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COMBINATORIAL CONSIDERATIONS FOR THE NUMBER OF DISTINCT EIGENVALUES OF A MATRIX[*]

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Abstract. We address the inverse eigenvalue problem of determining the potential number of distinct eigenvalues of a real matrix based on the zero-nonzero structure of the matrix. In particular, a nonzero pattern $\mathcal{A}$ is a matrix with entries in $\{*, 0\}$. The allow sequence of distinct eigenvalues for an $n \times n$ pattern $\mathcal{A}$ is a binary vector of length n with the kth entry equal to 1 if and only if there exists a real matrix with pattern $\mathcal{A}$ having exactly k distinct eigenvalues. We develop digraph techniques for identifying properties of the allow sequence and give some general results for cycle patterns. We obtain a classification for all the star patterns according to their allow sequence. We also determine the allow sequence for each $n \times n$ irreducible pattern with $n \leq 4$.

Key words. Nonzero pattern, Digraph, Inverse eigenvalue problem, Distinct eigenvalues, Strong multiplicity property.

AMS subject classifications. 15B35, 15A18, 05C50.

1. Introduction.

Combinatorial inverse eigenvalue problems involve determining properties of eigenvalues of a matrix based on the combinatorial structure of the nonzero entries (or the signs of the entries) in the matrix. Examples include determining when a matrix pattern allows nilpotence (see potentially nilpotent patterns, e.g. [10, 22]), determining when a pattern places no restrictions on the eigenvalues (see spectrally arbitrary patterns, e.g. [19, 21]), determining when a pattern requires distinct eigenvalues (see e.g. [11, 13, 16]), or determining the refined inertia allowed by a pattern (see e.g. [2, 21]).

There is a rich literature on matrix sign patterns, with applications to economic modeling and ecology [5], qualitative stability for biological models [1, 9], as well as applications to chemistry, sociology, cellular physiology, fluid dynamics, infectious disease models, and species competition models [21]. Nonzero patterns, perhaps more basic than sign patterns, can demonstrate structural constraints on eigenvalues independent of the signs of the matrix entries. An example is the fact that if a matrix has a proper Hessenberg nonzero pattern, then the geometric multiplicity of each eigenvalue is one [14]. The current paper, as well previous work (see, e.g. [2, 6, 12, 15, 17, 22]), develops other facts about matrix eigenvalues based on the placement of the nonzero entries. These facts can contribute to knowledge about sign patterns and also provide independent insights on practical problems.

Recent work in [3] explored the minimum number of distinct eigenvalues allowed by a sign pattern. In [4], the authors extended that work, introducing the allow sequence of a sign pattern which denotes the

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possibilities for the number of distinct eigenvalues that a pattern allows. We now introduce the idea of an allow sequence for nonzero patterns. We use digraph methods and the non-symmetric strong multiplicity property [7] to develop results. With these techniques, we are able to characterize allow sequences of all $n \times n$ nonzero matrix patterns for $n \leq 4$, as well as the allow sequence for any star pattern.

In Section 2, we present some of the main definitions used in the paper. In Section 3, we extend the methods developed in [3, 4] for sign patterns to nonzero patterns to determine the sequences (or its properties) that are realizable by some irreducible nonzero pattern. We also describe the strong multiplicity property developed in [7] and its usefulness for allow sequences. In Section 4, we classify allow sequences of the nonzero star patterns and provide some examples of nonrealizable allow sequences. In Section 4, we also develop some properties of the allow sequence associated with an n-cycle digraph based on the number of vertices with loops. We show that n-cycles do not allow exactly one eigenvalue if $n > 1$ and do not allow exactly two eigenvalues if $n > 4$. We characterize the allow sequences for n-cycles with $n \leq 6$. In Section 5, we determine the allow sequences of all the $n \times n$ irreducible nonzero patterns for $n \leq 4$. In Section 6, we provide some open problems for further consideration.

2. Main definitions.

A *nonzero pattern* (resp. *sign pattern*) $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ matrix with entries in $\{*, 0\}$ (resp. $\{+, -, 0\}$). A *superpattern* of $\mathcal{A}$ is an $n \times n$ nonzero pattern $\hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ that may have additional nonzero entries; in particular, $\hat{\alpha}_{ij} \neq 0$ whenever $\alpha_{ij} \neq 0$. A real matrix $A = [a_{ij}]$ is a *realization* of a pattern $\mathcal{A}$ if $a_{ij} \neq 0$ (resp. $a_{ij} > 0$, $a_{ij} < 0$, $a_{ij} = 0$) when the corresponding entry of $\mathcal{A}$ is $*$ (resp. $+$, $-$, 0). The *qualitative class* of $\mathcal{A}$, denoted by $Q(\mathcal{A})$, is the set of all realizations of $\mathcal{A}$. We say that a pattern $\mathcal{A}$ *allows* a particular property if there exists a matrix realization in $Q(\mathcal{A})$, which has the property. A pattern $\mathcal{A}$ *requires* a property if every realization in $Q(\mathcal{A})$ has the property. Let $q(\mathcal{A}) = \min\{q(A) \mid A \in Q(\mathcal{A})\}$, where $q(A)$ denotes the number of distinct eigenvalues of A. The *allow sequence for* $\mathcal{A}$ is the binary vector $q_{\mathrm{seq}}(\mathcal{A}) = \langle q_1(\mathcal{A}), q_2(\mathcal{A}), \ldots, q_n(\mathcal{A})\rangle$, where for $1 \leq k \leq n$, $q_k(\mathcal{A}) = 1$ if and only if $q(A) = k$ for some $A \in Q(\mathcal{A})$. A binary sequence s is *realizable* as an allow sequence for nonzero (resp. sign) patterns if there is a nonzero (resp. sign) pattern $\mathcal{A}$ with $q_{\mathrm{seq}}(\mathcal{A}) = s$. For a sign pattern $\mathcal{A}$, the problem of determining $q(\mathcal{A})$ was initiated in [3], while the problem of determining $q_{\mathrm{seq}}(\mathcal{A})$ was initiated in [4]. Note that the study of a corresponding parameter $q(G)$, for symmetric matrices corresponding to an undirected graph G, has a longer history (see e.g. [8]).

For an $n \times n$ pattern $\mathcal{A} = [\alpha_{ij}]$, its *digraph* $D(\mathcal{A})$ is the directed graph with vertex set $V(D(\mathcal{A})) = \{v_1, \ldots, v_n\}$ and arc set $E(D(\mathcal{A})) = \{(v_i, v_j) : \alpha_{ij} \neq 0\}$. The digraph $D(A)$ of a real matrix $A = [a_{ij}]$ is defined similarly, where each arc (v_i, v_j) is assigned a weight equal to a_{ij}. An arc of the form (v_i, v_i) is called a *loop*.

For $D = D(\mathcal{A})$ or $D = D(A)$, a *k-cycle* in D is a directed cycle of order k (on k vertices). The *weight* of a cycle in $D(A)$ is the product of the arc weights of the cycle. A *composite cycle* of order k is a collection of vertex-disjoint directed cycles in D such that they cover precisely k vertices of D. If $U = \{\sigma_1, \ldots, \sigma_r\}$ is a composite cycle with cycles σ_i, $1 \leq i \leq r$, then we define

$$V(U) = V(\sigma_1) \cup \cdots \cup V(\sigma_r), \quad E(U) = E(\sigma_1) \cup \cdots \cup E(\sigma_r),$$

and $|U| = r$. The characteristic polynomial of a real matrix A is denoted by $p_A(z)$. If $p_A(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$, then

$$c_k = \sum_{U \in \mathcal{U}_k} (-1)^{|U|} \prod_{(v_i, v_j) \in E(U)} a_{ij}, \tag{2.1}$$

where $\mathcal{U}_k$ denotes the set of all composite cycles of D that cover precisely k vertices (see, e.g. [10]). For an $n \times n$ pattern $\mathcal{A}$, we define $c(\mathcal{A})$ to be the maximum order of a composite cycle in $D(\mathcal{A})$ (with $c(\mathcal{A}) = 0$ if $D(\mathcal{A})$ has no cycles) and extend this notation to real matrices in the obvious manner.

Two patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if one can be obtained from the other by any combination of transposition and permutation similarity. Note that patterns with isomorphic digraphs are equivalent by permutation similarity and reversing all the arcs of a digraph will give an equivalent pattern via transposition. For sign patterns, we also include negation and signature similarity. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent, we also say that $D(\mathcal{A})$ and $D(\mathcal{B})$ are *equivalent.* For convenience, we refer to cycles of $D(\mathcal{A})$ as cycles of $\mathcal{A}$. We also let $q(D) = q(\mathcal{A})$ if $D = D(\mathcal{A})$ and we refer to the allow sequence of a digraph and the allow sequence of a pattern interchangeably.

A *star digraph* on n vertices is a strongly connected digraph (with loops permitted on any of its vertices) whose underlying graph is a star. Note that graph theory terms not defined here can be found in [23]. We call a pattern $\mathcal{A}$ a *star pattern* if $D(\mathcal{A})$ is a star digraph and a *cycle pattern* if $D(\mathcal{A})$ is an n-cycle with loops permitted on any of its vertices. An $n \times n$ pattern $\mathcal{A}$ is *potentially nilpotent* if $\mathcal{A}$ allows a nilpotent matrix and *spectrally arbitrary* if for any real monic polynomial $p(z)$ of degree n there exists $A \in Q(\mathcal{A})$ such that $p_A(z) = p(z)$. Note that $q_1(\mathcal{A}) = 1$ if $\mathcal{A}$ is potentially nilpotent and $q_{\text{seq}}(\mathcal{A}) = \langle 1, \ldots, 1 \rangle$ if $\mathcal{A}$ is spectrally arbitrary. The converses of the last two implications are false, as can be seen with the pattern in Corollary 5.1(i).

3. Methods.

In this section, we briefly describe some techniques from other sources that we employ in our analysis. The techniques include digraph considerations, a property of lacunary polynomials, and a strong matrix property. The following lemma can be found in [4], applied to sign patterns, but translates directly to nonzero patterns.

Lemma 3.1. *[4] Let $\mathcal{A}$ be an $n \times n$ pattern and $c = c(\mathcal{A})$. If $c \leq n-2$, then $q_{c+1}(\mathcal{A}) = 1$ and $q_k(\mathcal{A}) = 0$ for $c+2 \leq k \leq n$. Further, $q_n(\mathcal{A}) = 1$ if and only if $c \geq n-1$.*

Vertex duplication was used in [4] to construct higher order patterns with a specified allow sequence. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern and $v_k \in V(D(\mathcal{A}))$ for some $1 \leq k \leq n$. Define the $(n+1) \times (n+1)$ sign pattern $\mathcal{B} = [\beta_{ij}]$ with $\beta_{n+1,i} = \alpha_{k,i}$, $\beta_{i,n+1} = \alpha_{i,k}$ for $1 \leq i \leq n$, $\beta_{n+1,n+1} = \alpha_{k,k}$ and $\beta_{ij} = \alpha_{ij}$ otherwise. In this case, we say that the digraph $D(\mathcal{B})$ is obtained from $D(\mathcal{A})$ by *duplicating* v_k. The following lemma can be found in [4] and also applies to nonzero patterns.

Theorem 3.2. *[4] Let $\mathcal{A}$ be an $n \times n$ pattern and suppose $v \in V(D(\mathcal{A}))$ is not incident to a loop. Let D' be obtained from $D(\mathcal{A})$ by duplicating v and let $\mathcal{B}$ be the pattern whose digraph is D'.*

(i) If $c(\mathcal{A}) \leq n-1$, then $q_k(\mathcal{B}) \geq q_k(\mathcal{A})$ for $k = 1, 2, \ldots, n$.
(ii) If $\mathcal{A}$ requires nonsingularity, then $q_{k+1}(\mathcal{B}) \geq q_k(\mathcal{A})$ for $k = 1, 2, \ldots, n$.

A version of the following theorem was proven for sign patterns in [4, Theorem 3.13] using a derivative-Jacobian method, but is stated here for nonzero patterns, focusing on only two different cycles. It gives a condition for having a repeat eigenvalue.

Theorem 3.3. *Suppose $\mathcal{A}$ is an $n \times n$ nonzero pattern whose digraph contains two cycles, of different lengths, that intersect in at least one vertex. Then for every superpattern of $\mathcal{A}$, there is a matrix realization of the superpattern having a repeated real eigenvalue.*

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The proof of the following lemma about lacunary polynomials can be found in [20] (see also [4]).

LEMMA 3.4. *[20] Let* ℓ_1, ℓ_2, n *be integers with* $0 \le \ell_1 < \ell_2 \le n$ *and*

$$p(z) = z^n + \sum_{i=1}^{n-\ell_2} p_i z^{n-i} + \sum_{i=n-\ell_1}^{n} p_i z^{n-i},$$

for some $p_i \in \mathbb{R}$ *with* $p_{n-\ell_1} \neq 0$ *and* $p_{n-\ell_2} \neq 0$. *Then,* $p(z)$ *has at least* $\ell_2 - \ell_1$ *distinct nonzero roots (in* $\mathbb{C}$*).*

The difference $\ell_2 - \ell_1$ in Lemma 3.4 is called a *lacunary gap* in the polynomial. As such, the lemma shows that the number of distinct nonzero eigenvalues of a matrix is bounded below by the largest lacunary gap of its characteristic polynomial.

Another useful tool is a strong property developed in [7]: An $n \times n$ matrix A has the *non-symmetric strong multiplicity property* (nSMP) if $X = O$ is the only solution to the equations $A \circ X = O$, $AX^T = X^T A$, and the trace conditions $\mathrm{tr}(X^T A^k) = 0$ for $0 \le k \le n-1$, with $\circ$ being the Hadamard product. A matrix having this strong property has implications for the allow sequence of the matrix pattern and its superpatterns as demonstrated in the next theorem, which follows from the bifurcation theorem and the superpattern theorem in [7, Theorems 4.10 and 4.15].

THEOREM 3.5. *Let* $\mathcal{A}$ *be an* $n \times n$ *pattern and suppose* $A \in Q(\mathcal{A})$ *has the nSMP. If* A *has a repeated real eigenvalue, then every superpattern of* $\mathcal{A}$ *allows a matrix with* k *distinct eigenvalues for every* k *with* $q(A) \le k \le n$.

Note that a sentence in [7] mentions that the nSMP can be employed to get the conclusion of Theorem 3.5 but it fails to note the needed assumption of a repeated real eigenvalue.

4. Allow sequences of star patterns and cycle patterns.

In this section, we consider patterns with specific digraph structures: star digraphs and n-cycle digraphs. Allowable sequences for sign patterns with certain star digraphs were obtained in [4]. We characterize the allow sequences of nonzero patterns with a star digraph based on the placement of any loops. We make some observations about the realizability of allow sequences with many trailing zeros, characterizing some sequences via a star digraph structure. We also develop some general results about cycle patterns, before characterizing the allow sequences of cycles up to 6 vertices, according to the number of loops. Cycle patterns are of interest in part because they are the most sparse of all irreducible patterns. Further, knowledge of both the star patterns and cycle patterns helps with the characterization of allow sequences of small order in Section 5.

4.1. Star patterns.

Star sign patterns that are spectrally arbitrary are characterized in [19]. It follows from [19] that the $n \times n$ star nonzero patterns

$$\mathcal{A}_n = \begin{bmatrix} 0 & * & \cdots & * \\ * & * & & \\ \vdots & & \ddots & \\ * & & & * \end{bmatrix} \quad \text{and} \quad \mathcal{A}_n^* = \begin{bmatrix} * & * & \cdots & * \\ * & * & & \\ \vdots & & \ddots & \\ * & & & * \end{bmatrix}, \tag{4.2}$$

are spectrally arbitrary for $n \ge 3$ (and these are the only spectrally arbitrary star nonzero patterns for $n \ge 3$). If $n = 2$, then $\mathcal{A}_2^*$ is spectrally arbitrary, but $\mathcal{A}_2$ is not. Using $\mathcal{A}_n$ and $\mathcal{A}_n^*$, the allow sequence for some star patterns can be easily determined.

LEMMA 4.1. *Let* $m, n \geq 2$ *and* $\mathcal{S} = \begin{bmatrix} \mathcal{A} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{O} \end{bmatrix}$, *where* $\mathcal{A}$ *is an* $n \times n$ *spectrally arbitrary star pattern of form (4.2),* $\mathcal{O}$ *is the* $m \times m$ *zero pattern, and*

$$\mathcal{R} = \begin{bmatrix} * & \cdots & * \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix},$$

is the $n \times m$ *pattern with nonzero entries in its first row and zeros elsewhere. Then,*

$$q_{\text{seq}}(\mathcal{S}) = \langle \underbrace{1, \ldots, 1}_{n+2}, \underbrace{0, \ldots, 0}_{m-2} \rangle.$$

Proof. Since $\mathcal{A}$ is spectrally arbitrary, $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \ldots, 1 \rangle$ and for all $1 \leq k \leq n$, there exists $A_1, A_2 \in Q(\mathcal{A})$ such that $q(A_1) = q(A_2) = k$, A_1 is singular and A_2 is nonsingular. Let $R_1 \in Q(\mathcal{R})$ be the realization where all nonzero entries are equal to 1 and let $R_2 \in Q(\mathcal{R})$ be any realization where all nonzero entries sum to 0 (such an R_2 exists since $m \geq 2$). For $i = 1, 2$, let $S_i = \begin{bmatrix} A_i & R_1 \\ R_2^T & O \end{bmatrix}$, where O is the $m \times m$ zero matrix. Then, $p_{S_i}(z) = p_A(z)\, z^m$ for $i = 1, 2$, implying that $q_k(\mathcal{S}) = 1$ for $1 \leq k \leq n+1$. Since $c(\mathcal{S}) = n+1$, by Lemma 3.1, $q_{n+2}(\mathcal{S}) = 1$ and $q_k(\mathcal{S}) = 0$ for $k \geq n+3$. ☐

The next theorem gives a complete characterization of the star nonzero patterns according to their allow sequence. For an $n \times n$ star pattern with $n \geq 3$, the *center* of the star is the vertex adjacent to all the other vertices.

THEOREM 4.2. *Let* $n \geq 2$, $0 \leq \ell \leq n$ *and* $\mathcal{S}_{n,\ell}$ *be an* $n \times n$ *star nonzero pattern with* ℓ *loops. Then,*

$$q_{\text{seq}}(\mathcal{S}_{2,\ell}) = \begin{cases} \langle 0, 1 \rangle & \text{if } \ell = 0 \\ \langle 1, 1 \rangle & \text{otherwise,} \end{cases} \qquad q_{\text{seq}}(\mathcal{S}_{3,\ell}) = \begin{cases} \langle 1, 0, 1 \rangle & \text{if } \ell = 0 \\ \langle 0, 1, 1 \rangle & \text{if } \ell = 1 \text{ and } \mathcal{S}_{3,1} \text{ has a center loop} \\ \langle 1, 1, 1 \rangle & \text{otherwise,} \end{cases}$$

and for $n \geq 4$,

$$q_{\text{seq}}(\mathcal{S}_{n,\ell}) = \begin{cases} \langle 1, 0, 1, 0, 0, \ldots 0 \rangle & \text{if } \ell = 0 \\ \langle 0, 1, 1, 0, 0, \ldots 0 \rangle & \text{if } \ell = 1 \text{ and } \mathcal{S}_{n,1} \text{ has a center loop} \\ \langle 0, 1, 1, 1, 0, 0, \ldots 0 \rangle & \text{if } \ell = 1 \text{ and } \mathcal{S}_{n,1} \text{ has no center loop} \\ \langle \underbrace{1, \ldots, 1}_{t+3}, \underbrace{0, 0, \ldots, 0}_{n-t-3} \rangle & \text{if } 2 \leq \ell \leq n-3 \text{ and } \mathcal{S}_{n,\ell} \text{ has } t \text{ non-center loops} \\ \langle 1, \ldots, 1 \rangle & \text{if } \ell \geq n-2. \end{cases}$$

Proof. We consider three cases depending on the number of loops ℓ.

Case 1. Suppose $\ell = 0$. For $n \geq 2$, if $A \in Q(\mathcal{S}_{n,0})$ is not nilpotent, then A requires two distinct nonzero eigenvalues since the trace of A is zero (and also a zero eigenvalue if $n \geq 3$). By Lemma 3.1, $q_k(A) = 0$ for $k \geq 4$. Since $\mathcal{S}_{n,0}$ is potentially nilpotent for $n \geq 3$ (and not when $n = 2$), $q_{\text{seq}}(\mathcal{S}_{2,0}) = \langle 0, 1 \rangle$ and $q_{\text{seq}}(\mathcal{S}_{n,0}) = \langle 1, 0, 1, 0, 0, \ldots, 0 \rangle$ for $n \geq 3$.

Case 2.1. Suppose $\ell = 1$ and $\mathcal{S}_{n,1}$ has a center loop. Observe that $S_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ satisfies $q(S_2) = 1$, hence, $q_{\text{seq}}(\mathcal{S}_{2,1}) = \langle 1, 1 \rangle$ by Lemma 3.1. Repeatedly applying vertex duplication (Theorem 3.2), starting with the pattern corresponding to S_2, we obtain $q_2(\mathcal{S}_{n,1}) = q_3(\mathcal{S}_{n,1}) = 1$ for $n \geq 3$. Furthermore, for $n \geq 3$, every $A \in Q(\mathcal{S}_{n,1})$ has a zero eigenvalue and nonzero trace, thus $q_1(\mathcal{S}_{n,1}) = 0$. For $n \geq 4$, since $c(\mathcal{S}_{n,1}) = 2$, by Lemma 3.1, it follows that $q_k(\mathcal{S}_{n,1}) = 0$ for $k \geq 4$.

Case 2.2. Suppose $\ell = 1$ and $\mathcal{S}_{n,1}$ has no center loop. If $n = 2$, applying a permutation similarity to S_2 in Case 2.1 gives $q_{\text{seq}}(\mathcal{S}_{2,1}) = \langle 1, 1 \rangle$. Next observe that

$$S_3 = \begin{bmatrix} 0 & 1 & 1 \\ -8/3 & 3 & 0 \\ -1/3 & 0 & 0 \end{bmatrix},$$

satisfies $q(S_3) = 1$ and has the nSMP (with a repeated eigenvalue of 1), hence, $q_{\text{seq}}(\mathcal{S}_{3,1}) = \langle 1, 1, 1 \rangle$ by Theorem 3.5. Repeatedly applying vertex duplication (Theorem 3.2) to vertex v_3, starting with the pattern corresponding to S_3, shows that $q_2(\mathcal{S}_{n,1}) = q_3(\mathcal{S}_{n,1}) = q_4(\mathcal{S}_{n,1}) = 1$ for $n \geq 4$. Furthermore, for $n \geq 4$, every $A \in Q(\mathcal{S}_{n,1})$ has a zero eigenvalue and nonzero trace, thus $q_1(\mathcal{S}_{n,1}) = 0$. For $n \geq 5$, since $c(\mathcal{S}_{n,1}) = 3$, by Lemma 3.1 it follows that $q_k(\mathcal{S}_{n,1}) = 0$ for $k \geq 5$.

Case 3. Suppose $\ell \geq 2$. Since S_2 and S_3 have the nSMP with a repeated eigenvalue of 1, the result for $n = 2$ and $n = 3$ follows by Theorem 3.5. Thus, assume $n \geq 4$ and suppose $S_{n,\ell}$ has t non-center loops (note $1 \leq t \leq n-1$). If $1 \leq t \leq n-3$ or $t = n-1$, then the result follows by Lemma 4.1 (note if $t = 1$, then $\ell \geq 2$ implies that $S_{n,\ell}$ has a center loop). It remains to consider the two cases corresponding to $t = n-2$, i.e., when $\ell = n-1$ and $\mathcal{S}_{n,n-1}$ has a center loop and when $\ell = n-2$ and $\mathcal{S}_{n,n-2}$ has no center loop.

First consider when $\ell = n-2$ and $\mathcal{S}_{n,n-2}$ has no center loop. We generalize the matrix S_3 to give a class of matrices in $Q(\mathcal{S}_{n,n-2})$ that have the nSMP and characteristic polynomial $(z-1)^n$. We follow the proof of [19, Theorems 3.1 and 3.2]. Let $\alpha_1 = \alpha_2 = 0$ and $1 < \alpha_3 < \alpha_4 < \cdots < \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = n$. Note there exists α_i satisfying these restrictions. For $2 \leq i \leq n$, let

$$\beta_i = \frac{-(\alpha_i - 1)^n}{\displaystyle\prod_{\substack{2 \leq j \leq n \\ j \neq i}} (\alpha_i - \alpha_j)},$$

and observe that $\beta_i \in \mathbb{R}$ and $\beta_i \neq 0$ since $\alpha_i \neq 1$ for $2 \leq i \leq n$ and $\alpha_i \neq \alpha_j$ for $2 \leq i < j \leq n$. Consider the matrix

$$A = \begin{bmatrix} \alpha_1 & 1 & \cdots & 1 \\ \beta_2 & \alpha_2 & & \\ \vdots & & \ddots & \\ \beta_n & & & \alpha_n \end{bmatrix}.$$

Note that A is a realization of a star pattern with $n-2$ loops and no center loop. Let $g(z) = (z-1)^n$. We claim that $p_A(z) = g(z)$. By [19, Lemma 2.1],

$$p_A(z) = \prod_{j=1}^{n} (z - \alpha_j) - \sum_{i=2}^{n} \left(\beta_i \prod_{\substack{2 \leq j \leq n \\ j \neq i}} (z - \alpha_j) \right),$$

hence, for $2 \leq k \leq n$,

$$p_A(\alpha_k) = 0 - \beta_k \prod_{\substack{2 \leq j \leq n \\ j \neq k}} (\alpha_k - \alpha_j) = (\alpha_k - 1)^n = g(\alpha_k).$$

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Since $p_A(z)$ and $g(z)$ are both monic and have the same coefficients of z^{n-1} (namely, $-n$), the polynomial $p_A(z)-g(z)$ has degree at most $n-2$. As $\alpha_2,\dots,\alpha_n$ are distinct, $p_A(z)-g(z)$ has $n-1$ distinct zeroes. Therefore, $p_A(z)=g(z)=(z-1)^n$.

We next show that A has the nSMP. Let $A\circ X=O$ for some $X=[x_{ij}]$. Then, $x_{1j}=0$ for $2\le j\le n$, $x_{i1}=0$ for $2\le i\le n$, and $x_{ii}=0$ for $3\le i\le n$. Suppose $AX^T=X^TA$ and $\operatorname{tr}(X^TA^k)=0$ for $0\le k\le n-1$. Observe that $(AX^T-X^TA)_{ij}=\alpha_i x_{ji}-\alpha_j x_{ji}$ for $i,j\ge 2$. Since $\alpha_i\ne\alpha_j$ for $2\le i<j\le n$, it follows that $x_{ij}=0$ for $2\le i,j\le n$, with i,j not both equal to 2. Also, $x_{11}=0$ since $x_{11}=(X^TA)_{1n}=(AX^T)_{1n}=\sum_{i=2}^n x_{ni}=0$. Now $\operatorname{tr}(X^T)=x_{11}+x_{22}=x_{22}$, thus $\operatorname{tr}(X^T)=0$ implies $x_{22}=0$. Hence, $X=O$ and therefore A has the nSMP. By Theorem 3.5, $q_{\text{seq}}(\mathcal{S}_{n,n-2})=\langle 1,1,\dots,1\rangle$, and since the result holds for superpatterns, this also gives the allow sequence $\langle 1,1,\dots,1\rangle$ for the case when $\ell=n-1$ and $\mathcal{S}_{n,n-1}$ has a center loop. ◻

We end this section with one more result involving star patterns, focusing on allow sequences with trailing zeros. We show that the only irreducible patterns with allow sequences of length $n\ge 4$ that have exactly $n-3$ trailing zeros must be star patterns of a certain form. We first remark that the sign pattern results [4, Remark 4.8], [4, Theorem 4.9], and [4, Corollary 4.10], also hold for nonzero patterns. We restate these three results next.

Remark 4.3. Let $\mathcal{A}$ be an $n\times n$ nonzero pattern and $1\le t\le n-1$. Then, $q_{\text{seq}}(\mathcal{A})$ has exactly t trailing zeros if and only if $c(\mathcal{A})=n-t-1$.

Theorem 4.4. *Let* $n\ge 3$ *and* $\mathcal{A}$ *be an* $n\times n$ *nonzero pattern. Then,*

(i) $q_{\text{seq}}(\mathcal{A})=\langle 1,0,0,\dots,0\rangle$ *if and only if* $D(\mathcal{A})$ *is acyclic.*
(ii) $q_{\text{seq}}(\mathcal{A})=\langle 0,1,0,0,\dots,0\rangle$ *if and only if* $D(\mathcal{A})$ *has precisely one cycle whose length is* 1.
(iii) $q_{\text{seq}}(\mathcal{A})\ne\langle 1,1,0,0,\dots,0\rangle$.

Corollary 4.5. *Given* $n\ge 3$, *sequences* $\langle s_1,s_2,0,\dots,0\rangle$ *with* $s_1,s_2\in\{0,1\}$ *are not realizable by any* $n\times n$ *irreducible nonzero pattern.*

Unlike the previous three results, the corresponding relationship between sequences and digraphs is slightly different for nonzero patterns for sequences of length $n\ge 4$ with exactly $n-3$ trailing zeros. For sign patterns, [4, Theorem 4.11] shows that the sequences $\langle s_1,s_2,1,0,\dots,0\rangle$ of length $n\ge 4$, where $s_1,s_2\in\{0,1\}$, are realizable by an $n\times n$ irreducible sign pattern if and only if $(s_1,s_2)\ne(1,1)$, and furthermore, the realizable sequences require the digraph of the sign pattern to be a star digraph. An updated result for nonzero patterns is demonstrated next.

Theorem 4.6. *Let* $n\ge 4$ *and* $s=\langle s_1,s_2,1,0,\dots,0\rangle$ *have length* n *with* $s_1,s_2\in\{0,1\}$. *Then,* $\langle 1,1,1,0,0,\dots,0\rangle$ *and* $\langle 0,0,1,0,0,\dots,0\rangle$ *are the only sequences* s *not realizable by an* $n\times n$ *irreducible nonzero pattern. Furthermore, if* $\mathcal{A}$ *is an* $n\times n$ *irreducible nonzero pattern, then*

(i) $q_{\text{seq}}(\mathcal{A})=\langle 1,0,1,0,0,\dots,0\rangle$ *if and only if* $D(\mathcal{A})$ *is a star digraph with no loops.*
(ii) $q_{\text{seq}}(\mathcal{A})=\langle 0,1,1,0,0,\dots,0\rangle$ *if and only if* $D(\mathcal{A})$ *is a star digraph with exactly one loop that is located on the central vertex.*

Proof. Suppose $q_{\text{seq}}(\mathcal{A})=\langle s_1,s_2,1,0,\dots,0\rangle$ for some $s_1,s_2\in\{0,1\}$. Then, $c(\mathcal{A})=2$ by Remark 4.3. Since $\mathcal{A}$ is irreducible, $D(\mathcal{A})$ is a star digraph with either no loops or exactly one loop on the central vertex. The result now follows by Theorem 4.2. ◻

We remark that [4, Theorem 4.11] shows that the sequence $\langle 0,0,1,0,0,\ldots,0\rangle$ is realizable for irreducible star sign patterns; however, Theorem 4.6 shows that this sequence is not realizable for any irreducible nonzero patterns, giving us the following corollary.

Corollary 4.7. *There exists an allow sequence that is realizable by an irreducible sign pattern, but not by any irreducible nonzero pattern.*

We are left with the question as to whether there is an allow sequence that is realizable by some irreducible nonzero pattern, but is not realizable by an irreducible sign pattern.

4.2. Cycle patterns. In this subsection, we obtain some general results about the allow sequence for cycle digraphs and also characterize the allow sequences of cycles up to 6 vertices according to the number of nonzero diagonal entries (loops).

Theorem 4.8. *Suppose $\mathcal{A}$ is an $n \times n$ pattern with $n \geq 3$ such that $D(\mathcal{A})$ is an n-cycle with any number of loops. Then, $\mathcal{A}$ is not potentially nilpotent, and $q(\mathcal{A}) > 1$.*

Proof. We first claim that $\mathcal{A}$ is not potentially nilpotent. If $D(\mathcal{A})$ has n loops, then $\mathcal{A}$ is not potentially nilpotent by [22, Theorem 4.6]. Suppose $D(\mathcal{A})$ has fewer than n loops. Then, $D(\mathcal{A})$ has exactly one composite n-cycle, and any matrix in $Q(\mathcal{A})$ will have a nonzero determinant. Hence, $\mathcal{A}$ is not potentially nilpotent.

Suppose $q(\mathcal{A}) = 1$. Let $A \in Q(\mathcal{A})$ be a matrix with exactly one eigenvalue λ. Note that $B = A - \lambda I$ is nilpotent and $D(B)$ is an n-cycle when ignoring loops, contradicting the fact that no pattern with such a digraph is potentially nilpotent. Therefore, $q(\mathcal{A}) > 1$. ∎

The following lemma follows from Equation (2.1) and Lemma 3.4.

Lemma 4.9. *If $n \geq 2$ and $D(\mathcal{A})$ is an n-cycle with ℓ loops, then $q(A) \geq n - \ell$ for every $A \in Q(\mathcal{A})$.*

Theorem 4.10. *Let $n \geq 2$. Suppose $D(\mathcal{A})$ is an n-cycle when ignoring loops and let $\hat{\mathcal{A}}$ be any superpattern of $\mathcal{A}$.*

- *(i) If $D(\mathcal{A})$ has no loops, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,\ldots,0,1\rangle$ and $q_n(\hat{\mathcal{A}}) = 1$.*
- *(ii) If $n \geq 3$ and $D(\mathcal{A})$ has exactly one loop, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,\ldots,0,1,1\rangle$ and $q_{n-1}(\hat{\mathcal{A}}) = 1$.*
- *(iii) Let $n \geq 4$ be even. If $D(\mathcal{A})$ has exactly two loops, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,\ldots,0,1,1,1\rangle$, and if $D(\mathcal{A})$ has more than two loops, then $q_{n-2}(\mathcal{A}) = 1$.*

Proof. Suppose $D(\mathcal{A})$ is an n-cycle. By Lemma 4.9, $q_n(\hat{\mathcal{A}}) = 1$ for every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$. If $D(\mathcal{A})$ has no loops, then the result follows from Lemma 3.1. Suppose $D(\mathcal{A})$ has exactly one loop. Let $A \in Q(\mathcal{A})$. Then, A has characteristic polynomial $p_A(z) = z^n - az^{n-1} - b$ for some nonzero $a, b \in \mathbb{R}$. By Lemma 4.9, A has at least $n-1$ distinct eigenvalues. By Theorem 3.3, every superpattern of $\mathcal{A}$ allows a pattern with exactly $n-1$ eigenvalues. Thus, $q_{\text{seq}}(\mathcal{A}) = \langle 0,\ldots,0,1,1\rangle$ and $q_{n-1}(\hat{\mathcal{A}}) = 1$ for every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$.

Suppose $n \geq 4$ is even, and $D(\mathcal{A})$ has exactly two loops. By Lemma 4.9, every realization of $\mathcal{A}$ has at least $n-2$ distinct eigenvalues, thus, $q_1(\mathcal{A}) = \cdots = q_{n-3}(\mathcal{A}) = 0$. By Lemma 4.9, $q(\mathcal{A}) \geq n-2$.

First consider

$$B = \begin{bmatrix} w & 1 & 0 & \dots & 0 \\ 0 & -w & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 \\ b & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $w = \sqrt{n/(n-2)}$ and $b = \frac{-2}{n-2}$. Note that B has eigenvalues of 1 and -1 (each with multiplicity 2) since $p_B(z) = z^n - w^2 z^{n-2} - b$. Thus, $q(B) = n-2$. We next show that B has the nSMP.

Suppose $X \circ B = 0$ and $BX^T = X^T B$ and $\operatorname{tr}(X^T B^k) = 0$ for $0 \le k \le n-1$, for some $X = [x_{ij}]$. For $3 \le i \le n-1$, $(BX^T)_{i,i+1} = x_{i+1,i+1}$ and $(X^T B)_{i,i+1} = x_{ii}$. Thus, $x_{33} = x_{44} = \cdots = x_{nn}$. Since $x_{11} = x_{22} = 0$ and $\operatorname{tr}(X^T) = 0$, this implies $x_{33} = x_{44} = \cdots = x_{nn} = 0$.

Since $BX^T = X^T B$, and B is nonderogatory, $X^T = \displaystyle\sum_{k=0}^{n-1} c_k B^k$ by [14, Theorem 3.2.4.2] for some $c_0, \dots, c_{n-1} \in \mathbb{R}$. Then, $c_0 = 0$ since $(B^k)_{nn} = 0$ for $1 \le k < n$ and $x_{nn} = 0$. Likewise, considering the $(1,n)$ entry of X^T implies $c_{n-1} = 0$ since $(B^k)_{1n} = 0$ for $k < n-1$. Thus, $X^T = \displaystyle\sum_{k=1}^{n-2} c_k B^k$. Recognizing that the trace of B^k is the number of closed walks of length k in the digraph of B, it follows that

$$\operatorname{tr}(B^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2w^k & \text{if } k \text{ is even and } k \le n-1, \\ 2w^n + bn & \text{if } k \text{ is even and } k = n. \end{cases}$$

Thus, $\operatorname{tr}(X^T B^2) = w^2 \operatorname{tr}(X^T) + c_{n-2} bn$. Now, $\operatorname{tr}(X^T) = \operatorname{tr}(X^T B^2) = 0$ implies $c_{n-2} = 0$. Iterating for $3 \le t \le n-1$, $\operatorname{tr}(X^T B^t) = w^2 \operatorname{tr}(X^T B^{t-2}) + c_{n-t} bn$ implies $c_{n-t} = 0$. Therefore, $X = 0$ and so B has the nSMP.

By Theorem 3.5, $q_{n-2}(\hat{\mathcal{B}}) = 1$ for each superpattern $\hat{\mathcal{B}}$ of $\mathcal{B} = \operatorname{sgn}(B)$. Since the spectrum of a real matrix whose digraph is an n-cycle (with loops permitted) depends only on the weights of the n-cycle and loops (i.e., it is independent of the loop locations), it follows that $q_{n-2}(\mathcal{A}) = 1$. Thus, $q_{\text{seq}}(\mathcal{A}) = \langle 0, \dots, 0, 1, 1, 1 \rangle$.

Suppose $n \ge 4$ is even and $D(\mathcal{A})$ has $\ell \ge 3$ loops. As stated earlier, n-cycles with exactly ℓ loops are all "cospectral" to one another, thus, it suffices to consider the superpattern $\mathcal{A} = [\alpha_{ij}]$ of $\mathcal{B}$, with $\alpha_{n1} \ne 0$, $\alpha_{i,i+1} \ne 0$ for $1 \le i \le n-1$, $\alpha_{ii} \ne 0$ for $1 \le i \le \ell$ and $\alpha_{ij} = 0$ otherwise. Hence, $q_{n-2}(\mathcal{A}) = 1$ by Theorem 3.5. □

Note that the conclusion of Theorem 4.10 (iii) is restricted to even n. For $n = 5$, a pattern whose digraph is a 5-cycle with two loops cannot obtain exactly $n-2$ distinct eigenvalues as demonstrated in Theorem 4.12 (and we expect the same conclusion holds for all odd n). We first observe that $q(\mathcal{A}) \ge 3$ for directed cycles with at least five vertices.

THEOREM 4.11. *Given $n \ge 5$, if $D(\mathcal{A})$ is an n-cycle with any number of loops, then $q_2(\mathcal{A}) = 0$.*

Proof. Suppose $n \ge 5$ and $D(\mathcal{A})$ is an n-cycle with ℓ loops. If $\ell \le n-3$, then $q_2(\mathcal{A}) = 0$ by Lemma 4.9. Thus, suppose $\ell \ge n-2$. Let $A \in Q(\mathcal{A})$ be a matrix with nonzero diagonal entries $a_1, \dots, a_\ell$ and a n-cycle with weight b. Note that the characteristic polynomial of A is

$$p(z) = z^{n-\ell}\prod_{i=1}^{\ell}(z-a_i) - b. \tag{4.3}$$

Suppose $q_2(A) = 1$. Then, $p(z) = (z-\lambda_1)^{n_1}(z-\lambda_2)^{n_2}$ for some $\lambda_1 \neq \lambda_2$ with $n_1 + n_2 = n$. By the chain rule, $p'(z) = (z-\lambda_1)^{n_1-1}(z-\lambda_2)^{n_2-1}(nz - n_1\lambda_2 - n_2\lambda_1)$. In particular, $p(z) + h$ has at most three critical points for any $h \in \mathbb{R}$. As such, $p(z) + b$ can have at most four real roots. It follows that there are at most three distinct nonzero numbers in $\{a_1, a_2, \ldots, a_\ell\}$ if $\ell < n$ and at most four if $\ell = n$.

Suppose $\ell \in \{n-1, n\}$. Since $n > 4$, A has at least two diagonal entries that are equal to some nonzero scalar w. In this case, the matrix $B = A - wI$ has a digraph which is a n-cycle with at most $n-2$ loops and $q_2(B) = 1$. Thus, it will be sufficient to consider digraphs with $\ell = n-2$.

Thus, suppose $\ell = n-2$. Note that if $n \geq 9$, then the fact that A may have at most three distinct nonzero diagonal entries implies that at least three of the nonzero diagonal entries of A must be equal to some nonzero scalar w. In this case, $B = A - wI$ has at most $n-3$ loops and $q_2(B) = 1$, contradicting Lemma 4.9. Therefore $5 \leq n \leq 8$. For each of these remaining cases, the characteristic polynomial is $p(z) = z^2 g(z) - b$ for some polynomial $g(z)$ and nonzero $b \in \mathbb{R}$. By Equation (4.3), the roots of $g(z)$ are real diagonal entries of A. For each $5 \leq n \leq 8$ and $1 \leq n_1 \leq \lfloor n/2 \rfloor$, we derive a contradiction by showing that $g(z)$ has a nonreal root. As $D(\mathcal{A})$ has no composite $(n-1)$-cycles, the coefficient of z, namely $(-1)^{n-1}\left(n_1\lambda_1^{n_1-1}\lambda_2^{n_2} + n_2\lambda_1^{n_1}\lambda_2^{n_2-1}\right)$, is zero in $p(z)$, hence $\lambda_2 = -\frac{n_2}{n_1}\lambda_1$ (since $\lambda_1, \lambda_2 \neq 0$).

Table 1: Form of $g(z)$ for $p(z) = z^2 g(z) + b = (z-n_1)^{n_1}(z+n_2)^{n_2}$. Each $g(z)$ has a nonreal root.

n	(n_1, n_2)	$g(z)$
5	$(1,4)$	$z^3 + 15z^2 + 80z + 160$
	$(2,3)$	$z^3 + 5z^2 - 5z - 45$
6	$(1,5)$	$z^4 + 24z^3 + 225z^2 + 1000z + 1875$
	$(2,4)$	$z^4 + 12z^3 + 36z^2 - 64z - 384$
	$(3,3)$	$z^4 - 27z^2 + 243$
7	$(1,6)$	$z^5 + 35z^4 + 504z^3 + 3780z^2 + 15120z + 27216$
	$(2,5)$	$z^5 + 21z^4 + 154z^3 + 350z^2 - 875z - 4375$
	$(3,4)$	$z^5 + 7z^4 - 21z^3 - 203z^2 + 112z + 2016$
8	$(1,7)$	$z^6 + 48z^5 + 980z^4 + 10976z^3 + 72030z^2 + 268912z + 470596$
	$(2,6)$	$z^6 + 32z^5 + 400z^4 + 2304z^3 + 4320z^2 - 13824z - 62208$
	$(3,5)$	$z^6 + 16z^5 + 52z^4 - 352z^3 - 2050z^2 + 2000z + 22500$
	$(4,4)$	$z^6 - 64z^4 + 1536z^2 - 16384$

If $\lambda_1 \in \mathbb{R}$, by scaling A, we may assume $\lambda_1 = n_1$ and $\lambda_2 = -n_2$. Then, $g(z)$ has the form specified in Table 1, but this contradicts that $g(z)$ has only real roots. Thus, $\lambda_1 \notin \mathbb{R}$ (and $n = 6$ or $n = 8$). Then, $n_1 = n_2$ and $\lambda_1 = \overline{\lambda_2} = a + bi$ for some $a, b \in \mathbb{R}$. Since $\lambda_2 = -\frac{n_2}{n_1}\lambda_1$, it follows that $a = 0$. By scaling, we may then assume $b = 1$. But then $g(z) = z^4 + 3z^2 + 3$ when $n = 6$ and $g(z) = z^6 + 4z^4 + 6z^2 + 4$ when $n = 8$, both of which have nonreal roots. Thus, $q_2(\mathcal{A}) = 0$. $\square$

THEOREM 4.12. *Suppose $D(\mathcal{A})$ is an n-cycle.*

If $n = 2$, then

$$q_{\text{seq}}(\mathcal{A}) = \begin{cases} \langle 0,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has no loops} \\ \langle 1,1\rangle & \textit{otherwise.} \end{cases}$$

If $n = 3$, then

$$q_{\text{seq}}(\mathcal{A}) = \begin{cases} \langle 0,0,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has no loops} \\ \langle 0,1,1\rangle & \textit{otherwise.} \end{cases}$$

If $n = 4$, then

$$q_{\text{seq}}(\mathcal{A}) = \begin{cases} \langle 0,0,0,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has no loops} \\ \langle 0,0,1,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has 1 loop} \\ \langle 0,1,1,1\rangle & \textit{otherwise.} \end{cases}$$

If $n = 5$, then

$$q_{\text{seq}}(\mathcal{A}) = \begin{cases} \langle 0,0,0,0,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has no loops} \\ \langle 0,0,0,1,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has 1 or 2 loops} \\ \langle 0,0,1,1,1\rangle & \textit{otherwise.} \end{cases}$$

If $n = 6$, then

$$q_{\text{seq}}(\mathcal{A}) = \begin{cases} \langle 0,0,0,0,0,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has no loops} \\ \langle 0,0,0,0,1,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has 1 loop} \\ \langle 0,0,0,1,1,1\rangle & \textit{if } D(\mathcal{A}) \textit{ has 2 or 3 loops} \\ \langle 0,0,1,1,1,1\rangle & \textit{otherwise.} \end{cases}$$

Proof. If $n = 2$, the result follows by Theorem 4.2 since a 2-cycle is a star. Suppose $n \geq 3$. By Theorem 4.8, $q_1(\mathcal{A}) = 0$. By Theorem 4.10, if $D(\mathcal{A})$ has no loops, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,\dots,0,1\rangle$, and if $D(\mathcal{A})$ has at least one loop, then $q_{n-1}(\mathcal{A}) = q_n(\mathcal{A}) = 1$. Then by Lemma 4.9, if $D(\mathcal{A})$ has exactly one loop, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,\dots,0,1,1\rangle$.

If $n = 3$, then the result follows from the noted restrictions.

If $n = 4$ and $D(\mathcal{A})$ has at least two loops, then $q_{\text{seq}}(\mathcal{A}) = \langle 0,1,1,1\rangle$ by Theorem 4.10.

Now suppose $n > 4$ and $D(\mathcal{A})$ has $\ell \geq 2$ loops. By Theorem 4.11, $q_2(\mathcal{A}) = 0$, by Lemma 4.9, $q_k(\mathcal{A}) = 0$ for $1 < k < n - \ell$, and by Theorem 4.10, $q_{n-2}(\mathcal{A}) = 1$.

Suppose $n = 5$ and $D(\mathcal{A})$ has exactly two loops. Suppose $A \in Q(\mathcal{A})$ and $D(A)$ has loops with weights a and b and an n-cycle with weight w. Then, as in Equation (4.3),

$$p_A(z) = z^3(z-a)(z-b) - w. \tag{4.4}$$

Now suppose that A has exactly 3 eigenvalues, none of which can be zero since $w \neq 0$. By scaling, we may assume the eigenvalues of A are $1, \lambda_1, \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Then, we can assume that either $p_A(z) = (z-1)(z-\lambda_1)^2(z-\lambda_2)^2$ or $p_A(z) = (z-1)^3(z-\lambda_1)(z-\lambda_2)$. Suppose $p_A(z) = (z-1)(z-\lambda_1)^2(z-\lambda_2)^2$. Since the coefficients of z and z^2 are zero by (4.4), one can solve to obtain (without loss of generality) $\lambda_1 = \overline{\lambda_2} = \frac{-2}{3} + \frac{2\sqrt{5}i}{3}$. Considering the coefficients of z^3 and z^4, one can solve for a and b, observing that $a, b \notin \mathbb{R}$. This contradicts the fact that A is a real matrix. Thus, suppose $p_A(z) = (z-1)^3(z-\lambda_1)(z-\lambda_2)$. Using the fact that the coefficients of z and z^2 are zero, one can determine that (without loss of generality) $\lambda_1 = \overline{\lambda_2} = -\frac{1}{4} + \frac{\sqrt{15}i}{12}$. Then using the coefficients of z^3 and z^4 to solve for a and b leads to a nonreal matrix A as before. Therefore, $q_3(\mathcal{A}) = 0$ and $q_{\text{seq}}(\mathcal{A}) = \langle 0,0,0,1,1\rangle$.

Suppose $n = 5$ and $D(\mathcal{A})$ has exactly three loops. With $A \in Q(\mathcal{A})$ having loops with weights $\frac{1}{2}$, and $\frac{8\sqrt{2}-3\pm 7\sqrt{16\sqrt{2}-15}}{28}$, as well as having the weight of the 5-cycle being $\frac{57-40\sqrt{2}}{49}$, then A has exactly three distinct eigenvalues, namely 1 and $\frac{4\sqrt{2}-5\pm\sqrt{408\sqrt{2}-503}}{28}$. Hence, $q_3(\mathcal{A}) = 1$. Since $A - \frac{1}{2}I$ has four nonzero entries on the diagonal and $A + I$ has five nonzero diagonal entries, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 1, 1\rangle$ if $D(\mathcal{A})$ is a 5-cycle with three or more loops.

Suppose $n = 6$. By Theorem 4.10, if $D(\mathcal{A})$ has exactly two loops, then $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 0, 1, 1, 1\rangle$ and $q_4(\mathcal{A}) = 1$ if $D(\mathcal{A})$ has more than two loops.

Suppose $D(\mathcal{A})$ has exactly three loops. We claim that $q_3(\mathcal{A}) = 0$. Case 1. Suppose $p(z) = (z - \lambda_1)^4(z - \lambda_2)(z - \lambda_3)$. By scaling we can assume $\lambda_1 = 1$. Since the coefficient of both z and z^2 will be zero in $p(z)$ with only three loops, one can deduce that (without loss of generality) $\lambda_2 = \overline{\lambda_3} = \frac{-1}{5} + \frac{\sqrt{6}}{10}i$. But then it follows that $p(z) = z^3 g(z) + \frac{1}{10}$ for $g(z) = z^3 - \frac{18}{5}z^2 + \frac{9}{2}z - 2$. However, $g(z)$ has nonreal roots, in which case some of the diagonal entries of A must be nonreal, contradicting the fact that A is a real matrix. Case 2. Suppose $p(z) = (z - \lambda_1)^3(z - \lambda_2)^2(z - \lambda_3)$. Note that the roots must be real in this case. By scaling, we can assume that $\lambda_1 = 1$. The fact that the coefficients of both z and z^2 are zero implies that $\lambda_3 = \frac{-\lambda_2}{3\lambda_2+2}$ and $\lambda_2 = \frac{-1\pm i}{2}$, contradicting that roots are real. Case 3. Suppose $p(z) = (z - \lambda_1)^2(z - \lambda_2)^2(z - \lambda_3)^2$. Once again, assume by scaling that $\lambda_1 = 1$. Given that the coefficients of z and z^2 are zero in $p(z)$, it forces λ_2 and λ_3 to be non-unit cube roots of unity. In this case, the diagonal entries of A must be roots of $z^3 - 2$, contradicting the fact that A has real entries. Therefore, $q_3(\mathcal{A}) = 0$ and $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 0, 1, 1, 1\rangle$.

Finally, suppose $n = 6$ and $D(\mathcal{A})$ has exactly four loops. With $A \in Q(\mathcal{A})$ having loops with weights $-\frac{1}{2}, 1, 1, \frac{3}{2}$, as well as having the weight of the 6-cycle being $-\frac{1}{16}$, then A has three distinct eigenvalues, namely $\frac{1}{2}, \frac{1}{2} \pm \frac{\sqrt{3}}{2}$. Hence, $q_3(\mathcal{A}) = 1$. Since $A + \frac{1}{2}I$ has five nonzero entries on the diagonal and $A + I$ has six nonzero diagonal entries, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 0, 1, 1, 1\rangle$ if $D(\mathcal{A})$ is a 6-cycle with four or more loops. $\square$

5. Allow sequences of patterns of order at most four. In this section, we completely characterize the allow sequences of $n \times n$ irreducible nonzero patterns for $2 \leq n \leq 4$. For 3×3 and 4×4 irreducible nonzero patterns, we consider their associated digraphs (without loops) and adopt the labeling in [2] (note that the digraph of an irreducible nonzero pattern is strongly connected). A drawing of each corresponding digraph is displayed in [2, Appendix B]. We use the notation D:2 or D:23 to denote a digraph D with a loop at vertex 2 or loops on both vertices 2 and 3, respectively. For the analysis, it is necessary to obtain various matrix realizations with specific eigenvalue multiplicities, such as those listed in the Appendix. In order to construct these matrices, we made use of various results, such as the fact that we can assume at least $n - 1$ entries are one by diagonal similarity, as well as the relationship between a characteristic polynomial and its digraph as noted in (2.1), and for some, the combinatorial constraints given by the nSMP conditions, as well as the computational software Sage [18].

5.1. Order 2. Since every 2×2 irreducible nonzero pattern is a star pattern, Theorem 4.2 gives the characterization of the allow sequences for 2×2 irreducible nonzero patterns. We summarize this in the following corollary.

Corollary 5.1. *Let $\mathcal{A}$ be a 2×2 irreducible nonzero pattern. Then, $q_{\text{seq}}(\mathcal{A}) \in \{\langle 0, 1\rangle, \langle 1, 1\rangle\}$ and*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1\rangle$ if and only if $\mathcal{A}$ is equivalent to a superpattern of $\begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$.

(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1\rangle$ if and only if $\mathcal{A}$ is equivalent to $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$.

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5.2. Order 3. Let $\mathcal{A}$ be a 3×3 irreducible nonzero pattern. The possible digraphs of $\mathcal{A}$ are listed as digraphs D1–D5 in [2, Appendix A]. We list the allow sequence for $\mathcal{A}$ in Table 2 according to its digraph $D(\mathcal{A})$ and loop locations, as demonstrated in Theorem 5.2.

Table 2: Characterization of allow sequences for 3×3 irreducible nonzero patterns.

Digraph	**Loop location(s)**	**Sequence**
D1	no loops	$\langle 0,0,1\rangle$
D1	1, 2, 3, 12, 13, 23, 123	$\langle 0,1,1\rangle$
D2	2	$\langle 0,1,1\rangle$
D3–D4	no loops	$\langle 0,1,1\rangle$
D2	no loops	$\langle 1,0,1\rangle$
D2	1, 3, 12, 13, 23, 123	$\langle 1,1,1\rangle$
D3–D4	1, 2, 3, 12, 13, 23, 123	$\langle 1,1,1\rangle$
D5	no loops, 1, 2, 3, 12, 13, 23, 123	$\langle 1,1,1\rangle$

Theorem 5.2. *The allow sequence for each 3×3 irreducible nonzero pattern is given in Table 2. In particular, if $\mathcal{A}$ is a 3×3 irreducible nonzero pattern, then $q_{\text{seq}}(\mathcal{A})\in\{\langle 0,0,1\rangle,\langle 0,1,1\rangle,\langle 1,0,1\rangle,\langle 1,1,1\rangle\}$, and each sequence in this set is attainable by some irreducible nonzero pattern.*

Proof. Let D be a strongly connected digraph on 3 vertices with $q_{\text{seq}}(D)=\langle q_1,q_2,q_3\rangle$. If D is D1 (resp. D2) with any number of loops, then Theorem 4.12 (resp. Theorem 4.2) gives $q_{\text{seq}}(D)$. If $D=D(\mathcal{A})$ is one of D3 or D4 (with no loops), then since every $A\in Q(\mathcal{A})$ is nonsingular and has zero trace, it follows that $q_1(\mathcal{A})=0$. The realizations

$$\underset{\text{D3}}{\begin{bmatrix}0&1&0\\3&0&1\\-2&0&0\end{bmatrix}},\quad \underset{\text{D4}}{\begin{bmatrix}0&1&0\\1&0&1\\-2&2&0\end{bmatrix}},$$

show that $q_2(\text{D3})=q_2(\text{D4})=1$, thus, $q_{\text{seq}}(\text{D3})=q_{\text{seq}}(\text{D4})=\langle 0,1,1\rangle$. Suppose D is not equivalent to D1 or D2, nor a loopless D3 or D4. Then, D contains a subdigraph equivalent to either D3:1, D3:2, D3:3, or D5. Note that if A is one of the following four matrices, then $q(A)=1$ and A has the nSMP:

$$\underset{\text{D3:1}}{\begin{bmatrix}3&1&0\\-3&0&1\\1&0&0\end{bmatrix}},\quad \underset{\text{D3:2}}{\begin{bmatrix}0&1&0\\-3&3&1\\1&0&0\end{bmatrix}},\quad \underset{\text{D3:3}}{\begin{bmatrix}0&1&0\\-3&0&1\\-8&0&3\end{bmatrix}},\quad \underset{\text{D5}}{\begin{bmatrix}0&1&1\\\sqrt{3}-1&0&1\\2&-\sqrt{3}-1&0\end{bmatrix}}.$$

Thus, by Theorem 3.5, every superdigraph of D3:1, D3:2, D3:3, and D5 have allow sequence $\langle 1,1,1\rangle$. □

Following the notation in [3], define the following four nonzero patterns:

$$\mathcal{Z}_1^*=\begin{bmatrix}0&*&0\\ *&0&*\\0&*&0\end{bmatrix},\ \mathcal{Z}_2^*=\begin{bmatrix}0&*&*\\ *&0&*\\ *&*&0\end{bmatrix},\ \mathcal{Y}_1^*=\begin{bmatrix}*&*&0\\ *&0&*\\ *&0&0\end{bmatrix}\ \text{and}\ \mathcal{Y}_2^*=\begin{bmatrix}*&*&0\\0&0&*\\ *&*&0\end{bmatrix}.$$

The characterization in Table 2 gives the following characterization for 3×3 irreducible nonzero patterns $\mathcal{A}$ with $q(\mathcal{A})=1$.

Corollary 5.3. *Let $\mathcal{A}$ be a 3×3 irreducible nonzero pattern. Then, $q(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is equivalent to one of $\mathcal{Z}_1^*$ or $\mathcal{Z}_2^*$, or is equivalent to a superpattern of one of $\mathcal{Y}_1^*$ or $\mathcal{Y}_2^*$.*

5.3. Order 4. We list the 4×4 nonzero patterns classified by the nonequivalent strongly connected digraphs on 4 vertices. We use the labels G1–G61 for these digraphs as listed in [2, Appendix B] noting that the arc from vertex 1 to 3 in the drawing for G57 should be reversed in [2, Appendix B].

Lemma 5.4. *Let $\mathcal{A}$ be an irreducible 4×4 nonzero pattern. Then, $q_4(\mathcal{A}) = 0$ if and only if $D(\mathcal{A})$ is equivalent to one of the star digraphs G17 or G17:1.*

Proof. Let $q_{\text{seq}}(\mathcal{A}) = \langle q_1, q_2, q_3, q_4 \rangle$. By Lemma 3.1, $q_4 = 0$ if and only if $c(\mathcal{A}) \leq 2$. Since $D(\mathcal{A})$ is strongly connected, $D(\mathcal{A})$ is a star digraph with at most one loop on its central vertex; thus, $D(\mathcal{A})$ is equivalent to either G17 or G17:1. □

Lemma 5.5. *Let $\mathcal{A}$ be an irreducible 4×4 nonzero pattern. Then, $q_3(\mathcal{A}) = 0$ if and only if $D(\mathcal{A})$ is equivalent to one of G1, G4, G5, G12, or G15.*

Proof. We do a case analysis of each strongly connected digraph D on 4 vertices and let q_3 denote $q_3(D)$. For G1 and G4, $q_3 = 0$ by Theorem 4.12 and [4, Corollary 2.6], respectively. Each of G5, G12, and G15 is bipartite and do not allow singularity, thus, $q_3 = 0$ by [4, Corollary 2.13(ii)]. When $k \in \{16, 25, 46\}$ and Gk has no loops, then $q_3 = 1$ by [4, Lemma 5.3]. Theorem 4.2 gives the result for G17 with any number of loops. When $k \in \{1, 4, 5, 12, 15, 16, 25, 46\}$ and Gk has at least one loop, then it has a subdigraph equivalent to one of D2:1, D1:1, or G1:1 as outlined in Table 3. Since D2:1 (resp. D1:1 and G1:1) is equivalent to the digraph $D(\mathcal{H}_3)$ (resp. $D(\mathcal{H}_5)$ and $D(\mathcal{H}_{16})$) as defined in [4, Appendix B] with signs removed, $q_3 = 1$ by [4, Lemma 3.10]. Finally, when $k \notin \{1, 4, 5, 12, 15, 16, 17, 25, 46\}$, the digraph G$k$ with any number of loops has a subdigraph equivalent to one of D3, G2 or G3 as outlined in Table 3. Since D3 (resp. G2 and G3) is equivalent to the digraph $D(\mathcal{H}_6)$ (resp. $D(\mathcal{H}_9)$ and $D(\mathcal{H}_{17})$) as defined in [4, Appendix B] with signs removed, $q_3 = 1$ by [4, Lemma 3.10]. □

Table 3: Digraphs on 4 vertices that contain a particular subdigraph.

Digraph	**Loop location(s)**	**Subdigraph**
G1, G5, G12, G15, G16	at least one loop	G1:1
G4	at least one loop	D1:1
G15, G25, G46	at least one loop	D2:1
G6, G8–G11, G13, G18–G21, G23, G24, G26–G45, G47–G61	no restriction	D3
G2, G7, G22	no restriction	G2
G3, G14	no restriction	G3

Lemma 5.6. *Let $\mathcal{A}$ be an irreducible 4×4 nonzero pattern. Then, $q_2(\mathcal{A}) = 0$ if and only if $D(\mathcal{A})$ is equivalent to a (strongly connected) subdigraph of one of G1:1, G3, G4:1, G14, or G47.*

Proof. We do a case analysis of each strongly connected digraph D on 4 vertices and let q_2 denote $q_2(D)$. The digraphs in Table 4 and Table 5 have $q_2 = 1$ justified by a matrix realization in Appendix A or Appendix C, or by Theorem 3.5 via a matrix realization in Appendix B as indicated in the rationale of the tables. For the digraphs in Table 6, we note the following justifications. The digraph G17:1 is a star pattern, so $q_2 = 1$ by Theorem 4.2. For the digraphs G5, G12, and G15, realizations with $q_2 = 1$ are given in Appendix A.

We next consider the digraphs with $q_2 = 0$ as listed in Table 6. The characteristic polynomial of any matrix realization of a strongly connected subdigraph of G1:1, G3, or G14 has a lacunary gap of at least two and thus $q_2 = 0$ by Lemma 3.4. Any matrix realization of a strongly connected subdigraph of G4:1 or G47 is singular with its characteristic polynomial having a lacunary gap of at least one and thus $q_2 = 0$ by Lemma 3.4. Note that all the digraphs G1 with a single loop are equivalent and the digraph G4:3 is equivalent to G4:1. The digraphs G2, G10, G11, G13, G17, G30, G31, and G32 are all equivalent to a subdigraph of G47. Thus, $q_2 = 0$ for each subdigraph described. □

The next three remarks summarize some digraph tools to consider if a 4×4 pattern $\mathcal{A}$ has $q_1(\mathcal{A}) = 1$.

Remark 5.7. *If $D(\mathcal{A})$ has no composite k-cycle for some k, but has exactly one composite ℓ-cycle for some $\ell > k$, then $q_1(\mathcal{A}) = 0$ due to a lacunary gap (Lemma 3.4).*

Remark 5.8. *If $c(\mathcal{A}) < n$, then any $A \in Q(\mathcal{A})$ must have a zero eigenvalue. In this case, $\mathcal{A}$ must be potentially nilpotent if $q_1(\mathcal{A}) = 1$. However, if $D(\mathcal{A})$ has exactly one composite k-cycle for some k, then $\mathcal{A}$ can not be potentially nilpotent. In particular, if $c(\mathcal{A}) < n$ and $D(\mathcal{A})$ has exactly one loop, then $q_1(\mathcal{A}) = 0$.*

Remark 5.9. *If $q_1(\mathcal{A}) = 1$, then either $D(\mathcal{A})$ has a 2-cycle or $\mathcal{A}$ is a diagonal shift of a loopless potentially nilpotent pattern (see [3, Lemma 3.2]).*

We are now ready characterize the allow sequences for 4×4 nonzero patterns.

Theorem 5.10. *The allow sequences for 4×4 irreducible nonzero patterns are given in Tables 4, 5 and 6 of Appendix D. In particular, if $\mathcal{A}$ is a 4×4 irreducible nonzero pattern, then*

$$q_{\text{seq}}(\mathcal{A}) \in \{\langle 0,0,0,1\rangle, \langle 0,0,1,1\rangle, \langle 0,1,0,1\rangle, \langle 0,1,1,0\rangle, \langle 0,1,1,1\rangle, \langle 1,0,0,1\rangle\langle 1,0,1,0\rangle, \langle 1,0,1,1\rangle, \langle 1,1,1,1\rangle\},$$

and each sequence in this set is attainable by some irreducible nonzero pattern.

Proof. Let D be a strongly connected digraph on four vertices with $q_{\text{seq}}(D) = \langle q_1, q_2, q_3, q_4\rangle$. One can check that every possible strongly connected digraph with loops is listed in one of Tables 4, 5 and 6. Lemma 5.4 characterizes the patterns with $q_4 = 0$, while Lemmas 5.5 and 5.6 give the value of q_3 and q_2 respectively for each irreducible pattern. It suffices to determine q_1.

The digraphs with allow sequence $\langle 1,1,1,1\rangle$ are given in Table 4 and justified in column 3. The digraphs with allow sequence $\langle 0,1,1,1\rangle$ are given in Table 5 and justified in column 3. Justification for $q_1 = 0$ is provided in Table 5, except for five cases analyzed below: G2:1, G5:3, G9:1 (equivalent to G9:3), G9:2 (equivalent to G9:4), and G43.

We first note that the five digraphs G2:1, G5:3, G9:1, G9:2, and G12:4 have exactly one loop, and as such do not represent a potentially nilpotent pattern. Hence, for each, if $q_1 = 1$, then there is a matrix realization with characteristic polynomial $(z-1)^4$. For G2:1, this is not possible since the only composite 4-cycle is composed of the unique loop and the unique 3-cycle, which would make the constant coefficient involve the product of the coefficients of z^3 and z, which is not the case for $(z-1)^4$. The same argument applies for G5:3, except in this case applied to the unique 3-cycle being composed of the unique 2-cycle and unique loop. Similarly for G12:4, the coefficient of z is the product of the coefficients of z^3 and z^2, which is not the case for $(z-1)^4$.

Suppose now that A is real matrix whose digraph is G9:1. By diagonal similarity, we may assume

$$A = \begin{bmatrix} a & 1 & 1 & 0 \\ 0 & 0 & b & 0 \\ c & 0 & 0 & 1 \\ d & 0 & 0 & 0 \end{bmatrix},$$

for some nonzero $a, b, c, d \in \mathbb{R}$. Then, $p_A(z) = z^4 - az^3 - cz^2 - (d + bc)z - bd$. But no real choice of a, b, c, d will produce the polynomial $(z-1)^4$ since this requires $a = 4$, $c = -6$, $d = 6b + 4$ and hence $6b^2 + 4b + 1 = 0$ which has no real roots. Next suppose that A is real matrix whose digraph is G9:2. By diagonal similarity, we may assume

$$A = \begin{bmatrix} 0 & a & b & 0 \\ 0 & c & 1 & 0 \\ 1 & 0 & 0 & 1 \\ d & 0 & 0 & 0 \end{bmatrix},$$

for some nonzero $a, b, c, d \in \mathbb{R}$. Then, $p_A(z) = z^4 - cz^3 - bz^2 + [b(c - d) - a]z - ad$. But no real choice of a, b, c, d will produce the polynomial $(z-1)^4$ since this requires $c = 4$, $b = -6$, $a = 6d - 20$ and hence $6d^2 + 4d + 1 = 0$, which has no real roots. Thus, for these four digraphs, $q_1 = 0$. For the digraph G43, since G43 has no loops, if $q_1 = 1$, then G43 must allow a nilpotent matrix realization. If A is a real matrix whose digraph is G43, then by diagonal similarity, we may assume

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 1 \\ b & 0 & 0 & c \\ 0 & d & e & 0 \end{bmatrix},$$

for some nonzero $a, b, c, d, e \in \mathbb{R}$. Then, $p_A(z) = z^4 - (a + d + ce)z^2 - (b + cd)z + e(ac - b)$. But no nonzero choice of a, b, c, d, e will produce the polynomial z^4 since this requires $b = ac$, $d = -a$ and hence $c = 0$ or $e = 0$.

The remaining digraphs appear in Table 6. The value of q_1 for the star digraphs G17 and G17:1 is given by Theorem 4.2. Those with $q_1 = 0$ are justified by Remarks 5.7 and 5.8. Those with $q_1 = 1$ (G4, G30, G31, and G47) each have a nilpotent realization given in Appendix A. ☐

COROLLARY 5.11. *Let* $s = \langle s_1, s_2, s_3, s_4 \rangle$ *with* $s_i \in \{0, 1\}$. *There exists a* 4×4 *irreducible sign pattern* $\mathcal{A}$ *with* $q_{\text{seq}}(\mathcal{A}) = s$ *if and only if*

$$s \notin \{\langle 0,0,0,0\rangle, \langle 1,0,0,0\rangle, \langle 0,1,0,0\rangle, \langle 0,0,1,0\rangle, \langle 1,1,0,0\rangle, \langle 1,1,1,0\rangle, \langle 1,1,0,1\rangle\}.$$

6. Concluding remarks and open questions.

We end by raising some questions for further work.

In Theorem 4.2, we characterized the allow sequence for all irreducible star patterns. In Theorem 4.12, we characterized the allow sequences for $n \times n$ cycles for $n \leq 6$ and provided some general results for n-cycles (see Theorems 4.8, 4.10, and 4.11). The structures of the allow sequences for the $n \times n$ cycle patterns with $n \leq 6$ suggest a possible construction for the allow sequences of cycles in general with $n \geq 7$ based on the number of loops.

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QUESTION 6.1. *What is the allow sequence for the n-cycle based on the number of loops, for $n \geq 7$?*

A class of patterns that we have not considered here, but will also be of general interest, are the (strongly connected) path patterns. These patterns correspond to irreducible tridiagonal matrices. Path patterns without loops can be shown to have at least half of the entries in the allow sequence to be zero (see e.g. [4, Corollary 2.13]). Based on some initial case work, we expect that a path without loops has an alternating allow sequence, and most paths with loops have at most one zero in the allow sequence, with the zero appearing in the first entry of the allow sequence for the few cases that have a zero. See $\mathcal{S}_{2,\ell}$, $\mathcal{S}_{3,\ell}$, and G15 for small examples.

QUESTION 6.2. *What is the allow sequence for an $n \times n$ path pattern?*

We noted in Corollary 4.7 that there exists a sequence, namely $\langle 0, 0, 1, 0, 0, \ldots, 0\rangle$, that is realizable by an irreducible sign pattern, but is not realizable by an irreducible nonzero pattern.

QUESTION 6.3. *Is there is a sequence realizable by an irreducible nonzero pattern but not by any irreducible sign pattern?*

We observed that certain sequences with trailing zeros were not realizable in Theorem 4.4 and Theorem 4.6.

QUESTION 6.4. *Are there other classes of nonrealizable sequences that can be characterized?*

In Section 5, we saw that roughly half of the binary allow sequences are realizable by an irreducible pattern for $n < 5$.

QUESTION 6.5. *What is the proportion of realizable allow sequences for large n?*

In general, if a pattern places some restrictions on the number of distinct eigenvalues allowed by the pattern, then some superpatterns of the pattern will no longer have the restrictions. Let m_n be the number of nonzeros required in an $n \times n$ irreducible nonzero pattern to guarantee there are no restrictions on the number of distinct eigenvalues allowed by the pattern. By considering the tables in the Appendix, we can note that $m_2 = 3$, $m_3 = 7$, and $m_4 = 10$. For $n \geq 5$, consider a matrix pattern $\mathcal{A}$ having the first three columns being a nonzero multiple of the standard unit vector e_4 (having exactly one nonzero entry, placed in row 4) and the remaining columns all nonzero. This pattern will have $c(\mathcal{A}) = n - 2$ and hence $q_n(\mathcal{A}) = 0$. Thus, $m_n > n^2 - 3n + 3$.

QUESTION 6.6. *What is m_n, the minimum number of nonzero entries needed in an $n \times n$ irreducible nonzero pattern $\mathcal{A}$ to guarantee that $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \ldots, 1\rangle$?*

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Appendix A. Example matrices A with $q(A) = 1$ or $q(A) = 2$. Each matrix A listed has $q(A) = 1$ or $q(A) = 2$ but does not have the nSMP (thus, Theorem 3.5 does not apply). The label beneath indicates the corresponding digraph (from [2, Appendix B]) and loop locations follow the colon.

$$\underset{\text{G2:23 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G4 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}} \quad \underset{\text{G4:13 }(q=2)}{\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G4:1234 }(q=1)}{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}} \quad \underset{\text{G5 }(q=2)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}}$$

$$\underset{\text{G10:13 }(q=1)}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G11:13 }(q=1)}{\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}} \quad \underset{\text{G12 }(q=2)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G13:23 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G13:24 }(q=1)}{\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}}$$

$$\underset{\text{G15 }(q=2)}{\begin{bmatrix} 0 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G16 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G16 }(q=2)}{\begin{bmatrix} 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G25 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}} \quad \underset{\text{G25 }(q=2)}{\begin{bmatrix} 0 & -5 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}}$$

$$\underset{\text{G30 }(q=1)}{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}} \quad \underset{\text{G30:13 }(q=1)}{\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G31 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G31:13 }(q=1)}{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{bmatrix}} \quad \underset{\text{G32:23 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & -1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

$$\underset{\text{G46 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G46 }(q=2)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & -1 \\ 2+\sqrt{2} & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\text{G47 }(q=1)}{\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 2 \\ 1 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G47:23 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G49 }(q=1)}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}}$$

Appendix B. Matrices A with $q(A) = 1$ and having the nSMP. Each matrix A listed has $q(A) = 1$ and the nSMP (hence Theorem 3.5 applies). The label beneath indicates the corresponding digraph (from [2, Appendix B]) and loop locations. For G2:13, let $t \approx 1.71$ be the smallest real root of $t^4 - 8t^3 + 22t^2 - 28t + 15 = 0$, and $a = 4 - t$, $b = -t^2 + 4t - 6$, $c = -t^3 + 4t^2 - 6t + 4$.

$$\underset{\text{G2:12}}{\begin{bmatrix} 1 & -81 & 0 & 0 \\ 1 & 15 & 256 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G2:13}}{\begin{bmatrix} a & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 0 & 0 & t & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G2:34}}{\begin{bmatrix} 0 & -4\sqrt{2}-6 & 0 & 0 \\ 1 & 0 & -16\sqrt{2}-32 & 0 \\ 0 & 0 & \sqrt{4\sqrt{2}+2}+2\sqrt{2} & 1 \\ 0 & 1 & 0 & -\sqrt{4\sqrt{2}+2}+2\sqrt{2} \end{bmatrix}}$$

$$\underset{\text{G5:13}}{\begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G5:14}}{\begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}} \quad \underset{\text{G5:34}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix}} \quad \underset{\text{G6:1}}{\begin{bmatrix} 4 & -1 & 4 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G6:2}}{\begin{bmatrix} 0 & 1 & -20 & -6 \\ 0 & 4 & -81 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

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$$\underset{\text{G6:3}}{\begin{bmatrix}0&1&-20&-6\\0&0&-1&0\\0&0&4&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G6:4}}{\begin{bmatrix}0&-1&4&-6\\0&0&1&0\\0&0&0&1\\1&0&0&4\end{bmatrix}}\quad \underset{\text{G7:1}}{\begin{bmatrix}4&1&4&0\\-6&0&-1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G7:2}}{\begin{bmatrix}0&1&4&0\\-6&4&15&0\\0&0&0&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G7:3}}{\begin{bmatrix}0&1&-20&0\\-6&0&-1&0\\0&0&4&1\\1&0&0&0\end{bmatrix}}$$

$$\underset{\text{G7:4}}{\begin{bmatrix}0&1&-20&0\\-6&0&-1&0\\0&0&0&1\\1&0&0&4\end{bmatrix}}\quad \underset{\text{G8:1}}{\begin{bmatrix}4&1&0&-2\sqrt{2}-3\\0&0&1&0\\8\sqrt{2}-8&2\sqrt{2}-3&0&0\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G8:2}}{\begin{bmatrix}0&1&0&2\sqrt{2}-3\\0&4&1&0\\8\sqrt{2}-8&-2\sqrt{2}-3&0&0\\1&0&0&0\end{bmatrix}}$$

$$\underset{\text{G8:4}}{\begin{bmatrix}0&1&0&-2\sqrt{10}-11\\0&0&1&0\\8\sqrt{10}+24&2\sqrt{10}+5&0&0\\1&0&0&4\end{bmatrix}}\quad \underset{\text{G9:12}}{\begin{bmatrix}1&2&1&0\\0&3&1&0\\-3&0&0&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G9:13}}{\begin{bmatrix}1&-1&1&0\\0&0&1&0\\-3&0&3&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G9:24}}{\begin{bmatrix}0&-2&-3&0\\0&1&1&0\\1&0&0&1\\2&0&0&3\end{bmatrix}}$$

$$\underset{\text{G10:2}}{\begin{bmatrix}0&1&-24&1\\0&8&-162&0\\1&0&0&0\\0&0&2&0\end{bmatrix}}\quad \underset{\text{G11:2}}{\begin{bmatrix}0&1&0&1\\-24&8&30&0\\1&0&0&0\\0&0&2&0\end{bmatrix}}\quad \underset{\text{G11:4}}{\begin{bmatrix}0&1&0&-162\\-24&0&2&0\\1&0&0&0\\0&0&1&8\end{bmatrix}}\quad \underset{\text{G12:1}}{\begin{bmatrix}4&1&0&0\\-5&0&1&0\\0&-1&0&1\\-1&0&0&0\end{bmatrix}}\quad \underset{\text{G13:1}}{\begin{bmatrix}16&1&0&0\\-81&0&1&0\\0&-15&0&1\\0&16&0&0\end{bmatrix}}$$

$$\underset{\text{G15:1}}{\begin{bmatrix}20&-125&0&0\\1&0&-20&0\\0&1&0&-5\\0&0&1&0\end{bmatrix}}\quad \underset{\text{G15:2}}{\begin{bmatrix}0&-1&0&0\\1&4&-4&0\\0&1&0&-1\\0&0&1&0\end{bmatrix}}\quad \underset{\text{G16:1}}{\begin{bmatrix}4&4&0&-5\\0&0&1&0\\0&-1&0&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G29:1}}{\begin{bmatrix}4&-3&1&0\\2&0&0&1\\0&-1&0&9\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G34}}{\begin{bmatrix}0&1&1&0\\1&0&0&0\\0&1&0&1\\-1&-1&-1&0\end{bmatrix}}$$

$$\underset{\text{G36}}{\begin{bmatrix}0&1&0&0\\-4&0&3&1\\1&0&0&2\\-3&0&2&0\end{bmatrix}}\quad \underset{\text{G37}}{\begin{bmatrix}0&1&-1&-2\\2&0&2&0\\0&0&0&4\\1&1&0&0\end{bmatrix}}\quad \underset{\text{G39}}{\begin{bmatrix}0&-6&1&0\\1&0&0&1\\0&-18&0&-3\\2&0&-2&0\end{bmatrix}}\quad \underset{\text{G40}}{\begin{bmatrix}0&1&0&0\\2&0&2&1\\0&0&0&1\\-4&2&-4&0\end{bmatrix}}$$

Appendix C. Matrices A with $q(A) = 2$ and having the nSMP. Each matrix A listed has $q(A) = 2$, has the nSMP, and has a repeated real eigenvalue (and so Theorem 3.5 applies). The label beneath indicates the corresponding digraph (from [2, Appendix B]) and loop locations.

$$\underset{\text{G2:1}}{\begin{bmatrix}\sqrt{5}+3&2&0&0\\-3\sqrt{5}+3&0&2&0\\0&0&0&2\\0&3\sqrt{5}-7&0&0\end{bmatrix}}\quad \underset{\text{G2:2}}{\begin{bmatrix}0&1&0&0\\-3&3&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}}\quad \underset{\text{G2:3}}{\begin{bmatrix}0&1&0&0\\-3&0&1&0\\0&0&3&1\\0&-8&0&0\end{bmatrix}}\quad \underset{\text{G3:1}}{\begin{bmatrix}2&1&-2&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}}\quad \underset{\text{G3:2}}{\begin{bmatrix}0&3&2&0\\0&6&3&0\\0&0&0&3\\-9&0&0&0\end{bmatrix}}$$

$$\underset{\text{G3:4}}{\begin{bmatrix}0&1&-2&0\\0&0&1&0\\0&0&0&1\\1&0&0&2\end{bmatrix}}\quad \underset{\text{G4:2}}{\begin{bmatrix}0&2&0&2\\0&4&-3&0\\2&0&0&0\\0&0&-1&0\end{bmatrix}}\quad \underset{\text{G5:1}}{\begin{bmatrix}8&3&0&0\\-6&0&3&0\\0&0&0&1\\3&0&0&0\end{bmatrix}}\quad \underset{\text{G5:3}}{\begin{bmatrix}0&2&0&0\\3-\sqrt{5}&0&2&0\\0&0&2\sqrt{5}-2&2\\3\sqrt{5}-7&0&0&0\end{bmatrix}}\quad \underset{\text{G6}}{\begin{bmatrix}0&3&-8&6\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}}$$

$$\underset{\text{G7}}{\begin{bmatrix} 0 & 1 & -8 & 0 \\ 6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G8}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\sqrt{3}+3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -8 & -2\sqrt{3}+3 & 0 \end{bmatrix}} \quad \underset{\text{G9}}{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 8 & 0 & 0 & -8\sqrt{2} \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\text{G10:1}}{\begin{bmatrix} 3 & 1 & -3 & 2 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

Appendix D. Characterization of allow sequences for 4×4 irreducible nonzero patterns. We start with a characterization of the strongly connected digraphs with allow sequence $\langle 1,1,1,1\rangle$ in Table 4, and those with allow sequence $\langle 0,1,1,1\rangle$ in Table 5. The values $q_3 = q_4 = 1$ follow by Lemmas 5.4 and 5.5. In the columns labeled "Reasoning," we give justification as follows:

- Appendix B: This means that a realization A having $q(A) = 1$ and the nSMP is provided in Appendix B for this digraph. All superdigraphs of the corresponding digraph have allow sequence $\langle 1,1,1,1\rangle$ by Theorem 3.5.
- equiv to: This means that the digraph is equivalent to another digraph appearing earlier in the table and as such will have the same allow sequence.
- super of: This means that the digraph is equivalent to a superdigraph of the one listed in Appendix B or Appendix C, and thus, it inherits ones in its allow sequence from the digraph in the appendix by Theorem 3.5.
- Appendix A and Appendix C: This means that matrices with $q_1 = 1$ and $q_2 = 1$ are provided in these appendices.

For patterns appearing in Table 5, we also include reasoning to justify $q_1 = 0$: if it is a star digraph or cycle digraph, we cite Theorem 4.2 or Theorem 4.12; if we have a digraph technique, we cite Remarks 5.7, 5.8, or 5.9; and otherwise, for six of the digraphs (up to equivalence), Theorem 5.10.

Table 4: The 4×4 irreducible nonzero patterns with allow sequence $\langle 1,1,1,1\rangle$.

Digraph	**Loop location(s)**	**Reasoning**
G2	12, 13, 34	Appendix B
G2	23	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:2)
G2	14, 24	equiv to G2:13 or G2:23
G2	123, 124, 134, 234, 1234	super of G2:12, G2:13 or G2:34
G4	1234	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G4:2)
G5	13, 14, 34	Appendix B
G5	23, 24	equiv to G5:13 or G5:14
G5	123, 124, 134, 234, 1234	super of G5:13, G5:14 or G5:34
G6–G8	at least one loop	Appendix B and superdigraphs (G8:3 is equiv to G8:4)
G9	12, 13, 24	Appendix B
G9	14, 23, 34	equiv to G9:12
G9	123, 124, 134, 234, 1234	super of G9:12, G9:13 or G9:24
G10	2	Appendix B
G10	4	equiv to G10:2
G10	13	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G10:1)
G10	12, 14, 23, 24, 34 123, 124, 134, 234, 1234	super of G10:2 or G10:4

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Table 4: The 4×4 irreducible nonzero patterns with allow sequence $\langle 1, 1, 1, 1\rangle$.

Digraph	**Loop location(s)**	**Reasoning**
G11	2, 4	Appendix B
G11	13	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:3)
G11	12, 14, 23, 24, 34 123, 124, 134, 234, 1234	super of G11:2 or G11:4
G12	1	Appendix B
G12	3	equiv to G12:1
G12	24	super of G5:24
G12	12, 13, 14, 23, 34 123, 124, 134, 234, 1234	super of G12:1 or G12:3
G13	1	Appendix B
G13	23, 24	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:2)
G13	34	super of G2:34
G13	12, 13, 14, 123, 124, 134, 234, 1234	super of G2:34 or G13:1
G15	1, 2	Appendix B
G15	3, 4	equiv to G15:1 or G15:2
G15	at least two loops	super of G15:1, G15:2, G15:3 or G15:4
G16	no loops	Appendix A ($q_1 = q_2 = 1$)
G16	1	Appendix B
G16	2, 3, 4	equiv to G16:1
G16	at least two loops	super of G16:1, G16:2, G16:3 or G16:4
G17	at least two loops	Theorem 4.2
G18–G21	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G22	at least one loop	super of G7:1, G7:2, G7:3 or G7:4
G23	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G24	at least one loop	super of G8:1, G8:2, G8:3 or G8:4
G25	no loops	Appendix A ($q_1 = q_2 = 1$)
G25	at least one loop	super of G15:1, G15:2, G15:3 or G15:4
G26	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G27	at least one loop	super of G7:1, G7:2, G7:3 or G7:4
G28	at least one loop	super of G8:1, G8:2, G8:3 or G8:4
G29	1	Appendix B
G29	2, 3, 4	super of G11:2 or G11:4
G29	at least two loops	super of G29:1, G11:2 or G11:4
G30	2, 4, 12, 14, 23, 24, 34 123, 124, 134, 234, 1234	super of G10:2
G30	13	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:2)
G31	2, 4, 12, 14, 23, 24, 34 123, 124, 134, 234, 1234	super of G11:2
G31	13	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:3)
G32	1, 12, 13, 14,	super of G13:1

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Table 4: The 4×4 irreducible nonzero patterns with allow sequence $\langle 1, 1, 1, 1 \rangle$.

Digraph	**Loop location(s)**	**Reasoning**
	123, 124, 134, 1234	
G32	23	Appendices A ($q_1 = 1$) and C ($q_2 = 1$; super of G32:3)
G32	24	equiv to G32:23
G32	34, 234	super of G2:34
G33	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G34	no restriction	Appendix B and superdigraphs
G35	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G36–G37	no restriction	Appendix B and superdigraphs
G38	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G39–G40	no restriction	Appendix B and superdigraphs
G41	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G42	at least one loop	super of G8:1, G8:2, G8:3 or G8:4
G43–G44	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G45	at least one loop	super of G8:1, G8:2, G8:3 or G8:4
G46	no loops	Appendix A ($q_1 = q_2 = 1$)
G46	at least one loop	super of G15:1, G15:2, G15:3 or G15:4
G47	1, 4, 12, 13, 14, 24, 34	super of G10:2
	123, 124, 134, 234, 1234	
G47	23	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G2:2)
G48	no restriction	super of G37
G49	no loops	Appendices A ($q_1 = 1$) and C ($q_2 = 1$, super of G6)
G49	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G50	no restriction	super of G34
G51	at least one loop	super of G6:1, G6:2, G6:3 or G6:4
G52–G61	no restriction	super of G34, G37 or G40

Table 5: The 4×4 irreducible nonzero patterns with allow sequence $\langle 0, 1, 1, 1 \rangle$.

Digraph	**Loop location(s)**	**Reasoning**
G1	at least two loops	Theorem 4.12
G2	1	Appendix C; Theorem 5.10
G2	2, 3	Appendix C; Remark 5.8
G2	4	equiv to G2:3
G3	1, 2, 4	Appendix C; Remark 5.9
G3	3	equiv to G3:1
G3	at least two loops	equiv to super of G3:1 or G3:2; Remark 5.9
G4	2	Appendix C; Remark 5.9
G4	4, 12, 14, 23, 24, 34, 123, 124, 134, 234	super of G4:2; Remark 5.9
G4	13	Appendix A; Remark 5.9
G5	1	Appendix C; Remark 5.7

Table 5: The 4×4 irreducible nonzero patterns with allow sequence $\langle 0,1,1,1\rangle$.

Digraph	**Loop location(s)**	**Reasoning**
G5	3	Appendix C; Theorem 5.10
G5	2, 4	equiv to G5:1 or G5:3
G5	12	super of G5:1; Remark 5.7
G6–G8	no loops	Appendix C; Remark 5.7
G9	no loops	Appendix C; Remark 5.7
G9	1, 2	super of G9; Theorem 5.10
G9	3, 4	equiv to G9:1 or G9:2
G10	1	Appendix C; Remark 5.8
G10	3	equiv to G10:1
G11	1	super of G2:2; Remark 5.8
G11	3	super of G2:3; Remark 5.8
G12	2	super of G5:2; Remark 5.7
G12	4	super of G5:4; Theorem 5.10
G13	2	super of G2:2; Remark 5.8
G13	3, 4	super of G2:3, G2:4; Remark 5.8
G14	at least one loop	super of G3:1; Remark 5.9
G17	2, 3, 4	Theorem 4.2
G18–G21	no loops	super of G6; Remark 5.7
G22	no loops	super of G7; Remark 5.7
G23	no loops	super of G6; Remark 5.7
G24	no loops	super of G8; Remark 5.7
G26–G27	no loops	super of G6; Remark 5.7
G28	no loops	super of G8; Remark 5.7
G29	no loops	super of G9; Remark 5.7
G30	1	super of G10:1; Remark 5.8
G30	3	super of G2:2; Remark 5.8
G31	1	super of G2:2; Remark 5.8
G31	3	super of G2:3; Remark 5.8
G32	2	super of G2:2; Remark 5.8
G32	3, 4	super of G2:3; Remark 5.8
G33	no loops	super of G6; Remark 5.7
G35	no loops	super of G6; Remark 5.7
G38	no loops	super of G6; Remark 5.7
G41–42	no loops	super of G8; Remark 5.7
G43	no loops	super of G6; Theorem 5.10
G44	no loops	super of G6; Remark 5.7
G45	no loops	super of G8; Remark 5.7
G47	2	super of G2:2; Remark 5.8
G47	3	super of G32:3; Remark 5.8
G51	no loops	super of G6; Remark 5.7

Table 6: Sequences for 4×4 irreducible nonzero patterns omitting $\langle 0,1,1,1\rangle$ and $\langle 1,1,1,1\rangle$.

Digraph	**Loop location(s)**	**Sequence**
G1	no loops	$\langle 0,0,0,1\rangle$
G1	1, 2, 3, 4	$\langle 0,0,1,1\rangle$
G2–G3	no loops	$\langle 0,0,1,1\rangle$
G4	1, 3	$\langle 0,0,1,1\rangle$
G10–G11	no loops	$\langle 0,0,1,1\rangle$
G13–G14	no loops	$\langle 0,0,1,1\rangle$
G32	no loops	$\langle 0,0,1,1\rangle$
G5	no loops	$\langle 0,1,0,1\rangle$
G12	no loops	$\langle 0,1,0,1\rangle$
G15	no loops	$\langle 0,1,0,1\rangle$
G17	1	$\langle 0,1,1,0\rangle$
G4	no loops	$\langle 1,0,0,1\rangle$
G17	no loops	$\langle 1,0,1,0\rangle$
G30–G31	no loops	$\langle 1,0,1,1\rangle$
G47	no loops	$\langle 1,0,1,1\rangle$

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