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ON DIMENSIONS OF MAXIMAL FACES OF COMPLETELY POSITIVE CONES*

TRENT HOLMGREN† AND QINGHONG ZHANG†

Abstract. Because of the lack of characterizations of exposed extreme rays of the $n \times n$ copositive cone in general except for $n \leq 6$, by no means so far can we characterize all maximal faces of the $n \times n$ completely positive cone for $n \geq 7$. In this paper, we use the information of the maximal faces of lower order completely positive cones to study the dimensions of a class of maximal faces of higher order completely positive cones. Specifically, we establish a connection between the dimension of a maximal face of a lower order completely positive cone and the dimension of a maximal face of a higher order completely positive cone via a connection between exposed rays of a lower order copositive cone and a higher order copositive cone. Such a connection is used to find formulas for the dimensions of a certain class of maximal faces of higher order completely positive cones, which has not been studied in the related literature to the best of our knowledge.

Key words. Completely positive cones; Copositive cones; Maximal faces; Exposed rays.

AMS subject classifications. 15B48, 15A23, 15B99, 52A20.

1. Introduction.

An $n \times n$ real symmetric matrix is completely positive if there exist $\mathbf{x}_i \in \mathbb{R}^n_+$, $i = 1, 2, \ldots, p$, such that $A = \sum_{i=1}^{p} \mathbf{x}_i \mathbf{x}_i^\top$, where $\mathbb{R}^n_+$ represents the nonnegative orthant of $\mathbb{R}^n$. A matrix A is copositive if $\mathbf{x}^\top A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$. The set of all $n \times n$ completely positive matrices forms a cone, named completely positive cone and is denoted by $\mathcal{CP}^n$ in this paper. The set of all copositive matrices also forms a cone called copositive cone and is denoted by $\mathcal{COP}^n$, which is the dual cone of $\mathcal{CP}^n$. Let $\mathcal{S}^n$ denote the set of all real $n \times n$ symmetric matrices. We use $\mathcal{P}^n$ to represent the cone of all real $n \times n$ positive semidefinite matrices, $\mathcal{N}^n$ to represent the cone of $n \times n$ nonnegative symmetric matrices, that is, the cone of all symmetric matrices with nonnegative entries. The cone $\mathcal{P}^n \cap \mathcal{N}^n$ is the $n \times n$ doubly nonnegative cone consisting of all $n \times n$ symmetric matrices, which are nonnegative and positive semidefinite. It is a classical result that for $n \leq 4$, $\mathcal{COP}^n = \mathcal{P}^n + \mathcal{N}^n$ and $\mathcal{CP}^n = \mathcal{P}^n \cap \mathcal{N}^n$, and that all these cones are proper cones, see [2]. For $n \geq 5$, $(\mathcal{P}^n \cap \mathcal{N}^n) \backslash \mathcal{CP}^n$ and $\mathcal{COP}^n \backslash (\mathcal{P}^n + \mathcal{N}^n)$ are not empty.

The copositive cones and completely positive cones have many applications and have been topics of research for many years (see [2, 3]). Specifically in optimization, many hard problems ([3, 5]) can be reformulated as copositive optimization problems. Like linear semidefinite optimization, copositive optimization is a conic optimization problem. Instead of the cone of semidefinite matrices used in linear semidefinite optimization, the copositive cones and completely positive cones are used in the formulation of the primal-dual pair of a copositive optimization problem. Knowledge about the structures of the copositive and completely positive cones becomes very important in understanding copositive optimization problems. Hence, studying the structure of the copositive and completely positive cones becomes very critical not only in the study of the theory of copositive optimization but also in the design of algorithms to solve copositive optimization.

The copositive cones and completely positive cones have many other applications in addition to those in optimization. However, due to their complicated structures, knowledge about the geometric aspects of

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†Department of Mathematics and Computer Science, Northern Michigan University, Marquette, MI 49855, USA (tholmgre@nmu.edu, qzhang@nmu.edu).

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the copositive cones and completely positive cones are very limited. Because of the known representations of exposed rays of the completely positive cones, in [4], a way of representing all the maximal faces of the copositive cones along with a simple equation for the dimension of each one was given. Also in [4] some maximal faces (not all maximal faces) of the completely positive cones and their dimensions were discussed. Those maximal faces of the completely positive cones are determined by the representations of some known exposed rays of the copositive cones. The reason that not all maximal faces of the completely positive cones can be determined is due to the fact that in general characterizations of exposed rays of the $n \times n$ copositive cone are not available for $n \geq 7$. However, for $n \leq 6$, all extreme rays of the copositive cones have been well characterized. So all maximal faces of the $n \times n$ completely positive cone for $n \leq 6$ can be determined.

In this paper, we will use the well studied characterizations of some exposed rays of lower order copositive cones to study the dimensions of a class of maximal faces of higher order completely positive cones. This research was motivated by the question 'Finding the actual value of the tight lower bound for the dimension of a maximal face of the completely positive cone is still an open question' raised by Dickinson in [4]. Specifically, in this paper we will establish a connection of the dimensions of maximal faces between a lower order completely positive cone and a higher order completely positive cone. Then we use this result as a tool to study the dimension of a class of maximal faces of higher order completely positive cones using the knowledge of lower order completely positive cones. We will obtain the lower bound for the dimension of a maximal face of the $n \times n$ completely positive cone is between n and $\frac{n^2-5n+8}{2}$ for $n \geq 6$.

The paper is organized as follows: in Section 2 we provide some basic definitions and properties associated with a convex cone, for example, we give the definitions of cones, faces, rays, maximal faces, etc., then we state several results that will be used later in the paper. In Section 3, we establish a connection of the dimensions of maximal faces between a lower order completely positive cone and a higher order completely positive cone. The main results of the paper are given in this section. In Section 4, we provide three examples and show that the lower bound for the dimension of a maximal face of the $n \times n$ completely positive cone is between n and $\frac{n^2-5n+8}{2}$ for $n \geq 6$. Finally in Section 5, we provide some concluding remarks.

2. Preliminaries. Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space. $\mathbb{R}_+$ is the set of all nonnegative real numbers. $\mathbb{R}^n_+$ represents the nonnegative orthant of $\mathbb{R}^n$. If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x}^\top \mathbf{y}$ is used to represent the inner product of $\mathbf{x}$ and $\mathbf{y}$. For a set $\mathcal{L} \subseteq \mathbb{R}^n$, $Cone(\mathcal{L})$, and $Span(\mathcal{L})$ are the cone generated by $\mathcal{L}$ and the space spanned by $\mathcal{L}$, respectively. The reader is referred to [9] for the definitions of these terms. If we have $I \subset \{1, 2, ..., n\}$, we denote a principal submatrix of a matrix A whose elements have row and column indices in I to be A_I.

In this section, we provide some basic definitions and properties associated with a convex cone, for example, we give the definitions of cones, faces, rays, maximal faces, etc. Then we state some properties specifically to the copositive and completely positive cones. We need the following definitions.

Definition 2.1. *Let $\mathcal{K}$ be a convex cone in $\mathbb{R}^n$. The dual cone of $\mathcal{K}$ is defined as*

$$\mathcal{K}^* \equiv \{\mathbf{a} \in \mathbb{R}^n \mid \mathbf{a}^\top \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathcal{K}\}.$$

Definition 2.2. *Let $\mathcal{K}$ be a convex cone in $\mathbb{R}^n$. A convex subcone $\mathcal{F} \neq \{0^n\}$ of $\mathcal{K}$ is called a face of $\mathcal{K}$ if $\mathbf{x} \in \mathcal{K}$, $\mathbf{y} \in \mathcal{K}$, and $\mathbf{x} + \mathbf{y} \in \mathcal{F}$ implies $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \mathcal{F}$. A face $\mathcal{F}$ is exposed if it is the intersection of $\mathcal{K}$ and a nontrivial supporting hyperplane, in other words, there exists a nonzero $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a}^\top \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathcal{K}$ and $\mathcal{F} = \{\mathbf{x} \in \mathcal{K} \mid \mathbf{a}^\top \mathbf{x} = 0\}$.*

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Definition 2.3. *A face $\mathcal{F}_1$ is a maximal face of a closed convex cone $\mathcal{K}$ if $\mathcal{F}_1 \neq \mathcal{K}$, and there does not exist a face $\mathcal{F}_2$ with $\mathcal{F}_2 \neq \mathcal{K}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2$.*

For $\mathbf{a}^* \in \mathcal{K}^*$ and $\mathbf{a}^* \neq 0$, we define $\mathcal{F}(\mathcal{K}, \mathbf{a}^*) = \{\mathbf{x} \in \mathcal{K} \mid \mathbf{x}^\top \mathbf{a}^* = 0\}$. $\mathcal{F}(\mathcal{K}, \mathbf{a}^*)$ is an exposed face, which is called a face determined by $\mathbf{a}^*$ in this paper. Similarly, we define $\mathcal{F}(\mathcal{K}^*, \mathbf{a}) = \{\mathbf{y}^* \in \mathcal{K}^* \mid \mathbf{a}^\top \mathbf{y}^* = 0\}$ for $\mathbf{a} \in \mathcal{K}$. If a face $\mathcal{F}$ is generated by a nonzero vector, i.e., there exists a nonzero $\mathbf{a} \in \mathcal{K}$ such that $\mathcal{F} = \{\alpha \mathbf{a} \mid \alpha \in \mathbb{R}_+\}$, then we call $\mathcal{F}$ an extreme ray. If $\mathcal{F}$ is an extreme ray and it is also exposed, then we call $\mathcal{F}$ an exposed ray.

Theorem 2.4. *[4, Theorem 2.20] If $\mathcal{K}$ is a proper cone and $\mathbf{a}$ gives an exposed ray of $\mathcal{K}^*$, then $\mathcal{F}(\mathcal{K}, \mathbf{a})$ is a maximal face of $\mathcal{K}$.*

The definitions and theorem above are also valid for an inner product space that is isomorphic to $\mathbb{R}^n$. Since the set of $n \times n$ symmetric matrices with the Euclidean inner product is an inner product space, which is isomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}}$, a face, an extreme ray, an exposed ray, etc. are well defined for the copositive and completely positive cones. Based on Theorem 2.4, we know that the characterizations of exposed rays of the copositive cone are critical in determining maximal faces of the completely positive cone of the same order.

Next we give some known results about exposed rays of copositive cones. We also need a definition of a zero of a copositive matrix.

Definition 2.5. *Given an $n \times n$ copositive matrix A, a nonzero vector $\mathbf{v} \in \mathbb{R}^n_+$ is called a zero of A if $\mathbf{v}^\top A \mathbf{v} = 0$. The support of a zero of $\mathbf{v}$ is the index set* supp $\mathbf{v} \subseteq \{1, 2, \ldots, n\}$ *corresponding to the positive entries of $\mathbf{v}$. A zero $\mathbf{v}$ of A is called minimal if there does not exist another zero $\mathbf{u}$ of A such that* supp $\mathbf{u}$ *is a strict subset of* supp $\mathbf{v}$.

Lemma 2.6. *[7, Corollary 3.4] Let $\mathbf{v}$ be a zero of a copositive matrix A. Then $\mathbf{v}$ can be represented as a finite sum of minimal zeros of A.*

Lemma 2.7. *[4, Theorem 4.3 (vi)] For $n \geq 2$, if $\begin{pmatrix} M & \mathbf{m} \\ \mathbf{m}^\top & \mu \end{pmatrix} \notin \mathcal{N}^n$ gives an exposed ray of the $n \times n$ copositive cone, where $\mu \in \mathbb{R}_+$ and $\mathbf{m} \in \mathbb{R}^{n-1}$, then $\begin{pmatrix} M & \mathbf{m} & \mathbf{m} \\ \mathbf{m}^\top & \mu & \mu \\ \mathbf{m}^\top & \mu & \mu \end{pmatrix}$ gives an exposed ray of the $(n+1) \times (n+1)$ copositive cone.*

For a given $A \in \mathcal{COP}^n$, we define $\mathcal{V}^A = \{\mathbf{v} = (v_1, v_2, \ldots, v_n)^\top \in \mathbb{R}^n_+ \mid \mathbf{v}^\top A \mathbf{v} = 0\}$.

Lemma 2.8. *Let $A = \begin{pmatrix} M & \mathbf{m} \\ \mathbf{m}^\top & \mu \end{pmatrix} \notin \mathcal{N}^n$ give an exposed ray of the $n \times n$ copositive cone. Then $\mathbf{x} = (x_1, x_2, \ldots, x_n, x_{n+1})^\top \in \mathcal{V}^{A_{+1}}$ if and only if $\mathbf{x} = (x_1, x_2, \ldots, x_n + x_{n+1})^\top \in \mathcal{V}^A$, where $A_{+1} = \begin{pmatrix} M & \mathbf{m} & \mathbf{m} \\ \mathbf{m}^\top & \mu & \mu \\ \mathbf{m}^\top & \mu & \mu \end{pmatrix}$ gives an exposed ray of the $(n+1) \times (n+1)$ copositive cone.*

Proof. The lemma can be proved by showing the following, which is straightforward.

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \\ x_{n+1} \end{pmatrix}^\top \begin{pmatrix} M & \mathbf{m} & \mathbf{m} \\ \mathbf{m}^\top & \mu & \mu \\ \mathbf{m}^\top & \mu & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n + x_{n+1} \end{pmatrix}^\top \begin{pmatrix} M & \mathbf{m} \\ \mathbf{m}^\top & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n + x_{n+1} \end{pmatrix}. \tag{2.1}$$

□

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3. Connection of the dimensions of maximal faces between lower and higher order $\mathcal{CP}^n$. We begin this section by showing that the tight lower bound for the dimension of a maximal face of $\mathcal{CP}^n$ cannot be $n-1$ unless $n=2$.

LEMMA 3.1. *The tight lower bound for the dimension of a maximal face of the $n \times n$ completely positive cone is $n-1$ if and only if $n=2$.*

Proof. If $n=2$, then it is easy to see that the tight lower bound for the dimension of a maximal face of the 2×2 completely positive cone is 1. Indeed, we know that $\mathbf{x}\mathbf{x}^\top$, for $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an exposed ray of $\mathcal{COP}^2$. So $\mathcal{F}(\mathcal{CP}^2, \mathbf{x}\mathbf{x}^\top) = \{\alpha \mathbf{y}\mathbf{y}^\top \mid \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \geq 0\}$. Therefore, the maximal face $\mathcal{F}(\mathcal{CP}^2, \mathbf{x}\mathbf{x}^\top)$ has dimension 1.

Now we show that if $n > 2$, then the dimension of any maximal face of the $n \times n$ completely positive cone must be greater than $n-1$. We prove this by contradiction. Suppose there is a maximal face $\mathcal{F}$ of the $n \times n$ completely positive cone whose dimension k is less than or equal to $n-1$. Then there are k elements $\{\mathbf{a}_i\mathbf{a}_i^\top\}_{i=1}^k$ with $\mathbf{a}_i \in \mathbb{R}_+^n$, which are linearly independent such that $Span(\{\mathbf{a}_i\mathbf{a}_i^\top\}_{i=1}^k) \cap \mathcal{CP}^n = \mathcal{F}$. Since $k \leq n-1$, we can find a $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{b} \perp \mathbf{a}_i$ for $i = 1, 2, \ldots, k$. It is straightforward that $\mathcal{F} \subseteq \mathcal{F}(\mathcal{CP}^n, \mathbf{b}\mathbf{b}^\top)$. We next prove that $\mathcal{F}(\mathcal{CP}^n, \mathbf{b}\mathbf{b}^\top)$ is a maximal face by showing $\mathbf{b} \in \mathbb{R}^n \backslash (\mathbb{R}_+^n \cup (-\mathbb{R}_+^n))$. Suppose $\mathbf{b} \in \mathbb{R}^n \backslash (\mathbb{R}_+^n \cup (-\mathbb{R}_+^n))$ is not true. Without loss of generality, we can assume that $\mathbf{b} = (b_1, b_2, \ldots, b_n)^\top$ with $\mathbf{b} \in \mathbb{R}_+^n$ and $b_1 > 0$. Then we can see that the first row and first column of all matrices in $\mathcal{F}$ must be all 0 due to the fact that $\mathbf{b} \perp \mathbf{a}_i$ for $i = 1, 2, \ldots, k$ and $Span(\{\mathbf{a}_i\mathbf{a}_i^\top\}_{i=1}^k) \cap \mathcal{CP}^n = \mathcal{F}$. We let $\mathbf{a}_0 = (1, 0, \ldots, 0)^\top$. Then it is easy to verify $Cone(\mathbf{a}_0\mathbf{a}_0^\top, \mathcal{F})$ is a face of $\mathcal{CP}^n$ that is not equal to $\mathcal{CP}^n$, which contradicts the assumption that $\mathcal{F}$ is a maximal face. Therefore, $\mathbf{b} \in \mathbb{R}^n \backslash (\mathbb{R}_+^n \cup (-\mathbb{R}_+^n))$. By [4, Theorem 4.3 (ii)], we know $\mathbf{b}\mathbf{b}^\top$ gives an exposed ray of $\mathcal{COP}^n$, and hence, $\mathcal{F}(\mathcal{CP}^n, \mathbf{b}\mathbf{b}^\top)$ is maximal by Theorem 2.4. Because both $\mathcal{F}$ and $\mathcal{F}(\mathcal{CP}^n, \mathbf{b}\mathbf{b}^\top)$ are maximal, we obtain $\mathcal{F} = \mathcal{F}(\mathcal{CP}^n, \mathbf{b}\mathbf{b}^\top)$. By Theorem 6.3 in [4], we know that $n-1 \geq k = \frac{n(n-1)}{2}$, which is impossible unless $k = n-1 = 1$. ∎

The lemma below will be used several times in the proof of the main theorem in this paper.

LEMMA 3.2. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then the following is a set of $\frac{1}{2}m(m+1)$ linearly independent matrices,*

$$\mathcal{U} = \{(\mathbf{v}_i + \mathbf{v}_j)(\mathbf{v}_i + \mathbf{v}_j)^\top \mid i \leq j,\ i, j = 1, 2, \ldots, m\}.$$

Moreover, $\mathcal{U}$ is a basis of the matrix space

$$\mathcal{S} = Span\left(\left(\sum_{i \in I} \alpha_i \mathbf{v}_i\right)\left(\sum_{i \in I} \alpha_i \mathbf{v}_i\right)^\top \mid \alpha_i \in \mathbb{R}, i \in I \subseteq \{1, 2, \ldots, m\}\right).$$

Proof. By Lemma 6.2 in [4], we know that $\mathcal{U}$ is a linearly independent set. To show that $\mathcal{U}$ is a basis of $\mathcal{S}$, we only need to prove that for any element $A = \left(\sum_{i \in I} \alpha_i \mathbf{v}_i\right)\left(\sum_{i \in I} \alpha_i \mathbf{v}_i\right)^\top$ in $\mathcal{S}$, A can be written as a linear combination of the elements in $\mathcal{U}$, which is true since

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$$\left(\sum_{i\in I}\alpha_i\mathbf{v}_i\right)\left(\sum_{i\in I}\alpha_i\mathbf{v}_i\right)^\top=\sum_{i\in I}\alpha_i^2\mathbf{v}_i\mathbf{v}_i^\top+\sum_{i,j\in I,i<j}\alpha_i\alpha_j(\mathbf{v}_i\mathbf{v}_j^\top+\mathbf{v}_j\mathbf{v}_i^\top)$$
$$=\sum_{i\in I}\alpha_i^2\mathbf{v}_i\mathbf{v}_i^\top+\sum_{i,j\in I,i<j}\alpha_i\alpha_j\left[(\mathbf{v}_i+\mathbf{v}_j)(\mathbf{v}_i+\mathbf{v}_j)^\top-\mathbf{v}_i\mathbf{v}_i^\top-\mathbf{v}_j\mathbf{v}_j^\top\right]$$
$$=\sum_{i\in I}\left[\alpha_i^2-\sum_{j\in I,j\neq i}\alpha_i\alpha_j\right]\mathbf{v}_i\mathbf{v}_i^\top+\sum_{i,j\in I,i<j}\alpha_i\alpha_j(\mathbf{v}_i+\mathbf{v}_j)(\mathbf{v}_i+\mathbf{v}_j)^\top.$$

Therefore, $\mathcal{U}$ is a basis of $\mathcal{S}$. ∎

Now we are ready to prove the main theorem in this paper.

THEOREM 3.3. *Let* $A=\begin{pmatrix}M & \mathbf{m}\\ \mathbf{m}^\top & \mu\end{pmatrix}\notin\mathcal{N}^n$ *give an exposed ray of the* $n\times n$ *copositive cone, where* $\mu\in\mathbb{R}_+$ *and* $\mathbf{m}\in\mathbb{R}^{n-1}$. *We assume that* $\{\mathbf{v}_i\mathbf{v}_i^\top\}_{i=1}^k$ *with* $\mathbf{v}_i=(v_{i1},v_{i2},\ldots,v_{in})^\top\in\mathbb{R}_+^n$ *forms a basis of the matrix linear subspace* $Span(\mathcal{F}(\mathcal{CP}^n,A))$. *Let* $I=\{i\in\{1,2,\ldots,k\}\mid \mathbf{v}_i=(v_{i1},v_{i2},\ldots,v_{in})^\top, v_{in}\neq 0\}$. *If the dimension of the subspace* $Span(\{\mathbf{v}_i\}_{i\in I})$ *is* p, *then the dimension of the maximal face of the* $(n+1)\times(n+1)$ *completely positive cone determined by the exposed ray given by* $A_{+1}=\begin{pmatrix}M & \mathbf{m} & \mathbf{m}\\ \mathbf{m}^\top & \mu & \mu\\ \mathbf{m}^\top & \mu & \mu\end{pmatrix}$ *is* $k+p+1$.

Proof. By Lemma 2.8, $\mathbf{u}=(u_1,u_2,\ldots,u_n,u_{n+1})^\top\in\mathcal{V}^{A_{+1}}$ if and only if $\mathbf{v}=(u_1,u_2,\ldots,u_n+u_{n+1})^\top\in\mathcal{V}^A$. For each $\mathbf{v}_i=(v_{i1},v_{i2},\ldots,v_{in})^\top$, $i=1,2,\ldots,k$, we know that $\bar{\mathbf{v}}_i=(v_{i1},v_{i2},\ldots,v_{in},0)^\top\in\mathcal{V}^{A_{+1}}$. For each $i\in I$, because $0+v_{in}=v_{in}$ and $\frac{1}{2}v_{in}+\frac{1}{2}v_{in}=v_{in}$, we have both $\bar{\mathbf{u}}_i=(v_{i1},v_{i2},\ldots,v_{i(n-1)},0,v_{in})^\top\in\mathcal{V}^{A_{+1}}$ and $\bar{\mathbf{w}}_i=(v_{i1},v_{i2},\ldots,\frac{1}{2}v_{in},\frac{1}{2}v_{in})^\top\in\mathcal{V}^{A_{+1}}$. Since the dimension of the subspace $Span(\{\mathbf{v}_i\}_{i\in I})$ is p, without loss of generality, we can assume that $\mathbf{v}_j$, $j=1,2,\ldots,p$ are linearly independent and $Span(\{\mathbf{v}_j\}_{j=1}^p)=Span(\{\mathbf{v}_i\}_{i\in I})$. Therefore, $\{\bar{\mathbf{u}}_j\}_{j=1}^p$ is a linearly independent set and $Span(\{\bar{\mathbf{u}}_j\}_{j=1}^p)=Span(\{\bar{\mathbf{u}}_i\}_{i\in I})$. We now show that the $k+p+1$ matrices $\{\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top\}_{i=1}^k$, $\{\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top\}_{j=1}^p$, and $\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ in $\mathcal{S}^{n+1}$ are linearly independent.

Let $\sum_{i=1}^k\alpha_i\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top+\sum_{j=1}^p\beta_j\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top+\gamma\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ be the zero matrix. Noticing that the $(n,n+1)$-th element of $\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ is $\frac{1}{4}v_{1n}^2\neq 0$ due to the assumption that $1\in I$, and $(n,n+1)$-th elements of all $\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top$, $i=1,2,\ldots,k$ and $\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top$, $j=1,2,\ldots,p$ are 0, we obtain $\gamma=0$. Moreover, since $\bar{\mathbf{v}}_i=(v_{i1},v_{i2},\ldots,v_{in},0)^\top$ for $i=1,2,\ldots,k$ and $\bar{\mathbf{u}}_j=(v_{j1},v_{j2},\ldots,v_{j(n-1)},0,v_{jn})^\top$ for $j=1,2,\ldots,p$, we know that the $(n+1)$-th column of the matrix $\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top$ for $i=1,2,\ldots,k$ is always 0 vector, and the $(n+1)$-th column of the matrix $\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top$ for $j=1,2,\ldots,p$ is $v_{jn}\bar{\mathbf{u}}_j=v_{jn}(v_{j1},v_{j2},\ldots,v_{j(n-1)},0,v_{jn})^\top$. Because $\sum_{i=1}^k\alpha_i\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i{}^\top+\sum_{j=1}^p\beta_j\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j{}^\top+\gamma\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ is a zero matrix, we know that the last column must be the 0 vector. So we get $\sum_{j=1}^p\beta_jv_{jn}\bar{\mathbf{u}}_j=0$, which implies that $\beta_i=0$ for all $j=1,2,\ldots,p$ due to the fact that $\bar{\mathbf{u}}_j$, $j=1,2,\ldots,p$ are linearly independent and $v_{jn}\neq 0$ for $j=1,2,\ldots,p$ (because $\{1,2,\ldots,p\}\subseteq I$). Hence, $\alpha_i=0$, $\beta_j=0$, for $i=1,2,\ldots,k$ and $j=1,2,\ldots,p$, and $\gamma=0$, showing $\{\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top\}_{i=1}^k$, $\{\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top\}_{j=1}^p$, and $\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ in $\mathcal{S}^{n+1}$ are linearly independent.

Since each element of the completely positive cone $\mathcal{CP}^{n+1}$ can be written as $\sum_{i=1}^r\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^\top$ for $\bar{\mathbf{a}}_i\in\mathbb{R}_+^{n+1}$, $i=1,2,\ldots,r$, to show that the dimension of the maximal face determined by the exposed ray generated by A_{+1} is $p+k+1$, we only need to take any $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top\in\mathcal{F}(\mathcal{CP}^{n+1},A_{+1})$, and show that $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top$ can be written as a linear combination of the $k+p+1$ matrices $\{\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top\}_{i=1}^k$, $\{\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top\}_{j=1}^p$, and $\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$ in $\mathcal{S}^{n+1}$. We consider the following four cases.

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Case 1. Suppose $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_{n-1}, a_n, 0)^\top \in \mathcal{V}^{A_{+1}}$. Then $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1}, a_n)^\top \in \mathcal{V}^A$ by Lemma 2.8. Since $\{\mathbf{v}_i\mathbf{v}_i^\top\}_{i=1}^k$ forms a basis of the linear subspace $Span(\mathcal{F}(\mathcal{CP}^n, A))$, we know that $\mathbf{a}\mathbf{a}^\top = \sum_{i=1}^k \alpha_i \mathbf{v}_i\mathbf{v}_i^\top$ for some α_i, $i = 1, 2, \ldots, k$. And hence, $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top = \sum_{i=1}^k \alpha_i \bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top$, showing that $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top$ is a linear combination of these $k+p+1$ matrices.

Case 2. We now show that for each $q \in I\backslash\{1, 2, \ldots, p\}$, $\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top$ is a linear combination of these $k+p+1$ matrices. By the assumption that $\mathbf{v}_j$, $j = 1, 2, \ldots, p$ are linearly independent and $Span(\{\mathbf{v}_j\}_{j=1}^p) = Span(\{\mathbf{v}_i\}_{i\in I})$, we know that $\bar{\mathbf{v}}_q = \sum_{j=1}^p \alpha_j \bar{\mathbf{v}}_j$ for some α_j. We also have $\bar{\mathbf{u}}_q = \sum_{j=1}^p \alpha_j \bar{\mathbf{u}}_j$. Noticing that the last two columns and last two rows of the matrices $\frac{1}{v_{qn}}\bar{\mathbf{v}}_q\bar{\mathbf{v}}_q^\top - \sum_{j=1}^p \frac{\alpha_j}{v_{jn}}\bar{\mathbf{v}}_j\bar{\mathbf{v}}_j^\top$ and $\frac{1}{v_{qn}}\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top - \sum_{j=1}^p \frac{\alpha_j}{v_{jn}}\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top$ are all 0, and the first $n-1$ elements of $\bar{\mathbf{u}}_q$ and $\bar{\mathbf{v}}_q$ are the same, we obtain $\frac{1}{v_{qn}}\bar{\mathbf{v}}_q\bar{\mathbf{v}}_q^\top - \sum_{j=1}^p \frac{\alpha_j}{v_{jn}}\bar{\mathbf{v}}_j\bar{\mathbf{v}}_j^\top = \frac{1}{v_{qn}}\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top - \sum_{j=1}^p \frac{\alpha_j}{v_{jn}}\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top$. Therefore, $\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top = \bar{\mathbf{v}}_q\bar{\mathbf{v}}_q^\top - \sum_{j=1}^p \frac{\alpha_j v_{jn}}{v_{jn}}\bar{\mathbf{v}}_j\bar{\mathbf{v}}_j^\top + \sum_{j=1}^p \frac{\alpha_j v_{jn}}{v_{jn}}\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top$ showing $\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top$ is a linear combination of these $k+p+1$ matrices.

Case 3, We next let $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_{n-1}, 0, a_n)^\top \in \mathcal{V}^{A_{+1}}$. Then $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1}, a_n)^\top \in \mathcal{V}^A$ by Lemma 2.8. Since $\{\mathbf{v}_i\mathbf{v}_i^\top\}_{i=1}^k$ forms a basis of the linear subspace $Span(\mathcal{F}(\mathcal{CP}^n, A))$, we know that $\mathbf{a}\mathbf{a}^\top = \sum_{i=1}^k \alpha_i \mathbf{v}_i\mathbf{v}_i^\top$ for some α_i, $i = 1, 2, \ldots, k$. And hence, $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top = \sum_{i\in\{1,2,\ldots,k\}\backslash I} \alpha_i \bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top + \sum_{i=1}^p \alpha_i \bar{\mathbf{u}}_i\bar{\mathbf{u}}_i^\top + \sum_{q\in I\backslash\{1,2,\ldots,p\}} \alpha_q \bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top$. But for each $q \in I\backslash\{1, 2, \ldots, p\}$, by the proof of Case 2, we know $\bar{\mathbf{u}}_q\bar{\mathbf{u}}_q^\top$ is a linear combination of the $k+p+1$ matrices, which shows that $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top$ can be written as a linear combination of these $k+p+1$ matrices.

Case 4. Finally, we let $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_{n-1}, a, b)^\top \in \mathcal{V}^{A_{+1}}$ with $a > 0$ and $b > 0$. Then by Lemma 2.8, we know that $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1}, a+b)^\top \in \mathcal{V}^A$, and hence $\bar{\mathbf{a}}^* = (a_1, a_2, \ldots, a_{n-1}, a+b, 0)^\top \in \mathcal{V}^{A_{+1}}$ and $\bar{\mathbf{a}}^{**} = (a_1, a_2, \ldots, a_{n-1}, 0, a+b)^\top \in \mathcal{V}^{A_{+1}}$. Now we use a_n to denote $a+b$. Then $a = \lambda a_n$ and $b = (1-\lambda)a_n$ for some $0 < \lambda < 1$. Therefore, $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_{n-1}, \lambda a_n, (1-\lambda)a_n)^\top = \lambda\bar{\mathbf{a}}^* + (1-\lambda)\bar{\mathbf{a}}^{**}$. By Lemma 3.2, we know $\bar{\mathbf{a}}\bar{\mathbf{a}}^\top$ can be written as a linear combination of $\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top}$, $\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top}$, and $(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top$. By Case 1 and Case 3, we know that $\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top}$ and $\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top}$ can be written as linear combinations of $k+p+1$ matrices $\{\bar{\mathbf{v}}_i\bar{\mathbf{v}}_i^\top\}_{i=1}^k$, $\{\bar{\mathbf{u}}_j\bar{\mathbf{u}}_j^\top\}_{j=1}^p$, and $\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^\top$. So we only need to show $(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top$ is a linear combination of these $k+p+1$ matrices. Note that $\frac{1}{a_n}(\bar{\mathbf{a}}^* - \bar{\mathbf{a}}^{**}) = (0, 0, \ldots, 1, -1)^\top = \frac{1}{v_{1n}}(\bar{\mathbf{v}}_1 - \bar{\mathbf{u}}_1)$. By Lemma 3.2, we know that $(0, 0, \ldots, 1, -1)^\top(0, 0, \ldots, 1, -1)$ can be written as a linear combination of $\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top}$, $\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top}$, and $(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top$, and that $(0, 0, \ldots, 1, -1)^\top(0, 0, \ldots, 1, -1)$ can also be written as a linear combination of $\bar{\mathbf{v}}_1\bar{\mathbf{v}}_1^\top$, $\bar{\mathbf{u}}_1\bar{\mathbf{u}}_1^\top$, and $(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)^\top$. Therefore, there exist α_i and β_i, $i = 1, 2, 3$ such that

$$\begin{aligned}(0, 0, \ldots, 1, -1)^\top(0, 0, \ldots, 1, -1) &= \alpha_1\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top} + \alpha_2\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top} + \alpha_3(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top \\ &= \beta_1\bar{\mathbf{v}}_1\bar{\mathbf{v}}_1^\top + \beta_2\bar{\mathbf{u}}_1\bar{\mathbf{u}}_1^\top + \beta_3(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)^\top.\end{aligned}$$

Note that $\alpha_3 \neq 0$ and $\beta_3 \neq 0$. Otherwise the $(n, n+1)$-th entry of $(0, 0, \ldots, 1, -1)^\top(0, 0, \ldots, 1, -1)$ is not 0, but the $(n, n+1)$-th entry of $\alpha_1\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top} + \alpha_2\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top} + \alpha_3(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top$ or $\beta_1\bar{\mathbf{v}}_1\bar{\mathbf{v}}_1^\top + \beta\bar{\mathbf{u}}_1\bar{\mathbf{u}}_1^\top + \beta_3(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)^\top$ is 0. Hence, we have

$$(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top = \frac{1}{\alpha_3}\left(-\alpha_1\bar{\mathbf{a}}^*\bar{\mathbf{a}}^{*\top} - \alpha_2\bar{\mathbf{a}}^{**}\bar{\mathbf{a}}^{**\top} + \beta_1\bar{\mathbf{v}}_1\bar{\mathbf{v}}_1^\top + \beta_2\bar{\mathbf{u}}_1\bar{\mathbf{u}}_1^\top + \beta_3(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)(\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1)^\top\right).$$

Noticing that $\bar{\mathbf{v}}_1 + \bar{\mathbf{u}}_1 = 2\bar{\mathbf{w}}_1$, we obtain $(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})(\bar{\mathbf{a}}^* + \bar{\mathbf{a}}^{**})^\top$ can be written as a linear combination of these $k+p+1$ matrices. Therefore, the dimension of the maximal face determined by A_{+1} is $k+p+1$. ☐

The corollary below shows that if we know the dimension of a maximal face of a lower order completely positive cone determined by an exposed ray of the copositive cone of the same order, then we can apply Theorem 3.3 recursively to get the dimension of a maximal face of any higher order completely positive cone.

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Corollary 3.4. *Let* $A = \begin{pmatrix} M & \mathbf{m} \\ \mathbf{m}^\top & \mu \end{pmatrix} \in \mathcal{COP}^n \backslash \mathcal{N}^n$ *give an exposed ray of the* $n \times n$ *copositive cone, where* $\mu \in \mathbb{R}_+$ *and* $\mathbf{m} \in \mathbb{R}^{n-1}$*. Assume that* $\{\mathbf{v}_i \mathbf{v}_i^\top\}_{i=1}^k$ *with* $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in})^\top \in \mathbb{R}_+^n$ *forms a basis of the matrix linear subspace* $Span(\mathcal{F}(\mathcal{CP}^n, A))$*. Let* $I = \{i \in \{1, 2, \ldots, n\} \mid \mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in})^\top, v_{in} \neq 0\}$*. If the dimension of the subspace* $Span(\{\mathbf{v}_i\}_{i \in I})$ *is* p*, then the dimension of the maximal face in the* $(n+2) \times (n+2)$ *completely positive cone determined by the exposed ray generated by* A_{+2} *is* $k + (p+1) + (p+2)$*. Moreover, the dimension of the maximal face in the* $(n+m) \times (n+m)$ *completely positive cone determined by the exposed ray given by* A_{+m} *is* $k + mp + \frac{m(m+1)}{2}$*. Here,* $A_{+m} = (A_{+(m-1)})_{+1}$ *for* $m = 2, 3, 4, \ldots$*.*

Proof. Based on the proof of Theorem 3.3, we know that the $k+p+1$ matrices $\{\bar{\mathbf{v}}_i \bar{\mathbf{v}}_i^\top\}_{i=1}^k$, $\{\bar{\mathbf{u}}_j \bar{\mathbf{u}}_j^\top\}_{j=1}^p$, and $\bar{\mathbf{w}}_1 \bar{\mathbf{w}}_1^\top$ form a basis of the subspace $Span(\mathcal{F}(\mathcal{CP}^{n+1}, A_{+1}))$. By the definitions of $\bar{\mathbf{v}}_i$, $i = 1, 2, \ldots, k$, $\bar{\mathbf{u}}_j$, $j = 1, 2, \ldots, p$, and $\bar{\mathbf{w}}_1$, we know that the $(n+1)$-th entry of $\bar{\mathbf{v}}_i$ is always 0 for each $i = 1, 2, \ldots, k$, and the $(n+1)$-th entries of $\bar{\mathbf{u}}_j$ for each $j = 1, 2, \ldots, p$ and $\bar{\mathbf{w}}_1$ are not 0. Moreover, the $(p+1)$ vectors $\{\bar{\mathbf{u}}_j\}_{j=1}^p$ and $\bar{\mathbf{w}}_1$ are linearly independent due to the fact that the n-th entry of $\bar{\mathbf{w}}_1$ is not zero, and the n-th entry for each $\{\bar{\mathbf{u}}_j\}_{j=1}^p$ is 0, and $\{\bar{\mathbf{u}}_j\}_{j=1}^p$ are linearly independent. Therefore, by applying Theorem 3.3, we obtain the dimension of the maximal face determined by A_{+2} is $(k+p+1) + (p+1) + 1 = k + (p+1) + (p+2)$.

Continuing this process, we obtain the dimension of the maximal face in the $(n+m) \times (n+m)$ completely positive cone determined by A_{+m} is $k + \sum_{i=1}^m (p+i) = k + mp + \frac{m(m+1)}{2}$. ∎

The theorem and corollary above provide a tool to find the dimension of a class of maximal faces of higher order completely positive cones based on the knowledge of the lower order completely positive cone. In the next section, we provide three examples to show how the theorem and corollary are used to find the dimensions of some maximal faces.

4. Examples.

Although the dimensions of maximal faces of the $n \times n$ completely positive cone in all examples before Section 7 in [4] were greater than or equal to $\frac{1}{2}n(n-1)$, an example was given in Section 7 of [4] to show that the dimension of a maximal face of the completely positive cone can be less than $\frac{1}{2}n(n-1)$. The example in [4] was given for $n = 9$. Our first example is to use Theorem 3.3 and Corollary 3.4 to find dimensions of maximal faces of higher order completely positive cones based on the example given in Section 7 of [4].

Example 1: We start with the following theorem [4, Theorem 7.6].

Theorem 4.1. *If we let* A *be the matrix given below, then we have that* $\mathcal{F}(\mathcal{CP}^9, A)$ *is a maximal face of the completely positive cone and the dimension of* $\mathcal{F}(\mathcal{CP}^9, A)$ *is 27.*

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

The matrix A is referred to as a Hoffman–Pereira matrix, which generates an exposed extreme ray of $\mathcal{COP}^9$.

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The zeros of A are

$$\mathcal{V}^A = \bigcup_{\text{Cyclic Permutations}} \text{cone}\left\{ \begin{pmatrix}1\\1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\1\\1\\0\\0\end{pmatrix} \right\}.$$ We could list 27 matrices in the form $\mathbf{v}\mathbf{v}^\top$ or $(\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v})^\top$ with $\mathbf{u} \in \mathcal{V}^A$ and $\mathbf{v} \in \mathcal{V}^A$. However, in our discussion, we only need those vectors whose last entry is not 0. So we list all these vectors below.

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\0\\1\\1\\0\\0\\0\\1\\1\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}0\\0\\0\\1\\1\\0\\0\\1\\1\end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix}1\\0\\0\\1\\1\\0\\0\\0\\1\end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix}1\\0\\0\\0\\1\\1\\0\\0\\1\end{pmatrix},$$

$$\mathbf{v}_6 = \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\2\\1\end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix}2\\1\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}, \mathbf{v}_8 = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\1\end{pmatrix}, \mathbf{v}_9 = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\1\\2\end{pmatrix}.$$

It is easy to verify that $\mathbf{v}_8$ and $\mathbf{v}_9$ can be written as a linear combination of $\{\mathbf{v}_i\}_{i=1}^7$, respectively. Hence, p in the statements of Theorem 3.3 and Corollary 3.4 for this problem is 7. By Corollary 3.4, we know that the dimension of the maximal face of the $n \times n$ completely positive cone determined by A_{+q} with $q = n-9$ is $27 + 8 + 9 + \cdots + (n-2) = \frac{n(n-3)}{2}$, which is smaller than $\frac{1}{2}n(n-1)$.

Theorem 4.2. *The dimension of the maximal face of the $n \times n$ completely positive cone determined by A_{+q} with $q = n-9$ is $\frac{n(n-3)}{2}$.*

Example 2: Our second example is the one by taking A in Theorem 3.3 to be a Hildebrand matrix. We will provide a detailed discussion regarding the way a basis of the maximal face is determined. We now give necessary background materials about this type of matrices and their zeros.

In $\mathcal{COP}^5$, there are four types of exposed rays generated by four different types of matrices, see [10]: (1) semidefinite matrices of the form $\mathbf{x}\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^5$ has both positive and negative entries. (2) nonnegative

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matrices of the form $\mathbf{e}_i\mathbf{e}_j^\top + \mathbf{e}_j\mathbf{e}_i^\top$ with $i \neq j$, where $\mathbf{e}_i \in \mathbb{R}^5$ is a vector with the i-th entry being 1 and others being 0. (3) matrices in the orbit of the Horn matrix. (4) matrices in the orbit of the following matrix

$$T(\psi) = \begin{pmatrix} 1 & -\cos\psi_4 & \cos(\psi_4+\psi_5) & \cos(\psi_2+\psi_3) & -\cos\psi_3 \\ -\cos\psi_4 & 1 & -\cos\psi_5 & \cos(\psi_5+\psi_1) & \cos(\psi_3+\psi_4) \\ \cos(\psi_4+\psi_5) & -\cos\psi_5 & 1 & -\cos\psi_1 & \cos(\psi_1+\psi_2) \\ \cos(\psi_2+\psi_3) & \cos(\psi_5+\psi_1) & -\cos\psi_1 & 1 & -\cos\psi_2 \\ -\cos\psi_3 & \cos(\psi_3+\psi_4) & \cos(\psi_1+\psi_2) & -\cos\psi_2 & 1 \end{pmatrix},$$

with $\psi_j > 0$ and $\sum_{j=1}^5 \psi_j < \pi$, which will be called a Hildebrand matrix following [10].

While we may apply Theorem 3.3 to the first three types of exposed rays to get dimensions of maximal faces of higher order completely positive cones, we will work with a Hildebrand matrix. The reason we choose to work with a Hildebrand matrix is that 1) general results regarding the dimensions of maximal faces determined by the first three types of exposed rays were already given in [4]; 2) among these four types of exposed rays of $\mathcal{COP}^5$ a Hildebrand matrix leads to maximal faces of $\mathcal{CP}^n$ with smallest dimensions for $n \geq 5$.

We next state two known results regarding a Hildebrand matrix. The reader is referred to [6, 10] for the proofs of these lemmas.

Lemma 4.3. *Let $T(\psi)$ be a Hildebrand matrix and $\mathbf{v} \in \mathbb{R}^5_+$ be a zero of $T(\psi)$. Then $\mathbf{v}$ can only be a positive multiple of a column of the following matrix,*

$$B(\psi) = \begin{pmatrix} \sin\psi_5 & 0 & 0 & \sin\psi_2 & \sin(\psi_4+\psi_3) \\ \sin(\psi_5+\psi_4) & \sin\psi_1 & 0 & 0 & \sin\psi_3 \\ \sin\psi_4 & \sin(\psi_1+\psi_5) & \sin\psi_2 & 0 & 0 \\ 0 & \sin\psi_5 & \sin(\psi_2+\psi_1) & \sin\psi_3 & 0 \\ 0 & 0 & \sin\psi_1 & \sin(\psi_3+\psi_2) & \sin\psi_4 \end{pmatrix}.$$

Lemma 4.4. *The five column vectors of $B(\psi)$ are linearly independent.*

Since a Hildebrand matrix generates an exposed extreme ray (see the proof of Theorem 3.1 on Page 1545 of [6]), we know that the conjugate face of this extreme ray is maximal, which is $Cone(\mathbf{d}_i\mathbf{d}_i^\top, i = 1, 2, \ldots, 5)$, where $\mathbf{d}_i$ is the i-th column of the matrix $B(\psi)$ for $i = 1, 2, \ldots, 5$. Hence, the maximal face determined by a Hildebrand matrix has dimension $k = 5$. Since the fifth entries of $\mathbf{d}_1$ and $\mathbf{d}_2$ are 0, and the fifth entries of $\mathbf{d}_i$, $i = 3, 4, 5$ are not zero, we know p in Corollary 3.4 is 3. Therefore, the dimension of the maximal face of the $n \times n$ completely positive cone determined by the exposed ray generated by $T(\psi)_{+m}$ with $n = m + 5$ is $k + mp + \frac{m(m+1)}{2} = \frac{n(n-3)}{2}$.

Theorem 4.5. *The dimension of the maximal face of the $n \times n$ completely positive cone determined by $T(\psi)_{+m}$ with $n = m + 5$ is $\frac{n(n-3)}{2}$.*

We immediately have the following theorem by the theorem above and Theorem 3.1.

Theorem 4.6. *The tight lower bound on the dimension of maximal faces of the $n \times n$ completely positive cone is between n and $\frac{n(n-3)}{2}$ for $n \geq 5$.*

While these two examples suggest a lower bound on the dimension of a maximal face of $\mathcal{CP}^n$ is $\frac{n(n-3)}{2}$ for $n \geq 5$, there are maximal faces of $\mathcal{CP}^n$ for $n \geq 6$ whose dimension is smaller than $\frac{n(n-3)}{2}$. We will see this in the next example.

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Example 3: In [1], a complete classification of the extreme rays of $\mathcal{COP}^6$ was provided. There are 19 cases of exceptional extreme rays of $\mathcal{COP}^6$, that is, extreme rays of $\mathcal{COP}^6$ that are not positive semidefinite or nonnegative. We choose to work with Case 19 with the same reason as in the case for $n = 5$ that this type of exposed rays leads to maximal faces of $\mathcal{CP}^6$ with the smallest dimension compared to the ones determined by the other exposed rays including the ones in $\mathcal{P}^6$ and $\mathcal{N}^6$. To say matrices in Case 19 give exposed rays of $\mathcal{COP}^6$ is misleading because no discussion on the exposedness of these extreme rays was given in [1]. It suggests that our first task is to identify which of these 19 cases give exposed rays of $\mathcal{COP}^6$. While it is straightforward to check that matrices in some cases like Case 1 give non-exposed extreme rays, to verify whether matrices give exposed rays of $\mathcal{COP}^n$ in general could be very challenging even for small n like $n = 5$. Interested readers are referred to a proof of the exposedness of the Hildebrand matrices in [6].

Our main interest is to get a lower bound on the dimension of a maximal face of $\mathcal{CP}^6$. Since an exposed ray of $\mathcal{COP}^6$ determines a maximal face of $\mathcal{CP}^6$, we only need to make sure that there are some matrices in Case 19 that give exposed rays of $\mathcal{COP}^6$. Such a result can be proved by using the result that the closure of matrices that give exposed rays contains the set of matrices that give extreme rays, see [4, Theorem 2.12].

We recall matrices of Case 19 in [1]. We use $C(\psi, u)$ to represent the following matrix for each $\psi = (\psi_1, \psi_2, \ldots, \psi_6)$ and $u = (u_{62}, u_{63}, u_{64}, u_{66})$ with $u_{62} \neq 0, u_{63} \neq 0, u_{64} \neq 0$, and $u_{66} \neq 0$,

$$\begin{pmatrix} 1 & -\cos\psi_4 & \cos(\psi_4+\psi_5) & \cos(\psi_2+\psi_3) & -\cos\psi_3 & \cos(\psi_3+\psi_6) \\ -\cos\psi_4 & 1 & -\cos\psi_5 & a_{24} & \cos(\psi_3+\psi_4) & -\cos\psi_7 \\ \cos(\psi_4+\psi_5) & -\cos\psi_5 & 1 & -\cos\psi_1 & \cos(\psi_1+\psi_2) & a_{36} \\ \cos(\psi_2+\psi_3) & a_{24} & -\cos\psi_1 & 1 & -\cos\psi_2 & \cos(\psi_6-\psi_2) \\ -\cos\psi_3 & \cos(\psi_3+\psi_4) & \cos(\psi_1+\psi_2) & -\cos\psi_2 & 1 & -\cos\psi_6 \\ \cos(\psi_3+\psi_6) & -\cos\psi_7 & a_{36} & \cos(\psi_6-\psi_2) & -\cos\psi_6 & 1 \end{pmatrix}.$$

Then matrices in Case 19 are those $C(\psi, u)$ with the following properties.

1. $\psi_i \in (0, \pi), \sum_{i=1}^{5} \psi_i < \pi, \psi_2 < \psi_6, \psi_3 + \psi_4 + \psi_6 \leq \psi_7$.
2. $C(\psi, u)_{\{2,3,4\}}$ is positive definite. $C(\psi, u)_{\{2,3,4,6\}}$ has a positive kernel $u = (u_{62}, u_{63}, u_{64}, u_{66})^\top$. Therefore, ψ_7, a_{24}, and a_{36} are determined by ψ_i, $i = 1, 2, \ldots, 6$, u_{62}, u_{63}, u_{64} and u_{66}. Here $C(\psi, u)_{\{2,3,4\}}$ and $C(\psi, u)_{\{2,3,4,6\}}$ are the principal submatrices of $C(\psi, u)$ whose elements have row and column indices in $\{2, 3, 4\}$ and $\{2, 3, 4, 6\}$, respectively.
3. $C(\psi, u)$ is not the matrix which satisfy $\psi_3 + \psi_4 + \psi_6 < \psi_7$ and the relation
$$\begin{gathered} \cos(\psi_1+\psi_2+\psi_3+\psi_4-\psi_5)u_{62}^2 - 2\cos(\psi_1+\psi_2+\psi_3+\psi_4)u_{62}u_{63} + \\ \cos(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5)u_{63}^2 = \cos(\psi_1-\psi_2-\psi_3-\psi_4-\psi_5)u_{64}^2 + \\ 2\cos(\psi_1-\psi_3-\psi_4-\psi_5-\psi_6)u_{64}u_{66} + \cos(\psi_1+\psi_2-\psi_3-\psi_4-\psi_5-2\psi_6)u_{66}^2. \end{gathered}$$
4. $C(\psi, u)$ is not the matrix which satisfy $\psi_3 + \psi_4 + \psi_6 = \psi_7$, the above relation, and the relation
$$\begin{gathered} \cos(\psi_1+\psi_2+\psi_3+\psi_4-\psi_5)u_{62}^2 - 2\cos(\psi_1+\psi_2-\psi_5-\psi_6)u_{62}u_{66} + \\ \cos(\psi_1+\psi_2-\psi_3-\psi_4-\psi_5-2\psi_6)u_{66}^2 = \cos(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5)u_{63}^2 - \\ 2\cos(\psi_2+\psi_3+\psi_4+\psi_5)u_{63}u_{64} + \cos(\psi_1-\psi_2-\psi_3-\psi_4-\psi_5)u_{64}^2. \end{gathered}$$

We know that $C(\psi, u)$ satisfying the above properties gives an extreme ray of $\mathcal{COP}^6$ as discussed in [1]. The minimal zeros of $C(\psi, u)$ are positive multiples of the following six vectors

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$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ \sin\psi_2 \\ \sin(\psi_1+\psi_2) \\ \sin\psi_1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \sin\psi_2 \\ 0 \\ 0 \\ \sin\psi_3 \\ \sin(\psi_2+\psi_3) \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} \sin(\psi_3+\psi_4) \\ \sin\psi_3 \\ 0 \\ 0 \\ \sin\psi_4 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_4 = \begin{pmatrix} \sin\psi_5 \\ \sin(\psi_4+\psi_5) \\ \sin\psi_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} \sin\psi_6 \\ 0 \\ 0 \\ 0 \\ \sin(\psi_3+\psi_6) \\ \sin\psi_3 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ u_{62} \\ u_{63} \\ u_{64} \\ 0 \\ u_{66} \end{pmatrix},$$

where $\mathbf{v}_6$ is from the positive kernel vector above. By Lemma 2.6, a zero of $C(\psi,u)$ can always be written as a finite sum of minimal zeros of $C(\psi,u)$. By a direct verification, we know $\mathbf{v}_7 = \alpha\mathbf{v}_2 + \beta\mathbf{v}_5$ for $\alpha,\beta \in \mathbb{R}_+$ is the only other class of zeros other than the minimal zeros.

In [8], it was shown that Case 19 is not contained in the closure of another case. The next lemma shows that Case 19 is not contained in the closure of the union of the other cases. Therefore, by [4, Theorem 2.12] some extreme rays given by Case 19 are exposed rays of $\mathcal{COP}^6$, and hence, these exposed rays can be used to determine maximal faces of $\mathcal{CP}^6$.

Lemma 4.7. *There is a matrix $C(\psi,u)$ given by Case 19 in [1] as described above such that $C(\psi,u)$ generates an exposed ray of $\mathcal{COP}^6$.*

Proof. Suppose to the contrary that each matrix given by Case 19 gives a non-exposed ray. Let $C(\psi,u)$ be a matrix given by Case 19. To make our proof easy, we can choose some ψ and u such that a_{24} and a_{36} are not equal to 1 or -1 just like two matrices given at Page 190 of [1]. Therefore, $C(\psi,u)$ does not have off-diagonal entries 1 or -1 due to the requirement that $\psi_i \in (0,\pi), i = 1,2,\ldots,7$. Then $C(\psi,u)$ gives an extreme ray of $\mathcal{COP}^6$ but not an exposed ray. By the fact that the closure of matrices that give exposed rays of $\mathcal{COP}^6$ contains the set of all matrices that give extreme rays of $\mathcal{COP}^6$, we know that $C(\psi,u)$ must be the limit of matrices that give exposed rays. Let $\{B_n\}_{n=1}^{\infty} \subset \mathcal{COP}^6$ be a sequence of matrices that give exposed rays such that $\lim_{n\to\infty} B_n = C(\psi,u)$. Then by [1, Theorem 4], for each n there is a positive definite diagonal matrix D_n, a permutation matrix P_n, and a matrix A_n given by one of the forms NE, 05, or 1-19, such that $B_n = D_n P_n A_n P_n^\top D_n$. Recall from [1] the form NE gives all extreme rays that are either in $\mathcal{P}^6$ or in $\mathcal{N}^6$, the form 05 gives all extreme rays that contain a column and a row having all 0 entries, and the forms 1-19 are the matrices determined by Cases 1-19. Since there are finitely many elements in the permutation group and A_n can only take finitely many forms, we can simply assume $P_n = P$ for each n, and A_n for all n are of the same form out of the forms NE, 05, or 1–19. Otherwise, we can choose a subsequence and rename A_n accordingly. Note that the diagonal entries of $C(\psi,u)$ are all 1, we know that $D_n \to I$ as $n\to\infty$, where I is the identity matrix. Hence, we obtain $PA_nP^\top \to C(\psi,u)$.

We now show that A_n cannot be of the form NE, 05, or 1-18. In [8], it was shown that Cases 13.1, 13.2, 16-17, and 19 are the only ones that are not contained in the closure of another case. Therefore, we only need to show that A_n cannot be of Cases NE, 13.1, 13.2, 16-17.

Suppose that A_n for each n is of the form NE. Then it is easy to see that PA_nP^T converges to a matrix with at least one off-diagonal entries 1, which contradicts to the fact that $C(\psi,u)$ has no off-diagonal entries 1 or -1.

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Suppose A_n for all n are of one of the forms 13.1, 13.2, and 16–17. Since we assume that $PA_nP^\top \to C(\psi,u)$ and $C(\psi,u)$ does not have off-diagonal entries 1 or -1, we know each angle ψ_i^n appearing in A_n does not converge to 0 or π. Noticing that the cardinality of the support of each minimal zero of matrices being of the forms 13.1, 13.2, and 16-17 is 3, minimal zeros of A_n with the same support converge to a zero of $C(\psi,u)$ with 3 positive entries. Because each form of 13.1, 13.2, and 16–17 has at least 6 minimal zeros up to positive multiples whose support has cardinality 3, we see that $PA_nP^\top$ converges to a matrix with at least 6 minimal zeros up to positive multiples, and each minimal zero has exactly 3 non-zero entries. However, $C(\psi,u)$ has exactly 5 minimal zeros up to positive multiples with exactly 3 non-zero entries and one minimal zero up to positive multiples with exactly 4 non-zero entries. Therefore, $PA_nP^\top$ cannot converge to $C(\psi,u)$. Hence, A_n for each n cannot be of one of the forms 13.1, 13.2, and 16–17.

Thus if $C(\psi,u)$ does not give an exposed ray, then it must be the limit of a sequence of matrices of the form 19, which give exposed rays. We complete the proof. $\square$

Let $C(\psi,u)$ give an exposed ray of $\mathcal{COP}^6$. $\mathbf{v}_i$ $i=1,2,\ldots,6$ are defined as above. Then $Cone(\{\mathbf{v}_i\mathbf{v}_i^\top, i=1,2,\ldots,6\}\cup\{(\alpha\mathbf{v}_2+\beta\mathbf{v}_5)(\alpha\mathbf{v}_2+\beta\mathbf{v}_5)^\top,\alpha,\beta\in\mathbb{R}_+\})$ is a maximal face of $\mathcal{CP}^6$ determined by $C(\psi,u)$. By Lemma 3.2, we know that $\mathbf{v}_i\mathbf{v}_i^\top, i=1,2,\ldots,7$ form a basis of $Span(\mathcal{F}(\mathcal{CP}^6,C(\psi,u)))=Span(\{\mathbf{v}_i\mathbf{v}_i^\top, i=1,2,\ldots,6\}\cup\{(\alpha\mathbf{v}_2+\beta\mathbf{v}_5)(\alpha\mathbf{v}_2+\beta\mathbf{v}_5)^\top,\alpha,\beta\in\mathbb{R}_+\})$, where $\mathbf{v}_7=\mathbf{v}_2+\mathbf{v}_5$. Hence, the dimension of the subspace $Span(\mathcal{F}(\mathcal{CP}^6,C(\psi,u)))$ is 7. Since the last rows of $\mathbf{v}_5,\mathbf{v}_6,\mathbf{v}_7$ have nonzero entries, we get $p=3$ for Corollary 3.4. Therefore, the dimension of the maximal face of the $n\times n$ completely positive cone determined by $C(\psi,u)_{+m}$ with $n=m+6$ is $k+mp+\frac{m(m+1)}{2}=\frac{n^2-5n+8}{2}$, which is less than $\frac{n(n-3)}{2}$ for $n\geq 6$.

THEOREM 4.8. *The dimension of the maximal face of the $n\times n$ completely positive cone determined by $C(\psi,u)_{+m}$ with $n=m+6$ is $\frac{n^2-5n+8}{2}$ for $n\geq 6$.*

We now get a smaller bound on the dimensions of maximal faces of $\mathcal{CP}^n$ for $n\geq 6$ by Theorems 4.8 and 3.1.

THEOREM 4.9. *The tight lower bound on the dimension of maximal faces of the $n\times n$ completely positive cone is between n and $\frac{n^2-5n+8}{2}$ for $n\geq 6$.*

5. Conclusions. In this paper, motivated by an open problem raised by Dickinson in 2011, we have studied the lower bound for the dimension of a maximal face of the completely positive cone. We have established a connection of the dimensions of maximal faces between a lower order completely positive cone and a higher order completely positive cone. The connection has been established due to a fact that an exposed ray of $\mathcal{COP}^{n+1}$ can be obtained through an exposed ray of $\mathcal{COP}^n$. Using this connection as a tool, we have demonstrated by three examples how to use the information from a lower order completely positive cone to find the dimension of a class of maximal faces of a higher order completely positive cone. We have obtained the tight lower bound for the dimension of a maximal face of $\mathcal{CP}^n$ is between n and $\frac{n^2-5n+8}{2}$ for $n\geq 6$, which to the best of our knowledge is the smallest lower bound obtained so far.

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