmetric part and antisymmetric part. A basis for each part is given. The dimensions of the symmetric part and antisymmetric part are $D + 1$ and $\binom{D}{2}$, respectively.

Key words. A-like matrices, Distance-regular graphs, Hypercubes.

AMS subject classifications. 05E30, 05C50.

1. Introduction. This paper is about the A-like matrices for a hypercube. The concept of an A-like matrix originated in the study of tridiagonal pairs of linear transformations [4]. In order to explain the concept, let us briefly review the notion of a tridiagonal pair. Let V denote a vector space with finite positive dimension. Roughly speaking, a tridiagonal pair on V is a pair of diagonalizable linear transformations $A : V \to V$ and $A^* : V \to V$ that each act in a block-tridiagonal fashion on the eigenspaces of the other one [4, Definition 1.1]. For a tridiagonal pair A, A^* on V, we seek an attractive basis for V consisting of eigenvectors for A. In this search, the following type of linear transformation plays a role. A linear transformation $B : V \to V$ is called A-like whenever it commutes with A and acts on the eigenspaces of A^* in the above block-tridiagonal fashion. The simultaneous diagonalization of a large collection of mutually commuting A-like linear transformations is expected to yield the desired basis. The study of A-like linear transformations has just begun and our results so far are tentative. In order to gain more insight on these transformations, in the present paper we consider the following related problem.

Let Γ denote a finite undirected graph with vertex set X. Let $A \in \mathrm{Mat}_X(\mathbb{R})$ denote the adjacency matrix of Γ. For $B \in \mathrm{Mat}_X(\mathbb{R})$, we define B to be A-*like*

$$\boxed{\textbf{ELA}}$$

whenever both

(i) $BA = AB$;

(ii) for all $x, y \in X$ that are not equal or adjacent, the (x, y)-entry of B is zero.

Let $\mathcal{L}$ denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ consisting of the A-like elements for Γ. We will discuss $\mathcal{L}$ after a few comments.

For $B \in \mathrm{Mat}_X(\mathbb{R})$, let B^t denote the transpose of B. Recall that B is *symmetric* (resp., *antisymmetric*) whenever $B^t = B$ (resp., $B^t = -B$). For a subspace $H \subseteq \mathrm{Mat}_X(\mathbb{R})$, the set of symmetric (resp., antisymmetric) matrices in H is a subspace of H called the *symmetric part* (resp., *antisymmetric part*) of H. Note that the following are equivalent: (i) H is the direct sum of its symmetric and antisymmetric part; (ii) H is closed under the transpose map.

Let $\mathcal{L}^{sym}$ (resp., $\mathcal{L}^{asym}$) denote the symmetric (resp., antisymmetric) part of $\mathcal{L}$. Note that $\mathcal{L}$ is closed under the transpose map, so the sum $\mathcal{L} = \mathcal{L}^{sym} + \mathcal{L}^{asym}$ is direct.

For a positive integer D, let Q_D denote the graph of the D-dimensional hypercube (see Section 4 for formal definitions). For $\Gamma = Q_D$, we find a basis for $\mathcal{L}^{sym}$ and $\mathcal{L}^{asym}$. In particular, we show that the dimensions of $\mathcal{L}^{sym}$ and $\mathcal{L}^{asym}$ are $D+1$ and $\binom{D}{2}$, respectively.

2. Preliminaries. Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by X. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with entries in $\mathbb{R}$ and rows indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. We refer to V as the *standard module* of $\mathrm{Mat}_X(\mathbb{R})$. For $v \in V$, let v^t denote the transpose of v. We endow V with the bilinear form $\langle u, v \rangle = u^t v$ $(u, v \in V)$. The vector space V together with $\langle \, , \, \rangle$ is a Euclidean space. For $x \in X$, let $\hat{x}$ denote the vector in V that has x-coordinate 1 and all other coordinates 0. Observe that $\{\hat{x} | x \in X\}$ is an orthonormal basis for V. For $B \in \mathrm{Mat}_X(\mathbb{R})$, we have $\langle Bu, v \rangle = \langle u, B^t v \rangle$ for all $u, v \in V$. Therefore, B is symmetric (resp., antisymmetric) if and only if $\langle Bu, v \rangle = \langle u, Bv \rangle$ (resp., $\langle Bu, v \rangle = -\langle u, Bv \rangle$) for all $u, v \in V$.

Given a subspace $U \subseteq V$, let $\mathrm{End}(U)$ denote the $\mathbb{R}$-algebra consisting of the linear transformations from U to U. We identify $\mathrm{End}(V)$ with $\mathrm{Mat}_X(\mathbb{R})$.

LEMMA 2.1. *For a subspace $U \subseteq V$ and $B \in \mathrm{End}(U)$, there exists a unique $B^\dagger \in \mathrm{End}(U)$ such that $\langle Bu, v \rangle = \langle u, B^\dagger v \rangle$ for all $u, v \in U$. We call $B^\dagger$ the* adjoint *of B relative to $\langle \, , \, \rangle$.*

Proof. By the Fischer-Riesz theorem [5, Theorem 9.18], for all $v \in U$, there exists

ELA

a unique $z_v \in U$ such that $\langle Bu, v \rangle = \langle u, z_v \rangle$ for all $u \in U$. Observe that the map $U \to U$, $v \mapsto z_v$ is an element of $\mathrm{End}(U)$ which we denote by $B^\dagger$. By construction $\langle Bu, v \rangle = \langle u, B^\dagger v \rangle$ for all $u, v \in U$. We have shown that $B^\dagger$ exists. The uniqueness of $B^\dagger$ follows from the uniqueness of the vector z_v in our preliminary remarks. $\square$

EXAMPLE 2.2. Referring to Lemma 2.1, assume $U = V$. Then $B^\dagger = B^t$.

The lemma and corollaries below follow from Lemma 2.1.

LEMMA 2.3. *Let U denote a subspace of V and fix an orthonormal basis for U. Pick $B \in \mathrm{End}(U)$ and consider the matrices representing B and $B^\dagger$ with respect to this basis. Then these matrices are transposes of each other.*

COROLLARY 2.4. *For a subspace $U \subseteq V$ and $B \in \mathrm{End}(U)$, the following* (i)–(iv) *are equivalent.*

 (i) *There exists an orthonormal basis for U with respect to which the matrix representing B is symmetric.*
 (ii) *With respect to any orthonormal basis for U the matrix representing B is symmetric.*
(iii) $\langle Bu, v \rangle = \langle u, Bv \rangle$ *for all $u, v \in U$.*
(iv) $B = B^\dagger$.

COROLLARY 2.5. *For a subspace $U \subseteq V$ and $B \in \mathrm{End}(U)$, the following* (i)–(iv) *are equivalent.*

 (i) *There exists an orthonormal basis for U with respect to which the matrix representing B is antisymmetric.*
 (ii) *With respect to any orthonormal basis for U the matrix representing B is antisymmetric.*
(iii) $\langle Bu, v \rangle = -\langle u, Bv \rangle$ *for all $u, v \in U$.*
(iv) $B = -B^\dagger$.

Motivated by Corollaries 2.4 and 2.5 we make a definition.

DEFINITION 2.6. For a subspace $U \subseteq V$ and $B \in \mathrm{End}(U)$, we call B *symmetric* (resp., *antisymmetric*) whenever the equivalent conditions (i)–(iv) hold in Corollary 2.4 (resp., Corollary 2.5).

DEFINITION 2.7. Pick a subspace $U \subseteq V$ and a subspace $H \subseteq \mathrm{End}(U)$. By the *symmetric part* of H we mean the subspace of H consisting of its symmetric elements. By the *antisymmetric part* of H we mean the subspace of H consisting of its antisymmetric elements.

We make two observations.

ELA

LEMMA 2.8. *For a subspace $U \subseteq V$ and a subspace $H \subseteq \mathrm{End}(U)$, the following (i), (ii) are equivalent.*

(i) *H is the direct sum of its symmetric and antisymmetric part.*
(ii) *H is closed under the adjoint map.*

LEMMA 2.9. *Let $B \in \mathrm{Mat}_X(\mathbb{R})$ and assume B is symmetric (resp., antisymmetric). Then for any B-invariant subspace $U \subseteq V$, the restriction of B to U is symmetric (resp., antisymmetric) in the sense of Definition 2.6.*

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $D := \max\{\partial(x,y)| \ x, y \in X\}$. For a vertex $x \in X$ and an integer $i \geq 0$, let $\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, we say Γ is *regular with valency k* whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers $0 \leq h, i, j \leq D$ and all $x, y \in X$ with $\partial(x,y) = h$, the number $p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x, y. The constants p_{ij}^h are known as the *intersection numbers* of Γ. From now on assume Γ is distance-regular with $D \geq 1$. Observe that Γ is regular with valency $k = p_{11}^0$.

We now recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We abbreviate $A = A_1$ and call this the *adjacency matrix of Γ*. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by A. By [3, Theorem 11.2.2] the matrices $\{A_i\}_{i=0}^D$ form a basis for M. We call M the *Bose-Mesner algebra of Γ*. Observe that M is commutative and semi-simple. By [3, Theorem 12.2.1] there exists a basis $\{E_i\}_{i=0}^D$ for M such that (i) $E_0 = |X|^{-1}J$, (ii) $I = \sum_{i=0}^D E_i$, (iii) $E_i^t = E_i$ $(0 \leq i \leq D)$, (iv) $E_iE_j = \delta_{ij}E_i$ $(0 \leq i, j \leq D)$, where I and J denote the identity and the all-ones matrix of $\mathrm{Mat}_X(\mathbb{R})$, respectively. The matrices $\{E_i\}_{i=0}^D$ are known as the *primitive idempotents of Γ*, and E_0 is called the *trivial* idempotent. We recall the eigenvalues of Γ. Since $\{E_i\}_{i=0}^D$ is a basis for M, there exist real scalars $\{\theta_i\}_{i=0}^D$ such that

$$A = \sum_{i=0}^D \theta_i E_i.$$

Combining this with (iv) above we obtain $AE_i = E_iA = \theta_iE_i$ for $0 \leq i \leq D$. Using (i) above we find $\theta_0 = k$. For $0 \leq i \leq D$, we call θ_i the *eigenvalue* of Γ associated with E_i. The eigenvalues $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M. For $0 \leq i \leq D$, let m_i denote the rank of E_i. We call m_i the *multiplicity* of θ_i.

ELA

By (ii)–(iv) above,

$$(2.1) \qquad V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum)}.$$

For $0 \leq i \leq D$, the space $E_i V$ is the eigenspace of A associated with θ_i.

3. The tensor product. In this section, we recall the tensor product of vectors and matrices.

For $v \in \mathbb{R}^X$ and $v' \in \mathbb{R}^{X'}$, let $v \otimes v'$ denote the vector in $\mathbb{R}^{X \times X'}$ with a (x, x')-entry equal to the x-entry of v times the x'-entry of v'. We call $v \otimes v'$ the *tensor product* of v and v'.

For $B \in \text{Mat}_X(\mathbb{R})$ and $B' \in \text{Mat}_{X'}(\mathbb{R})$, let $B \otimes B'$ denote the matrix in $\text{Mat}_{X \times X'}(\mathbb{R})$ with a $((x, x'), (y, y'))$-entry equal to the (x, y)-entry of B times the (x', y')-entry of B'. We call $B \otimes B'$ the *tensor product* of B and B'. Pick $B_1, B_2 \in \text{Mat}_X(\mathbb{R})$ and $B_1', B_2' \in \text{Mat}_{X'}(\mathbb{R})$. Then by [2, p. 107],

$$(3.1) \qquad (B_1 \otimes B_1')(B_2 \otimes B_2') = (B_1 B_2) \otimes (B_1' B_2').$$

Similarly, for $v_1, v_2 \in \mathbb{R}^X$ and $v_1', v_2' \in \mathbb{R}^{X'}$, we have

$$(3.2) \qquad \langle v_1 \otimes v_1', v_2 \otimes v_2' \rangle = \langle v_1, v_2 \rangle \langle v_1', v_2' \rangle,$$

$$(3.3) \qquad (B_1 \otimes B_1')(v_1 \otimes v_1') = (B_1 v_1) \otimes (B_1' v_1').$$

4. The hypercubes and the matrices α_i. In this section, we recall the hypercube graph and some of its basic properties. Fix a positive integer D. Let X denote the Cartesian product $\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}$ (D copies). Thus, X is the set of sequences $(x_1, x_2, \ldots, x_D)$ such that $x_i \in \{0,1\}$ for $1 \leq i \leq D$. For $x \in X$ and $1 \leq i \leq D$, let x_i denote the i-th coordinate of x. We interpret $\text{Mat}_X(\mathbb{R})$ and the standard module $V = \mathbb{R}^X$ as follows. We abbreviate $\text{Mat}_2(\mathbb{R}) = \text{Mat}_{\{0,1\}}(\mathbb{R})$ and identify $\text{Mat}_X(\mathbb{R}) = (\text{Mat}_2(\mathbb{R}))^{\otimes D}$. We abbreviate $\mathbb{R}^2 = \mathbb{R}^{\{0,1\}}$ and identify $V = (\mathbb{R}^2)^{\otimes D}$. Let Q_D denote the graph with vertex set X, and where two vertices are adjacent if and only if they differ in exactly one coordinate. We call Q_D the D-*cube* or the D-dimensional *hypercube*. The graph Q_D is connected, and for $x, y \in X$, the distance $\partial(x, y)$ is the number of coordinates at which x and y differ. The diameter of Q_D equals D. The graph Q_D is bipartite with bipartition $X = X^+ \cup X^-$, where X^+ (resp., X^-) is the set of vertices of Q_D with an even (resp., odd) number of positive coordinates. By [1, p. 261] Q_D is distance-regular.

Let $\theta_0 > \cdots > \theta_D$ denote the eigenvalues of Q_D. By [1, p. 261] these eigenvalues and their multiplicities are given by

$$\theta_i = D - 2i, \qquad m_i = \binom{D}{i} \qquad (0 \leq i \leq D).$$

ELA

DEFINITION 4.1. For $1 \leq i \leq D$, vertices $x, y \in X$ are said to be *i-adjacent* whenever they differ in the i-th coordinate and are equal in all other coordinates. Define $\alpha_i \in \mathrm{Mat}_X(\mathbb{R})$ by

$$(\alpha_i)_{xy} = \begin{cases} 1 & \text{if } x, y \text{ are } i\text{-adjacent,} \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

From Definition 4.1 we routinely obtain the following result.

LEMMA 4.2. *With reference to Definition 4.1 the following* (i)–(iii) *hold.*

   (i) $\alpha_i \alpha_j = \alpha_j \alpha_i$ $(1 \leq i, j \leq D)$;
  (ii) $\alpha_i^2 = I$ $(1 \leq i \leq D)$;
 (iii) $A = \sum_{i=1}^{D} \alpha_i$.

We now describe the $\{\alpha_i\}_{i=1}^{D}$ from another point of view.

DEFINITION 4.3. Define $\alpha \in \mathrm{Mat}_2(\mathbb{R})$ by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $\alpha^2 = \mathbb{1}$, where $\mathbb{1}$ denotes the identity in $\mathrm{Mat}_2(\mathbb{R})$.

LEMMA 4.4. *For $1 \leq i \leq D$, the matrix α_i from Definition 4.1 satisfies*

$$(4.1) \qquad \alpha_i = \mathbb{1}^{\otimes(i-1)} \otimes \alpha \otimes \mathbb{1}^{\otimes(D-i)}.$$

Proof. Using the definition of tensor product in Section 3, along with Definition 4.1 and Definition 4.3, we find that for $x, y \in X$, the (x, y)-entry of the left-hand side of (4.1) equals the (x, y)-entry of the right-hand side of (4.1). $\square$

5. The matrices α_i^*. We continue to discuss the hypercube Q_D from Section 4. In Section 4 we defined the matrices $\{\alpha_i\}_{i=1}^{D}$. We now define some matrices $\{\alpha_i^*\}_{i=1}^{D}$.

DEFINITION 5.1. For $1 \leq i \leq D$ let $\alpha_i^* \in \mathrm{Mat}_X(\mathbb{R})$, denote the diagonal matrix with (x, x)-entry

$$(\alpha_i^*)_{xx} = \begin{cases} 1 & \text{if } x_i = 0, \\ -1 & \text{if } x_i = 1 \end{cases} \qquad (x \in X).$$

From Definition 5.1 we routinely obtain the following result.

LEMMA 5.2. *With reference to Definition 5.1 the following* (i), (ii) *hold.*

   (i) $\alpha_i^* \alpha_j^* = \alpha_j^* \alpha_i^*$ $(1 \leq i, j \leq D)$;

$$\boxed{\textbf{ELA}}$$

(ii) $(\alpha_i^*)^2 = I$ $(1 \leq i \leq D)$.

We now show that the matrices $\{\alpha_i^*\}_{i=1}^D$ satisfy an analog of Lemma 4.4.

DEFINITION 5.3. Define $\alpha^* \in \mathrm{Mat}_2(\mathbb{R})$ by

$$\alpha^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that $(\alpha^*)^2 = \mathbb{1}$ and $\alpha\alpha^* = -\alpha^*\alpha$.

LEMMA 5.4. *For $1 \leq i \leq D$, the matrix α_i^* from Definition 5.1 satisfies*

(5.1) $$\alpha_i^* = \mathbb{1}^{\otimes(i-1)} \otimes \alpha^* \otimes \mathbb{1}^{\otimes(D-i)}.$$

Proof. Using the definition of tensor product in Section 3, along with Definition 5.1 and Definition 5.3, we find that for $x, y \in X$, the (x, y)-entry of the left-hand side of (5.1) equals the (x, y)-entry of the right-hand side of (5.1). $\square$

LEMMA 5.5. *With reference to Definitions 4.1 and 5.1 the following* (i), (ii) *hold.*

(i) $\alpha_i \alpha_j^* = \alpha_j^* \alpha_i$ *if $i \neq j$ $(1 \leq i, j \leq D)$;*
(ii) $\alpha_i \alpha_i^* = -\alpha_i^* \alpha_i$ $(1 \leq i \leq D)$.

Proof. Straightforward from Lemma 4.4 and Lemma 5.4, using (3.1) and $\alpha\alpha^* = -\alpha^*\alpha$. $\square$

6. An orthonormal A-eigenbasis for V. We continue to discuss the hypercube Q_D from Section 4. In this section, we display an orthonormal basis for the standard module V that consists of eigenvectors for A.

DEFINITION 6.1. Define $u, v \in \mathbb{R}^2$ by

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Observe that u, v form an orthonormal basis for $\mathbb{R}^2$.

We have a comment.

LEMMA 6.2. *The vectors u, v from Definition 6.1 satisfy*

$$\alpha u = u, \qquad \alpha v = -v, \qquad \alpha^* u = v, \qquad \alpha^* v = u,$$

where α is from Definition 4.3 and α^* is from Definition 5.3.

DEFINITION 6.3. For a subset $S \subseteq \{1, 2, \ldots, D\}$, define $w_S \in V$ by

$$w_S = w_1 \otimes w_2 \otimes \cdots \otimes w_D,$$

where

$$w_i = \begin{cases} u & \text{if } i \notin S, \\ v & \text{if } i \in S \end{cases} \quad (1 \le i \le D).$$

LEMMA 6.4. *The vectors*

$$(6.1) \qquad\qquad w_S, \qquad S \subseteq \{1, 2, \ldots, D\}$$

form an orthonormal basis for V.

Proof. The number of vectors in (6.1) is 2^D, and this number is the dimension of V. Therefore, it suffices to show that the vectors (6.1) have square norm 1 and are mutually orthogonal. But this is the case by (3.2) and the observation below Definition 6.1. $\square$

We now consider the actions of $\{\alpha_i\}_{i=1}^D$, $\{\alpha_i^*\}_{i=1}^D$ on the basis (6.1).

PROPOSITION 6.5. *For $1 \le i \le D$ and $S \subseteq \{1, 2, \ldots, D\}$, the action of α_i and α_i^* on w_S is given by*

$$\alpha_i w_S = \begin{cases} w_S & \text{if } i \notin S, \\ -w_S & \text{if } i \in S, \end{cases} \qquad \alpha_i^* w_S = \begin{cases} w_{S \cup i} & \text{if } i \notin S, \\ w_{S \setminus i} & \text{if } i \in S. \end{cases}$$

Proof. To compute $\alpha_i w_S$ use (3.3), Lemma 4.4, Lemma 6.2 and Definition 6.3. To compute $\alpha_i^* w_S$ use (3.3), Lemma 5.4, Lemma 6.2 and Definition 6.3. $\square$

COROLLARY 6.6. *For $0 \le i \le D$, the vectors*

$$(6.2) \qquad\qquad w_S, \qquad S \subseteq \{1, 2, \ldots, D\}, \qquad |S| = i$$

form a basis for $E_i V$.

Proof. In view of (2.1) and Lemma 6.4 it suffices to show that each vector w_S from (6.2) is contained in $E_i V$. Recall that $E_i V$ is the eigenspace of A for the eigenvalue $\theta_i = D - 2i$. Using Lemma 4.2(iii) and Proposition 6.5,

$$A w_S = (D - |S|) w_S - |S| w_S = (D - 2i) w_S.$$

Therefore, $w_S \in E_i V$ and the result follows. $\square$

7. A characterization of $\mathcal{L}$. We continue to discuss the hypercube Q_D from Section 4. For this graph, we now give a characterization of $\mathcal{L}$.

LEMMA 7.1. *Pick distinct integers i, j $(1 \le i, j \le D)$ and $B \in \text{Mat}_X(\mathbb{R})$. Consider the following expression:*

$$(7.1) \qquad\qquad \alpha_i^* \alpha_j^* B - \alpha_i^* B \alpha_j^* - \alpha_j^* B \alpha_i^* + B \alpha_i^* \alpha_j^*.$$

ELA

Then for $x, y \in X$, the following (i), (ii) *hold.*

(i) *The (x, y)-entry of (7.1) is equal to*

(7.2) $$((\alpha_i^*)_{xx} - (\alpha_i^*)_{yy})((\alpha_j^*)_{xx} - (\alpha_j^*)_{yy})B_{xy}.$$

(ii) *The (x, y)-entry of (7.1) is 0 whenever $x = y$ or x, y are adjacent.*

Proof. (i) Use matrix multiplication, together with the fact that the matrices α_i^* and α_j^* are diagonal.

(ii) First assume $x = y$. Then the first two factors in (7.2) are zero, so (7.2) is zero. Next assume that x, y are adjacent. Then there exists a unique integer r $(1 \leq r \leq D)$ such that x, y are r-adjacent. By assumption $i \neq j$, so $r \neq i$ or $r \neq j$. If $r \neq i$ then the first factor in (7.2) is zero. If $r \neq j$ then the second factor in (7.2) is zero. In any case (7.2) is zero. □

LEMMA 7.2. *For $B \in \mathrm{Mat}_X(\mathbb{R})$, the following* (i), (ii) *are equivalent.*

(i) *For all $x, y \in X$ that are not equal or adjacent, the (x, y)-entry of B is zero.*
(ii) *For $1 \leq i < j \leq D$,*

$$\alpha_i^* \alpha_j^* B - \alpha_i^* B \alpha_j^* - \alpha_j^* B \alpha_i^* + B \alpha_i^* \alpha_j^* = 0.$$

Proof. (i) $\rightarrow$ (ii): Immediate from Lemma 7.1.

(ii) $\rightarrow$ (i): Write $r = \partial(x, y)$ and note that $r \geq 2$. By construction x, y differ in exactly r coordinates. So there exist two distinct coordinates i, j $(i < j)$ at which x, y differ. For these values of i, j, we apply Lemma 7.1. By assumption (7.1) is zero so (7.2) is zero. But in (7.2) the first two factors are nonzero so the last factor B_{xy} is zero. □

PROPOSITION 7.3. *For $B \in \mathrm{Mat}_X(\mathbb{R})$, the following* (i), (ii) *are equivalent.*

(i) *B is A-like;*
(ii) *B commutes with A and*

$$\alpha_i^* \alpha_j^* B - \alpha_i^* B \alpha_j^* - \alpha_j^* B \alpha_i^* + B \alpha_i^* \alpha_j^* = 0 \qquad (1 \leq i < j \leq D).$$

Proof. By Lemma 7.2 and the definition of an A-like matrix. □

8. The symmetric A-like matrices for Q_D. We continue to discuss the hypercube Q_D from Section 4. For this graph, we now describe the vector space $\mathcal{L}^{sym}$. We will give a basis for $\mathcal{L}^{sym}$ and show that the dimension is $D + 1$.

LEMMA 8.1. *The following* (i)–(iii) *hold.*

(i) *$I \in \mathcal{L}^{sym}$;*

ELA

(ii) $\alpha_i \in \mathcal{L}^{sym}$ for $1 \le i \le D$;

(iii) The matrices $I, \alpha_1, \alpha_2, \ldots, \alpha_D$ are linearly independent.

Proof. (i) This is clear.

(ii) The matrix α_i is symmetric by Definition 4.1, so it suffices to show that $\alpha_i \in \mathcal{L}$. By Lemma 4.2(i),(iii) we have $\alpha_i A = A\alpha_i$. For all $x, y \in X$ that are not equal or adjacent, x and y are not i-adjacent so the (x, y)-entry of α_i is zero. We have shown $\alpha_i \in \mathcal{L}$ and the result follows.

(iii) For each matrix in this list, define the *support* to be the set of ordered pairs (x, y) of vertices such that the (x, y)-entry is nonzero. These supports are nonempty and mutually disjoint. Therefore, the matrices are linearly independent. □

LEMMA 8.2. *The following* (i), (ii) *hold for* $B \in \mathcal{L}^{sym}$.

(i) $B_{xx} = B_{yy}$ for all $x, y \in X$.
(ii) Pick $x, z \in X$ such that $\partial(x, z) = 2$, and let y, w denote the two vertices in $\Gamma(x) \cap \Gamma(z)$. Then $B_{xy} = B_{zw}$ and $B_{yz} = B_{wx}$.

Proof. (i) Since Q_D is connected we may assume without loss that x, y are adjacent. We have $BA = AB$ so $(BA)_{xy} = (AB)_{xy}$. By matrix multiplication,

$$(BA)_{xy} = \sum_{v \in X} B_{xv} A_{vy} = \sum_{v \in \Gamma(y)} B_{xv}.$$

By construction $x \in \Gamma(y)$. For all $v \in \Gamma(y) \backslash x$, we have $\partial(x, v) = 2$, so $B_{xv} = 0$. Therefore, $(BA)_{xy} = B_{xx}$. By a similar argument $(AB)_{xy} = B_{yy}$. The result follows.

(ii) Recall $BA = AB$ so $(BA)_{xz} = (AB)_{xz}$. In this equation, we expand each side using matrix multiplication and simplify the result using the fact that B is in $\mathcal{L}^{sym}$. This yields $B_{xy} + B_{wx} = B_{yz} + B_{zw}$. In the above argument we replace (x, y, z, w) by (y, z, w, x) to obtain $B_{yz} + B_{xy} = B_{zw} + B_{wx}$. Combining the above two equations we obtain $B_{xy} = B_{zw}$ and $B_{yz} = B_{wx}$. □

LEMMA 8.3. *Fix $x \in X$. Let B denote a matrix in $\mathcal{L}^{sym}$ such that $B_{xy} = 0$ for all $y \in X$. Then $B = 0$.*

Proof. For $z \in X$, we have $B_{zz} = 0$ by Lemma 8.2(i) and since $B_{xx} = 0$. We now show that $B_{zw} = 0$ for all edges zw. We proceed as follows. For the moment pick an edge zw. Since Q_D is bipartite, the distances $\partial(x, z)$ and $\partial(x, w)$ differ by 1. We claim that for all integers m ($1 \le m \le D$), $B_{zw} = 0$ for all edges zw such that $z \in \Gamma_{m-1}(x)$ and $w \in \Gamma_m(x)$. To prove the claim we use induction on m. First assume $m = 1$. Then the claim holds by the assumptions of the lemma. Next assume $m \ge 2$. Pick $v \in \Gamma_{m-2}(x) \cap \Gamma(z)$. Note that $\partial(v, w) = 2$; let u denote the unique vertex in $\Gamma(v) \cap \Gamma(w)$ other than z. Applying Lemma 8.2(ii) to v, z, w, u we find $B_{zw} = B_{vu}$.

ELA

In this equation, the right-hand side is zero by induction on m, so the left-hand side is zero, as desired. The claim is proved and the result follows. ▯

COROLLARY 8.4. *The matrices*

$$(8.1) \qquad\qquad I, \alpha_1, \alpha_2, \ldots, \alpha_D$$

form a basis for $\mathcal{L}^{sym}$. *Moreover, the dimension of* $\mathcal{L}^{sym}$ *is* $D + 1$.

Proof. By Lemma 8.1 the matrices (8.1) are linearly independent and contained in $\mathcal{L}^{sym}$. It remains to show that the matrices (8.1) span $\mathcal{L}^{sym}$. Pick $B \in \mathcal{L}^{sym}$. We show that B is in the span of (8.1). Fix $x \in X$ and define real scalars $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_D$ as follows. Define $\varepsilon_0 = B_{xx}$. For $1 \le i \le D$, define $\varepsilon_i = B_{xy}$, where $y = y_i$ is the unique vertex in X that is i-adjacent to x. We show

$$(8.2) \qquad\qquad B = \varepsilon_0 I + \sum_{i=1}^{D} \varepsilon_i \alpha_i.$$

Let C denote the left-hand side of (8.2) minus the right-hand side of (8.2), and note that $C \in \mathcal{L}^{sym}$. By construction $C_{xy} = 0$ for all $y \in X$, so $C = 0$ in view of Lemma 8.3. We have shown (8.2). Therefore, the matrices (8.1) span $\mathcal{L}^{sym}$ and the result follows. ▯

9. The antisymmetric A-like matrices for Q_D. We continue to discuss the hypercube Q_D from Section 4. For this graph, we now describe the vector space $\mathcal{L}^{asym}$. We will give a basis for $\mathcal{L}^{asym}$ and show that the dimension is $\binom{D}{2}$. We start with a comment. For $B \in \mathcal{L}$ and $0 \le i \le D$, the space $E_i V$ is B-invariant since B commutes with A.

LEMMA 9.1. *For all $B \in \mathcal{L}^{asym}$, we have $BE_0 V = 0$.*

Proof. Note that $BE_0 V \subseteq E_0 V$. Since $E_0 V$ has dimension 1, there exists $\lambda \in \mathbb{R}$ such that $(B - \lambda I)E_0 V = 0$. We show that $\lambda = 0$. Pick a nonzero $v \in E_0 V$ and note that $\langle v, v \rangle \ne 0$. Since B is antisymmetric, we get

$$\lambda \langle v, v \rangle = \langle Bv, v \rangle = -\langle v, Bv \rangle = -\lambda \langle v, v \rangle.$$

Therefore, $\lambda = 0$ and the result follows. ▯

To motivate our next result, pick $B \in \mathcal{L}^{asym}$. Note that $E_1 V$ is B-invariant. Consider the restriction $B|_{E_1 V}$. By Lemma 2.9 this restriction is contained in the antisymmetric part of $\text{End}(E_1 V)$. Denote this part by $\text{End}(E_1 V)^{asym}$ and consider the restriction map $\mathcal{L}^{asym} \to \text{End}(E_1 V)^{asym}$, $B \mapsto B|_{E_1 V}$. We show that this restriction map is an injection.

LEMMA 9.2. *The restriction map*

$$
\begin{array}{rcl}
\mathcal{L}^{asym} & \to & \mathrm{End}(E_1V)^{asym} \\
B & \mapsto & B|_{E_1V}
\end{array}
$$

is an injection.

Proof. Pick $B \in \mathcal{L}^{asym}$ such that $BE_1V = 0$. We show that $B = 0$. We will do this in steps as follows. We claim that $BE_\ell V = 0$ for $0 \le \ell \le D$. Our proof is by induction on ℓ. For $\ell = 0$, the claim follows from Lemma 9.1, and for $\ell = 1$, the claim follows from our assumptions. Next assume $\ell \ge 2$. To show $BE_\ell V = 0$, by Corollary 6.6 it suffices to show $Bw_S = 0$ for all subsets $S \subseteq \{1, 2, \ldots, D\}$ such that $|S| = \ell$. Let S be given and pick distinct $i, j \in S$ with $i < j$. Define $P = S \setminus \{i, j\}$, $Q = S \setminus i$, $R = S \setminus j$. Note that $|P| = \ell - 2$, so $w_P \in E_{\ell-2}V$ by Corollary 6.6. Similarly $w_Q \in E_{\ell-1}V$ and $w_R \in E_{\ell-1}V$. By these comments and the induction hypothesis, B vanishes on each of w_P, w_Q, w_R. By Proposition 6.5 we have $w_S = \alpha_i^* \alpha_j^* w_P$, $w_Q = \alpha_j^* w_P$, $w_R = \alpha_i^* w_P$. By Proposition 7.3,

$$
\alpha_i^* \alpha_j^* B - \alpha_i^* B \alpha_j^* - \alpha_j^* B \alpha_i^* + B \alpha_i^* \alpha_j^* = 0.
$$

In this equation, we apply each side to w_P and evaluate the result using the above comments to get $Bw_S = 0$. We have shown $BE_\ell V = 0$ and the claim is proved. It follows that $B = 0$. $\square$

We now show that $\alpha_i^* A \alpha_j^* - \alpha_j^* A \alpha_i^*$ $(1 \le i < j \le D)$ form a basis for $\mathcal{L}^{asym}$. We start with a few comments about these expressions. To simplify the notation we abbreviate

$$
(9.1) \qquad\qquad B_{ij} = \alpha_i^* A \alpha_j^* - \alpha_j^* A \alpha_i^* \qquad (1 \le i < j \le D).
$$

LEMMA 9.3. *For $1 \le i < j \le D$,*

$$
B_{ij} = 2\alpha_i^* \alpha_j^* (\alpha_i - \alpha_j).
$$

Proof. Routine using Lemma 4.2(iii), Lemma 5.2 and Lemma 5.5. $\square$

LEMMA 9.4. *The following* (i), (ii) *hold for $1 \le i < j \le D$ and $1 \le \ell \le D$.*

(i) *Assume $\ell = i$ or $\ell = j$. Then $B_{ij}\alpha_\ell = -\alpha_\ell B_{ij}$.*
(ii) *Assume $\ell \ne i$ and $\ell \ne j$. Then $B_{ij}\alpha_\ell = \alpha_\ell B_{ij}$.*

Proof. Use Lemma 4.2(i), Lemma 5.5, and Lemma 9.3. $\square$

LEMMA 9.5. *For $1 \le i < j \le D$, we have $B_{ij} \in \mathcal{L}^{asym}$.*

Proof. The matrix B_{ij} is antisymmetric by (9.1) and since each of A, $\alpha_i^* \; \alpha_j^*$ is symmetric. We show B_{ij} commutes with A. Using Lemma 4.2(iii), Lemma 9.3 and Lemma 9.4 we find

$$B_{ij}A - AB_{ij} = 2B_{ij}(\alpha_i + \alpha_j) = 4\alpha_i^*\alpha_j^*(\alpha_i^2 - \alpha_j^2) = 0,$$

with the last equality holding since $\alpha_i^2 = \alpha_j^2 = I$. Therefore, B_{ij} commutes with A.

Pick $x, y \in X$ that are not equal or adjacent. We show that the (x, y)-entry of B_{ij} is zero. By (9.1) and since α_i^*, α_j^* are diagonal,

$$(B_{ij})_{xy} = (\alpha_i^*)_{xx}A_{xy}(\alpha_j^*)_{yy} - (\alpha_j^*)_{xx}A_{xy}(\alpha_i^*)_{yy}.$$

But $A_{xy} = 0$ since x, y are not adjacent, so $(B_{ij})_{xy} = 0$. The result follows. $\square$

For $1 \leq i < j \leq D$, we now give the action of B_{ij} on the basis (6.1).

LEMMA 9.6. *For* $1 \leq i < j \leq D$ *and* $S \subseteq \{1, 2, \ldots, D\}$,

$$B_{ij}w_S = \begin{cases} -4w_{(S\cup j)\setminus i} & \text{if } i \in S \text{ and } j \notin S, \\ 4w_{(S\cup i)\setminus j} & \text{if } i \notin S \text{ and } j \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Use Proposition 6.5 and Lemma 9.3. $\square$

THEOREM 9.7. *The matrices*

$$(9.2) \qquad\qquad \alpha_i^* A \alpha_j^* - \alpha_j^* A \alpha_i^*, \qquad 1 \leq i < j \leq D$$

form a basis for $\mathcal{L}^{asym}$. *Moreover, the dimension of* $\mathcal{L}^{asym}$ *is* $\binom{D}{2}$.

Proof. The number of elements in (9.2) is equal to $\binom{D}{2}$. The elements in (9.2) are contained in $\mathcal{L}^{asym}$ by Lemma 9.5. The elements in (9.2) are linearly independent; this can be verified using Lemma 9.6 with $|S| = 1$. Therefore, the dimension of $\mathcal{L}^{asym}$ is at least $\binom{D}{2}$. To finish the proof it suffices to show that the dimension of $\mathcal{L}^{asym}$ is at most $\binom{D}{2}$. By Lemma 9.2 the dimension of $\mathcal{L}^{asym}$ is at most the dimension of $\text{End}(E_1V)^{asym}$. The dimension of $\text{End}(E_1V)^{asym}$ is $\binom{D}{2}$ since the dimension of E_1V is D. Therefore, the dimension of $\mathcal{L}^{asym}$ is at most $\binom{D}{2}$. The result follows. $\square$

COROLLARY 9.8. *The following is a basis for* $\mathcal{L}$:

$$\{I, \alpha_1, \alpha_2, \ldots, \alpha_D\} \cup \{\alpha_i^* A \alpha_j^* - \alpha_j^* A \alpha_i^* \mid 1 \leq i < j \leq D\}.$$

Moreover, the dimension of $\mathcal{L}$ *is* $1 + D + \binom{D}{2}$.

Proof. Recall that $\mathcal{L}$ is a direct sum of $\mathcal{L}^{sym}$ and $\mathcal{L}^{asym}$. The result now follows from Corollary 8.4 and Theorem 9.7. $\square$

$$\boxed{\textbf{ELA}}$$

The following result might be of independent interest.

PROPOSITION 9.9. *The restriction map*

$$
\begin{array}{ccc}
\mathcal{L}^{asym} & \to & \mathrm{End}(E_1 V)^{asym} \\
B & \mapsto & B|_{E_1 V}
\end{array}
$$

is a bijection.

Proof. By Lemma 9.2 and since the dimensions of $\mathcal{L}^{asym}$ and $\mathrm{End}(E_1 V)^{asym}$ are equal. $\square$

REFERENCES

[1] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin, Heidelberg, 1989.

[2] H. Eves. *Elementary Matrix Theory.* Allyn and Bacon Inc., Boston, 1966.

[3] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, New York, 1993.

[4] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P-and Q-polynomial association schemes. *DIMACS Workshop on Discrete Optimization*, Piscataway, NJ, USA, 167–192, 1999.

[5] S. Roman. *Advanced Linear Algebra*, 3rd edition. Graduate Texts in Mathematics 135, Springer, New York, 2008.