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PRODUCTS OF DIAGONALIZABLE TRIANGULAR MATRICES*

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Abstract. It is known that a $n \times n$ matrix defined over a field can be expressed as a product of diagonalizable matrices. In this work, this problem is studied for triangular matrices of sizes $n \times n$ ($n \in \mathbb{N}$) and $\mathbb{N} \times \mathbb{N}$. It is proved that each such matrix that is invertible can be expressed as a product of at most four diagonalizable triangular matrices. Additionally, some partial results for noninvertible matrices are presented.

Key words. Diagonalizable matrix, Product of diagonalizable matrices, Triangular matrix, Infinite matrix, Square-zero matrix, Nilpotent matrix.

AMS subject classifications. 15A23, 15A18.

1. Introduction.

In paper [12], Sourour showed that if A is a nonsingular matrix with $\det(A) = \prod_{j=1}^{n} \beta_j \gamma_j$, then A can be expressed as a product BC, where B is lower triangular with eigenvalues β_j and C is upper triangular with eigenvalues γ_j. For the case of complex field, Horn and Johnson extended this result to nonsingular matrices assuming that $n - \operatorname{rank}(A)$ of the elements β_j, γ_j are equal to 0 [10, Thm.4.5.4]. Shortly thereafter, Sourour and Tang [13] generalized this result proving that every $n \times n$ singular matrix, except of 2×2 nilpotent, can be factored as above if and only if the number of zeros among the elements β_j and γ_j is not less than nullity of A. Botha showed [4, 5] that every matrix over a field can be expressed as a product of diagonalizable matrices. Recent study [7] shows that similar result also hold for matrices over division rings.

It is known that even if elements of an algebra possess some specific factorization, the same property does not have to hold for its subset consisting of elements sharing some property (see for instance [8]). Furthermore, even if such factorization exists, it may involve other, usually greater, number of factors.

In this work, we would like to study the problem of factorization of triangular matrices into diagonalizable matrices. We are mainly interested in $\mathbb{N} \times \mathbb{N}$ matrices over field, whose set we will denote by $\mathcal{T}_\infty(F)$. Nevertheless, from the proofs carried out for $\mathbb{N} \times \mathbb{N}$ matrices, one can naturally deduce analogous results for $n \times n$ triangular matrices.

Let us explain that when we write in this paper that a triangular matrix A diagonalizable (adding *in* $\mathcal{T}_\infty(F)$ or ommitting this phrase), we mean $A = X^{-1}DX$ for some diagonal D and some triangular X.

Note that, as the only invertible diagonalizable matrix over two-element field is the identity, the only product of diagonalizable matrices is identity as well. Therefore, in the invertible case, we will focus on the matrices defined over fields consisting of at least three elements. We will prove the following.

Theorem 1.1. *Let F be a field of at least three elements and let $n \in \mathbb{N}$. Then the following claims hold.*

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1. *Every invertible matrix from $\mathcal{T}_\infty(F)$ can be expressed as a product of at most four diagonalizable matrices from $\mathcal{T}_\infty(F)$.*
2. *Every invertible matrix from $\mathcal{T}_n(F)$ can be expressed as a product of at most four diagonalizable matrices from $\mathcal{T}_n(F)$.*

We would also like to consider the noninvertible matrices. Let $\mathcal{NT}_\infty(F)$ ($\mathcal{NT}_n(F)$ resp.) denote the set of all matrices $A = [a_{ij}] \in \mathcal{T}_\infty(F)$ such that $a_{ii} = 0$ for all i. As we will see, this case is far more complicated. Yet, we will present some observations on it in Section 3. In particular, we will show

THEOREM 1.2. *Let F be a field of at least three elements and let $k, n \in \mathbb{N}$.*

1. *If $A \in \mathcal{T}_\infty(F)$ is such that exactly k of its diagonal entries are equal to 0, then A is a product of at most* $\max(k, 4)$ *diagonalizable matrices from $\mathcal{T}_\infty(F)$.*
2. *If $A \in \mathcal{T}_n(F)$ is such that exactly k of its diagonal entries are equal to 0, then A is a product of at most* $\max(k, 4)$ *diagonalizable matrices from $\mathcal{T}_n(F)$.*

2. Proof of the main result.

Assuming that A_i, where either $i = 1, 2, \ldots, n$ or $i =\in \mathbb{N}$, are square matrices, we use (standard) notation $\bigoplus_i A_i$ for the direct sum of A_i-s:

$$\bigoplus_i A_i = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & \ddots \end{bmatrix}. \tag{2.1}$$

Moreover, similarly as in [18], and later in [15] (see also [16]), we introduce the following generalization of the above notation.

DEFINITION 2.1 ([15, Def.1.1]). *Let $\cup_{n\in I} I_n$ be a decomposition of $\mathbb{N}$, i.e. $\cup_{n\in I} I_n = \mathbb{N}$ and $I_n \cap I_m = \emptyset$ for $n \neq m$. If $A_n \in M_{|I_n|\times|I_n|}(F)$, then we call A a generalized direct sum of A_n and write*

$$A = \bigoplus_{n\in I} (A_n)_{I_n},$$

if

$$A_{ij} = \begin{cases} 0 & \text{if } i \in I_n,\ j \in I_m \text{ for some } n \neq m, \\ (A_n)_{xy} & \text{if } i = i_x,\ j = i_y,\ i, j \in I_n = \{i_1, i_2, i_3, \ldots\} \quad \text{where } i_p < i_{p+1}. \end{cases}$$

Strictly speaking, the notation $(A_n)_{I_n}$ with $I_n = \{i_1, i_2, i_3, \ldots\}$ is used for the matrix that in the i_1-th row has the entries from the first row of A_n, in the i_2-th row has the entries from the second row of A_n and so on; analogously for the columns. Surely, the direct sum of the matrices is a special case of their generalized direct sum.

The notion from Definition 2.1 appears in the following result.

PROPOSITION 2.2 ([15, Prop.2.1]). *If F is any field and $A \in \mathcal{T}_\infty(F)$, then there exists $X \in T_\infty(F)$ such that*

$$A^X = \bigoplus_{n\in I} (A_n)_{I_n},$$

with A_n satisfying

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- $(A_n)_{ii} \neq (A_m)_{jj}$ *for all* i, j *and* $m \neq n$;
- $(A_n)_{ii} = (A_n)_{jj}$ *for all* i, j, n.

Clearly, we can note the following.

REMARK 2.3. *Let* A_n $(n \in I)$ *be matrices of finite or infinite size. If each of* A_n *can be expressed as a product of diagonalizable matrices, then so can be their generalized direct sum* $\bigoplus_{n\in I} A_n$.

Thus, in order to prove the first result, we may focus on the matrices with a constant main diagonal. First let us cite the result that was used in some earlier works.

LEMMA 2.4 ([3, Lemma 2.1]). *Let* λ *be an arbitrary element in* F. *If* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are two matrices in* $\mathcal{T}_{\mathfrak{n}}(F)$ *such that* $a_{ii} = b_{ii} = \lambda$ *and* $a_{i,i+1} = b_{i,i+1} \neq 0$ *for all* $i \in \mathbb{N}$, *then* A *and* B *are conjugates in* $\mathcal{T}_\infty(F)$.

Using the above fact we can prove

LEMMA 2.5. *Let* F *be a field such that* $|F| > 2$ *and let* $\lambda \in F^*$. *Then every* $[a_{ij}] = A \in \mathcal{T}_\infty(F)$ *with* $a_{ii} = \lambda$, $a_{i,i+1} \neq 0$ *for all* $i \in \mathbb{N}$ *can be expressed as a product of two diagonalizable matrices from* $\mathcal{T}_\infty(F)$.

Proof. Since $|F| > 2$, there exists $\alpha \in F^*$ that is different from 1. Clearly, in this case α^{-1} is also different from 1. Thus, all the matrices of forms

$$\begin{bmatrix} 1 & \beta \\ 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \alpha^{-1} & \beta' \\ 0 & 1 \end{bmatrix},$$

are diagonalizable, as well as their scalar multiplicities. Hence, the direct sums

$$A_1 = \bigoplus_{n\in\mathbb{N}} \begin{bmatrix} \lambda & \lambda\alpha a_{2n-1,2n} \\ 0 & \lambda\alpha \end{bmatrix}, \quad A_2 = [1] \bigoplus_{n\in\mathbb{N}} \begin{bmatrix} \alpha^{-1} & \alpha^{-1} a_{2n,2n+1} \\ 0 & 1 \end{bmatrix},$$

are diagonalizable as well. Their product, say $\tilde{A} = [\tilde{a}_{ij}]$, is a matrix such that $\tilde{a}_{ii} = \lambda$ and $\tilde{a}_{i,i+1} = a_{i,i+1}$. Since, $a_{i,i+1} \neq 0$, by Lemma 2.4, $\tilde{A}$ and A are conjugates. Thus, A is a product of diagonalizable matrices. □

Now we can prove our main result.

Proof of Thm.1.1. Let us note that this proof is similar to the proof of [19, Thm 1.1]. Namely, let $[a_{ij}] = A \in \mathcal{T}_\infty(F)$ be a matrix whose all diagonal entries are nonzero. From Proposition 2.2 and Remark 2.3, it follows that it suffices to prove our claim for the case when the main diagonal of A is constant. Suppose then $a_{ii} = \alpha$ for all $i \in \mathbb{N}$.

Define now the following set of indices

$$I(A) = \{i \in \mathbb{N} : \ a_{i,i+1} = 0\}.$$

Since $|F| > 2$, there exists $\beta \in F \setminus \{0\}$ such that $\beta \neq \alpha$. We will make use of this element in the matrices $B = [b_{ij}]$, $C = [c_{ij}]$, which we now define as follows:

$$B = \bigoplus_i \begin{bmatrix} \alpha & b_{2i-1,2i} \\ 0 & \beta \end{bmatrix}, \quad \text{where} \quad b_{2i-1,2i} = \begin{cases} -\alpha\beta^{-1} & \text{if } 2i-1 \in I(A) \\ 0 & \text{otherwise,} \end{cases}$$

$$C = [1] \oplus \bigoplus_i \begin{bmatrix} \alpha\beta^{-1} & c_{2i,2i+1} \\ 0 & 1 \end{bmatrix}, \quad \text{where} \quad c_{2i,2i+1} = \begin{cases} -\beta^{-1} & \text{if } 2i \in I(A) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha \neq \beta$, B and C are diagonalizable.

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Volume 41, pp. 484-490, September 2025.

Consider now $[\tilde{a}_{ij}] = \tilde{A} := (BC)^{-1}A$. By the definitions of B and C

$$\tilde{a}_{i,i+1} = \begin{cases} a_{i,i+1} & \text{if } i \notin I(A) \\ 1 & \text{if } i \in I(A), \end{cases}$$

that is, $\tilde{a}_{i,i+1} \neq 0$ for all $i \in \mathbb{N}$. Moreover, $\tilde{a}_{ii} = 1$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.5, $\tilde{A}$ is a product of two diagonalizable triangular matrices. Consequently, as $A = BC\tilde{A}$, our A is a product of four diagonalizable matrices. □

Obviously, sometimes the number of factors can be lowered.

EXAMPLE 2.6. *Suppose that F is an infinite field*[1] *and d_1, d_2, ... are nonzero, pairwise different elements of F. Then, every matrix whose main diagonal is either d_1, d_2, ... or d_1^{-1}, d_2^{-1}, ... is diagonalizable, as well as their scalar multiplicities. Thus, the matrices*

$$B = \begin{bmatrix} d_1 & 0 & 0 & * & * & \cdots \\ & d_2 & 0 & 0 & * & \\ & & d_3 & 0 & 0 & \\ & & & d_4 & 0 & \\ & & & & d_5 & \\ & & & & & \ddots \end{bmatrix}, \quad C = \begin{bmatrix} \lambda d_1^{-1} & 0 & d_1^{-1} & * & * & \cdots \\ & \lambda d_2^{-1} & 0 & d_2^{-1} & * & \\ & & \lambda d_3^{-1} & 0 & d_3^{-1} & \\ & & & \lambda d_4^{-1} & 0 & \\ & & & & \lambda d_5^{-1} & \\ & & & & & \ddots \end{bmatrix},$$

where the $*$ entries denote arbitrary elements, are diagonalizable, and therefore the matrix

$$A = BC = \begin{bmatrix} \lambda & 0 & 1 & \star & \star & \cdots \\ & \lambda & 0 & 1 & \star & \\ & & \lambda & 0 & 1 & \\ & & & \lambda & 0 & \\ & & & & \lambda & \\ & & & & & \ddots \end{bmatrix},$$

is a product of two diagonalizable matrices[2].

EXAMPLE 2.7. *Suppose $A = [a_{ij}]$ is a matrix with two alternating elements in the main diagonal, that is, $(a_{ii}) = (\lambda, \delta, \lambda, \delta, \ldots)$, and $a_{2i,2i+1} = 0$ for all i. Let*

$$C = \begin{bmatrix} \lambda & 0 & & & & \cdots \\ & \delta & -1 & & & \\ & & \lambda & 0 & & \\ & & & \delta & -1 & \\ & & & & \lambda & \\ & & & & & \ddots \end{bmatrix}, \quad \tilde{A} = C^{-1}A = \begin{bmatrix} 1 & \lambda^{-1}a_{12} & * & * & * & \cdots \\ & 1 & \delta^{-1} & * & * & \\ & & 1 & \lambda^{-1}a_{23} & * & \\ & & & 1 & \delta^{-1} & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

Then C is diagonalizable, whereas $\tilde{A}$ is, by Lemma 2.5, a product of two diagonalizable matrices. Hence, A is a product of three diagonalizable matrices.

[1] Observe that the same result will hold for $n \times n$ matrices when we assume $|F| > n$.

[2] Let us acknowledge that the matrices with a constant main diagonal, zero first diagonal(s), and completely nonzero next diagonal have some special properties (see for instance [14]).

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3. Discussion on the noninvertible case. As we have acknowledged, the description of products of noninvertible triangular matrices is not known and seem to be quite complex. Although every singular $n \times n$ matrix can be factorized into product of idempotents [6, 2, 11], not every singular triangular matrix can be factorized into a product of triangular idempotents [9]. In the case of $\mathbb{N} \times \mathbb{N}$, this problem gets even more difficult [17]. In particular, we have already showed that the $\mathbb{N} \times \mathbb{N}$ analog of the Jordan cell

$$J_\infty(0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ & 0 & 1 & 0 & 0 & \\ & & 0 & 1 & 0 & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}$$

can not be expressed as a product of idempotents. Here, we can repeat those arguments and prove the following.

Proposition 3.1. *Suppose F is a field. The matrix $J_\infty(0)$ cannot be expressed a products of diagonalizable matrices from $\mathcal{T}_\infty(F)$.*

Let us first observe that if $A = [a_{ij}]$ is a matrix satisfying $a_{ii} = 0$ for all i, then it is diagonalizable only in the case when $A = 0$. Obviously, in this case every its conjugate is 0 matrix as well. Combining this fact with Proposition 2.2 we obtain

Corollary 3.2. *Suppose F is a field and A is a matrix from either $\mathcal{T}_\infty(F)$ or $\mathcal{T}_n(F)$ for some $n \in \mathbb{N}$. If A is diagonalizable, then A can be written as a generalized direct sum of a zero matrix and a diagonalizable invertible matrix.*

Moreover, let us notice that we have

Remark 3.3. *If $A = [a_{ij}] \in \mathcal{NT}_\infty(F)$ is a product $A = D_1 D_2 \cdots D_k$ of k diagonalizable matrices from $\mathcal{T}_\infty(F)$, then for each i there exists (at least one) j, $1 \leqslant j \leqslant k$, such that $(D_j)_{ii} = 0$.*

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Assume that $J_\infty(0) = \prod_{j=1}^n D_j$, where all D_j are diagonalizable in $\mathcal{T}_\infty(F)$. Let $V \in \mathcal{T}_\infty(F)$ be a matrix such that $D_1^V = V^{-1} D_1 V$ is diagonal. Consider $(J_\infty(0))^V$. From Lemma 2.4, we know that $\left[(J_\infty(0))^V\right]_{i,i+1} \neq 0$.

From Remark 3.3, it follows that there exists $k \in \mathbb{N}$ such that $(D_1)_{kk} = 0$, so by Corollary 3.2 $(D_1^V)_{ki} = 0$ for all $i \in \mathbb{N}$. Consequently,

$$\left[(J_\infty(0))^V\right]_{ik} = \left[D_1^V \cdot \prod_{j=2}^n D_j\right]_{ki} = 0 \quad \text{for all } i \in \mathbb{N}.$$

In particular, we would have $\left[(J_\infty(0))^V\right]_{i,i+1} = 0$ although $(J_\infty(0))_{i,i+1} \neq 0$. That is an obvious contradiction with Lemma 2.4. Thus, the desired factorization does not exist. ∎

It should be underlined that although $J_\infty(0)$ cannot be expressed as a product of diagonalizable matrices, there exist matrices in $\mathcal{NT}_\infty(F)$ that are not nilpotent and can be written as such product – see for instance

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Example 3.7 in [17]. Thus, this problem is more difficult than it seems. However, one may observe some particular rules, for instance as below.

PROPOSITION 3.4. *Let F be a field and let $n \in \mathbb{N}$. Then*

1. *A matrix from $\mathcal{NT}_\infty(F)$ can be expressed as a product of two diagonalizable matrices from $\mathcal{T}_\infty(F)$ if and only if it is square-zero.*
2. *A matrix from $\mathcal{NT}_n(F)$ can be expressed as a product of two diagonalizable matrices from $\mathcal{T}_n(F)$ if and only if it is square-zero.*

Let us add that square-zero matrices can often also be written as products of 3, or more, diagonalizable matrices. Yet, at the same time they can still be factorized into product of two such matrices – in this sense, we mean the 'only if' part of the above proposition.

Proof. Let $A \in \mathcal{NT}_\infty(F)$ be equal to D_1D_2, where D_1, D_2 are diagonalizable. Then, by Remark 3.3, there exists I, $\emptyset \subsetneq I \subset \mathbb{N}$, such that $(D_1)_{ii} = 0$ for all $i \in I$, and $(D_2)_{kk} = 0$ for all $k \notin I$. From Corollary 3.2 and fact that every conjugate of a zero matrix is zero matrix, it follows then that

$$D_1 = 0_I \oplus B_{\mathbb{N}\setminus I} \quad \text{and} \quad D_2 = C_I \oplus 0_{\mathbb{N}\setminus I}.$$

Thus, $a_{ij} = (D_1D_2)_{ij} = 0$ for all pairs $(i,j) \in I^2 \cup (\mathbb{N}\setminus I)^2$. Since

$$(A^2)_{k\ell} = \sum_{k\leqslant r\leqslant \ell} a_{kr}a_{r\ell} = \sum_{\substack{k\leqslant r\leqslant \ell\\ r\in I}} a_{kr}a_{r\ell} + \sum_{\substack{k\leqslant r\leqslant \ell\\ r\in \mathbb{N}\setminus I}} a_{kr}a_{r\ell} = 0 + 0 = 0,$$

A is square-zero.

Suppose now that $A \in \mathcal{NT}_\infty(F)$ is a square-zero matrix. From [18, Thm.1.5], we know that for every square-zero matrix $\tilde{A} \in \mathcal{NT}_n(F)$ there exists invertible $\tilde{V} \in \mathcal{T}_n(F)$ such that $\tilde{V}^{-1}\tilde{A}\tilde{V}$ is the generalized Jordan form of $\tilde{A}$. Moreover, from the proof it follows that $\tilde{V}$ can be constructed inductively by columns. Thus, if $A \in \mathcal{NT}_\infty(F)$ is square-zero, in the same manner one can construct $V \in \mathcal{T}_\infty(F)$ such that $V^{-1}AV$ is the generalized Jordan form of A. Hence,

$$V^{-1}AV = \bigoplus_i \begin{bmatrix} 0 & a_i \\ 0 & 0 \end{bmatrix}. \tag{3.2}$$

Define then B_1 and B_2 by

$$B_1 = V\left(\bigoplus_i \begin{bmatrix} 1 & a_i \\ 0 & 0 \end{bmatrix}\right)V^{-1}, \quad B_2 = V\left(\bigoplus_i \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)V^{-1}.$$

As D_1 and D_2 are diagonalizable and $B_1B_2 = V^{-1}AV$, we have our desired factorization. □

Although the factorization of matrices from $\mathcal{NT}_\infty(F)$ into product of diagonalizable matrices is not possible, the case of $\mathcal{NT}_n(F)$ is different. In fact, it was showed that the following holds.

PROPOSITION 3.5 ([1, Prop.2.7]). *Let R be a ring and let $n \in \mathbb{N}$. Any strictly upper triangular matrix (i.e. having zeros on the main diagonal) $A \in \mathcal{T}_n(R)$ is a product of n idempotent matrices.*

Clearly, an idempotent matrix is diagonalizable in $\mathcal{T}_n(F)$. Therefore, we easily derive

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Proof of Theorem 1.2. Consider $A \in \mathcal{T}_\infty(F)$ satisfying given assumptions. By Proposition 2.2 A, is similar to a direct sum

$$N_I \oplus B_{\mathbb{N}\setminus I} \quad \text{where } I = i_1, i_2, \ldots, i_k,$$

where $N \in \mathcal{NT}_k(F)$ and $B \in \mathcal{T}_\infty(F)$ is invertible. By Proposition 3.5, N is a product of at most k diagonalizable matrices, whereas by Theorem 1.1 B is a product of at most four diagonalizable matrices. Thus, A is a product of at most $\max(4, k)$ diagonalizable matrices. ☐

4. Closing comments. In this work, we investigated factorization of triangular matrices into products of diagonalizable triangular matrices. We have seen that in those factorizations the elements in the main diagonals may often be chosen (quite) arbitrarily. In particular, one may limit themselves to two elements λ_1, λ_2. In such case, this diagonalizable matrix satisfies the quadratic equation $q(x) = (x - \lambda_1)(x - \lambda_2) = 0$ and therefore is called quadratic with respect to $q(x)$. Products of such matrices were studied in [3], whereas generalization of this problem to products of matrices annihilated by polynomial is treated in [19].

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