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WEIGHTED SUBDIRECT SUM OF NEKRASOV MATRICES*

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Abstract. The weighted subdirect sum of matrices plays a fundamental role in applications involving overlapping structures, such as multilayer network analysis and overlapping block iterative methods for solving linear systems. In this paper, some sufficient conditions are presented to ensure that the weighted subdirect sum of Nekrasov matrices is in the class of Nekrasov matrices. We obtain, in particular, that the weighted 1-subdirect sum of a Nekrasov matrix and a strictly diagonally dominant matrix remains a Nekrasov matrix. Theoretical results are complemented by numerical examples demonstrating the validity and applicability of the proposed conditions.

Key words. Subdirect sum, Weighted subdirect sum, Nekrasov matrices, Strictly diagonally dominant matrices, Overlapping blocks.

AMS subject classifications. 15A06, 15A42, 15B48.

1. Introduction. The concept of weighted subdirect sum of matrices, which is a generalization of the subdirect sum of matrices, was introduced and studied by Pedroche [25]. Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$. Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1.1}$$

where A_{22} and B_{11} are square matrices of size $k \times k$. The following matrix of size $(n_1+n_2-k)\times(n_1+n_2-k)$

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & \alpha A_{22} + \beta B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix},$$

is called the weighted k-subdirect sum of A and B, with weights $\alpha \geq 0$ and $\beta \geq 0$, and it is denoted by $C = A \oplus_k^{\alpha,\beta} B$.

The weighted subdirect sum arises in applications involving overlapping regions, such as overlapping graphs in multilayer networks or iterative methods for solving linear systems of equations based on overlapping blocks with varying weights [25]. When $\alpha = 1$ and $\beta = 1$, the concept of weighted subdirect sum of matrices is reduced to the well-known subdirect sum of matrices introduced by Fallat and Johnson [8], which has many applications in various problems, such as matrix completion problems [15], overlapping subdomains in domain decomposition methods [5, 9, 28], and global stiffness matrices in finite elements [27, 28]. For the weighted subdirect sum of matrices, it is common to focus on the following four questions [25]:

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Q-I If A and B lie in a class S, are there any $\alpha > 0$ and $\beta > 0$ such that $C = A \oplus_k^{\alpha,\beta} B$ lies in the same class S for $k = 1$?

Q-II A matrix of the form

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix},$$

belonging to a class of S can be written as $A \oplus_k^{\alpha,\beta} B$, for $k = 1$, with some $\alpha > 0$ and $\beta > 0$, and with A and B in the same class as C?

Q-III Question Q-I with $k > 1$.

Q-IV Question Q-II with $k > 1$.

Recently, Pedroche [25] provided affirmative answers to questions Q-I to Q-IV regarding some positivity classes of matrices. The author proved that (i) question Q-I (and question Q-III, taking $k > 1$) for positive definite (PD) and positive semidefinite (PSD) matrices, symmetric M-matrices (SM), completely positive (CP) matrices, and double nonnegative (DN) matrices have affirmative answer for the weighted subdirect sum when $\alpha \geq 1$ and $\beta \geq 1$; (ii) question Q-I for M-matrices, P-matrices, P_0-matrices, and totally nonnegative (TN) matrices have affirmative answer for the weighted subdirect sum when $\alpha \geq 1$ and $\beta \geq 1$; (iii) question Q-II (and question Q-IV, taking $k > 1$) for PD, PSD, SM, CP, M, P, P_0, and TN class have affirmative answer for the weighted subdirect sum when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$; (iv) question Q-II for DN class has affirmative answer for the weighted subdirect sum when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$.

The above results are extensions of the previous results on positivity classes for subdirect sums provided by Fallat and Johnson [8]. In the case of subdirect sums, in addition to positivity classes, other classes of structure matrices were also studied, such as inverses of M-matrices [6], inverse-positive matrices [1, 16], doubly diagonally dominant matrices [33], and accretive, dissipative, Benzi–Golub matrices [14], S-strictly diagonally dominant matrices [7, 4], α_1 and α_2-matrices [3], Nekrasov matrices [18, 23, 29, 30], weakly chained diagonally dominant matrices [19], quasi-Nekrasov (QN) matrices [10], H-matrices [34], B-matrices and doubly B-matrices [2], $\{P_1, P_2\}$-Nekrasov matrices [11], Ostrowski–Brauer sparse (OBS) matrices [12], SDD(p) matrices [22], Dashnic–Zusmanovich matrices [21], strong SDD_1 matrices [32], SDD_1 matrices [26], partially doubly strictly diagonally matrices [31], and generalization to linear operators on Hilbert spaces [17]; for more details, see [24] and references therein. Among them, the subdirect sum of Nekrasov matrices has attracted much attention and studied [10, 18, 23, 29]. However, the weighted subdirect sum of Nekrasov matrices remains unclear, and it may also appear when analyzing overlapping subgraphs in complex networks. For example, two weighted graphs with an overlapping clique (i.e., a complete subgraph) of four vertices are shown in Fig. 1. The weighted adjacency matrices are given by

$$A_1 = \left[\begin{array}{cc:cccc} 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 3 \\ \hdashline 2 & 0 & 14 & 5 & 1 & 6 \\ 0 & 0 & 5 & 15 & 7 & 3 \\ 0 & 0 & 1 & 7 & 12 & 4 \\ 0 & 3 & 6 & 3 & 4 & 16 \end{array}\right], \quad A_2 = \left[\begin{array}{cccc:cc} 8 & 2 & 3 & 2 & 0 & 0 \\ 2 & 9 & 2 & 1 & 3 & 0 \\ 3 & 2 & 8 & 1 & 1 & 0 \\ 2 & 1 & 1 & 5 & 0 & 0 \\ \hdashline 0 & 3 & 1 & 0 & 9 & 4 \\ 0 & 0 & 0 & 0 & 4 & 5 \end{array}\right].$$

As reported in [25], the union of the above two graphs can be constructed by assigning different weights to the cliques based on their origin: those from G_1 will be weighted by α, whereas those from G_2 will be

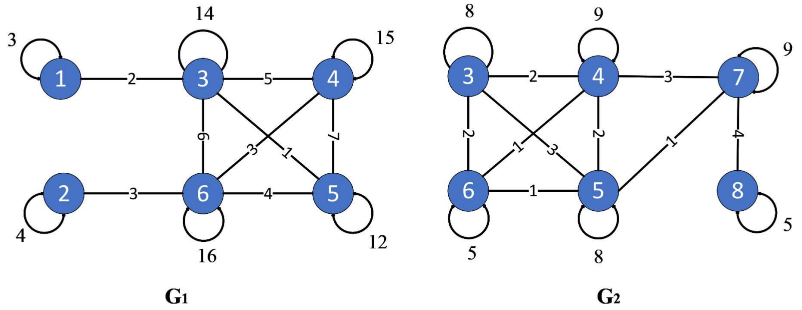

FIG. 1. *Two graphs with an overlapping clique.*

assigned a weight of β. The adjacency matrix of the union $G_1 \cup G_2$ is given by

$$A_1 \oplus_4^{\alpha,\beta} A_2 = \left[\begin{array}{cc|cccc|cc} 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 2 & 0 & 14\alpha+8\beta & 5\alpha+2\beta & \alpha+3\beta & 6\alpha+2\beta & 0 & 0 \\ 0 & 0 & 5\alpha+2\beta & 15\alpha+9\beta & 7\alpha+2\beta & 3\alpha+\beta & 3 & 0 \\ 0 & 0 & \alpha+3\beta & 7\alpha+2\beta & 12\alpha+8\beta & 4\alpha+\beta & 1 & 0 \\ 0 & 3 & 6\alpha+2\beta & 3\alpha+\beta & 4\alpha+\beta & 16\alpha+5\beta & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 1 & 0 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \end{array}\right].$$

Observe that A_1 and A_2 are Nekrasov matrices. An interesting problem arises: are there any $\alpha > 0$ and $\beta > 0$ such that $A_1 \oplus_4^{\alpha,\beta} A_2$ is a Nekrasov matrix? To address this type of problem, this paper focuses on the weighted subdirect sum of Nekrasov matrices. Several sufficient conditions are provided to ensure that the weighted subdirect sum of Nekrasov matrices lies in the class of Nekrasov matrices. These conditions provide affirmative answers to questions Q-I and Q-III for Nekrasov matrices. Numerical examples are presented to illustrate the corresponding theoretical results.

2. Preliminary.

In this section, several definitions and notations are introduced that will be used in the sequel. For any arbitrary positive integer n, $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices, and let $N := \{1, 2, \ldots, n\}$. Let $A = [a_{ij}] \in \mathbb{C}^{n_1\times n_1}$, $B = [b_{ij}] \in \mathbb{C}^{n_2\times n_2}$, and let k be an integer such that $1 \le k \le \min\{n_1, n_2\}$. Then, $C = A \oplus_k^{\alpha,\beta} B = [c_{ij}]$, where

$$c_{ij} = \begin{cases} a_{ij}, & i \in S_1,\ j \in S_1 \cup S_2, \\ 0, & i \in S_1,\ j \in S_3, \\ a_{ij}, & i \in S_2,\ j \in S_1, \\ \alpha a_{ij} + \beta b_{i-t,j-t}, & i \in S_2,\ j \in S_2, \\ b_{i-t,j-t}, & i \in S_2,\ j \in S_3, \\ 0, & i \in S_3,\ j \in S_1, \\ b_{i-t,j-t}, & i \in S_3,\ j \in S_2 \cup S_3, \end{cases},$$

with

$$S_1 = \{1, 2, \ldots, n_1 - k\},\ S_2 = \{n_1 - k + 1, \ldots, n_1\},\ S_3 = \{n_1 + 1, \ldots, n\}, \tag{2.1}$$

$t = n_1 - k$ and $n = n_1 + n_2 - k$. Obviously, $S_1 \cup S_2 \cup S_3 = N$. It is worth noting that the weighted k-subdirect sum is reduced to the k-subdirect sum when $\alpha = \beta = 1$. In this case, "$\oplus_k^{\alpha,\beta}$" will be abbreviated as "$\oplus_k$".

Definition 1. *[13] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a strictly diagonally dominant (SDD) matrix if for each $i \in N$,*

$$|a_{ii}| > r_i(A) := \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

Definition 2. *[20] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a Nekrasov matrix if for each $i \in N$,*

$$|a_{ii}| > h_i(A),$$

where $h_1(A) = r_1(A),\ h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|,\ i = 2, 3, \ldots, n.$

Let $A = [a_{ij}] \in \mathbb{C}^{n_1\times n_1}$, $B = [b_{ij}] \in \mathbb{C}^{n_2\times n_2}$, and let S_2 be an index set defined in (2.1). Given positive real parameters $\alpha > 0$ and $\beta > 0$, we define the parameter constraint set

$$I_{(\alpha,\beta)} := \left\{ (\alpha,\beta) | (\alpha - 1) \sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} \geq (1-\beta) \sum_{j=n_1+1}^{n} |b_{i-t,j-t}|, i \in S_2 \right\}. \tag{2.2}$$

3. Weighted subdirect sum of Nekrasov matrices. In this section, we consider the weighted subdirect sum of Nekrasov matrices and give affirmative answers for questions Q-I and Q-III under certain conditions. Before that, an example is given to show that the weighted subdirect sum of a Nekrasov matrix of order n_1 and an SDD matrix of order n_2 may not be a Nekrasov matrix.

Example 3.1. *Consider the following matrices from [18]:*

$$A = \left[\begin{array}{ccc:c} 7.9 & -0.5 & -0.5 & -0.5 \\ -9 & 16 & -5 & -5 \\ -4 & -7 & 9.6 & -3 \\ \hdashline -4.9 & -0.9 & -4.76 & 6 \end{array}\right], \quad B = \left[\begin{array}{c:ccc} 1.51 & -0.4 & -0.5 & -0.6 \\ \hdashline 0 & 1.63 & -0.8 & -0.8 \\ -0.5 & -0.1 & 2.4 & -0.9 \\ -0.5 & -0.8 & -0.2 & 2.9 \end{array}\right],$$

where A is a Nekrasov matrix and B is an SDD matrix. If we take $\alpha = 0.5, \beta = 0.5$, then

$$C = A \oplus_1^{0.5,0.5} B = \left[\begin{array}{ccc:c:ccc} 7.9 & -0.5 & -0.5 & -0.5 & 0 & 0 & 0 \\ -9 & 16 & -5 & -5 & 0 & 0 & 0 \\ -4 & -7 & 9.6 & -3 & 0 & 0 & 0 \\ \hdashline -4.9 & -0.9 & -4.76 & 3.755 & -0.4 & -0.5 & -0.6 \\ \hdashline 0 & 0 & 0 & 0 & 1.63 & -0.8 & -0.8 \\ 0 & 0 & 0 & -0.5 & -0.1 & 2.4 & -0.9 \\ 0 & 0 & 0 & -0.5 & -0.8 & -0.2 & 2.9 \end{array}\right].$$

By calculation, we know that $C = A \oplus_1^{0.5,0.5} B$ is not a Nekrasov matrix as $h_4(C) = 7.4931 > 3.755 = |c_{44}|$.

Example 3.1 shows that, in general, we cannot ensure that the weighted subdirect sum of Nekrasov matrices is also in the Nekrasov class. However, under certain conditions, we can guarantee that the weighted k-subdirect sum of a Nekrasov matrix and an SDD matrix is in the class of Nekrasov matrices. The weighted 1-subdirect sum is first considered.

Theorem 3. *Let $A = [a_{ij}] \in \mathbb{C}^{n_1\times n_1}$ and $B = [b_{ij}] \in \mathbb{C}^{n_2\times n_2}$ be two matrices partitioned as in (1.1). Let $k = 1$, S_1, S_2 and S_3 be sets defined by (2.1). Assume that A is a Nekrasov matrix and B is an SDD matrix. If all diagonal entries of A_{22} and B_{11} are positive (or negative), then, for $(\alpha,\beta) \in I_{(\alpha,\beta)}$ given by (2.2), the weighted 1-subdirect sum $C = A \oplus_1^{\alpha,\beta} B$ is a Nekrasov matrix.*

Proof. Since A is a Nekrasov matrix, it follows that $|a_{ii}| > h_i(A)$ for all $i \in S_1 \cup S_2$. We divide three cases to prove that $|c_{ii}| > h_i(C)$ for all $i \in S_1 \cup S_2 \cup S_3$.

Case 1: $i \in S_1$. We have

$$\begin{aligned}|c_{ii}| = |a_{ii}| > h_i(A) &= \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| \\ &= \sum_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^{n} |c_{ij}| = h_i(C).\end{aligned}$$

Case 2: $i = n_1 \in S_2$. By the assumptions, we have

$$\begin{aligned}|c_{n_1,n_1}| = \alpha |a_{n_1,n_1}| + \beta |b_{11}| &> \alpha h_{n_1}(A) + \beta r_1(B) \\ &= \alpha \sum_{j=1}^{n_1-1} |a_{n_1,j}| \frac{h_j(A)}{|a_{jj}|} + \beta \sum_{j=2}^{n_2} |b_{1j}| \\ &\geq \sum_{j=1}^{n_1-1} |a_{n_1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=2}^{n_2} |b_{1j}| \text{ (by (2.2))} \\ &= \sum_{j=1}^{n_1-1} |c_{n_1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+1}^{n} |c_{n_1,j}| \\ &= h_{n_1}(C).\end{aligned}$$

Case 3: $i \in S_3$. For $i = n_1 + 1$, we have

$$\begin{aligned}|c_{n_1+1,n_1+1}| = |b_{22}| > r_2(B) &= |b_{21}| + \sum_{j=3}^{n_2} |b_{2j}| \\ &= \sum_{j=1}^{n_1} |c_{n_1+1,j}| + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\ &\geq \sum_{j=1}^{n_1} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\ &= h_{n_1+1}(C).\end{aligned}$$

Suppose that $|c_{ii}| > h_i(C)$ for all $i < n_1 + m$, where m is a positive integer and $1 < m \leq n_2 - 1$. Note that B is an SDD matrix and $t = n_1 - 1$. It follows that

$$
\begin{aligned}
|c_{n_1+m,n_1+m}| = |b_{n_1+m-t,n_1+m-t}| > r_{n_1+m-t}(B) &\\
&= \sum_{j=1}^{n_1+m-t-1} |b_{n_1+m-t,j}| + \sum_{j=n_1+m-t+1}^{n_2} |b_{n_1+m-t,j}| \\
&= \sum_{j=1}^{n_1+m-1} |c_{n_1+m,j}| + \sum_{j=n_1+m+1}^{n} |c_{n_1+m,j}| \\
&\geq \sum_{j=1}^{n_1+m-1} |c_{n_1+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+m+1}^{n} |c_{n_1+m,j}| \\
&= h_{n_1+m}(C).
\end{aligned}
$$

Therefore, we can conclude that $|c_{ii}| > h_i(C)$ for all $i \in S_3$. From Cases 1, 2, and 3, it follows that for any $i \in S_1 \cup S_2 \cup S_3$, $|c_{ii}| > h_i(C)$, which implies that $C = A \oplus_1^{\alpha,\beta} B$ is a Nekrasov matrix. ☐

Example 3.2. *Consider matrices A and B again in Example 3.1, where A is a Nekrasov matrix and B is an SDD matrix. Without loss of generality, if we take $\alpha = 1.1$, then from (2.2) we have*

$$
\begin{aligned}
\beta &\geq \max\left\{0, \max_{i\in S_2}\left\{1 + \frac{(1-\alpha)\sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|}}{\sum_{j=n_1+1}^{n} |b_{i-t,j-t}|}\right\}\right\} \\
&= \max\left\{0, 1 + \frac{(1-\alpha)\sum_{j=1}^{3} |a_{4j}| \frac{h_j(A)}{|a_{jj}|}}{\sum_{j=5}^{7} |b_{1,j-3}|}\right\} \\
&= 0.6005.
\end{aligned}
$$

Hence, by Theorem 3, it follows that $C = A \oplus_1^{1.1,\beta} B$ is a Nekrasov matrix for any $\beta \geq 0.6005$. In fact, for $\beta = 0.8$, we have

$$
C = A \oplus_1^{1.1,0.8} B = \left[\begin{array}{ccc:c:ccc}
7.9 & -0.5 & -0.5 & -0.5 & 0 & 0 & 0 \\
-9 & 16 & -5 & -5 & 0 & 0 & 0 \\
-4 & -7 & 9.6 & -3 & 0 & 0 & 0 \\
\hdashline
-4.9 & -0.9 & -4.76 & 7.808 & -0.4 & -0.5 & -0.6 \\
\hdashline
0 & 0 & 0 & 0 & 1.63 & -0.8 & -0.8 \\
0 & 0 & 0 & -0.5 & -0.1 & 2.4 & -0.9 \\
0 & 0 & 0 & -0.5 & -0.8 & -0.2 & 2.9
\end{array}\right].
$$

By calculation, we know that $|c_{ii}| > h_i(C)$ holds for $i = 1, 2, \ldots, 7$, which implies that $C = A \oplus_1^{1.1,0.8} B$ is a Nekrasov matrix. However, when $k \geq 2$, $C = A \oplus_k^{\alpha,\beta} B$ may not be a Nekrasov matrix for $(\alpha, \beta) \in I_{(\alpha,\beta)}$. For example, for $k = 2, \alpha = 1, \beta = 1$ ($(\alpha, \beta) \in I_{(\alpha,\beta)}$), one has

$$
C = A \oplus_2^{1,1} B = \left[\begin{array}{cc:cc:cc}
7.9 & -0.5 & -0.5 & -0.5 & 0 & 0 \\
-9 & 16 & -5 & -5 & 0 & 0 \\
\hdashline
-4 & -7 & 11.11 & -3.4 & -0.5 & -0.6 \\
-4.9 & -0.9 & -4.76 & 7.63 & -0.8 & -0.8 \\
\hdashline
0 & 0 & -0.5 & -0.1 & 2.4 & -0.9 \\
0 & 0 & -0.5 & -0.8 & -0.2 & 2.9
\end{array}\right].
$$

Obviously, $C = A \oplus_2^{1,1} B$ *is not a Nekrasov matrix because* $|c_{44}| = 7.63 < 7.6371 = h_4(C)$.

Example 3.2 motivates the search for other conditions such that $C = A \oplus_k^{\alpha,\beta} B$ for $k \geq 2$ is a Nekrasov matrix, when A is a Nekrasov matrix and B is an SDD matrix.

Theorem 4. *Let* $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ *and* $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ *be two matrices partitioned as in* (1.1). *Let* k *be an integer such that* $1 \leq k \leq \min\{n_1, n_2\}$, *and let* S_1, S_2, *and* S_3 *be sets defined by (2.1). Assume that* A *is a Nekrasov matrix and* B *is an SDD matrix. If all diagonal entries of* A_{22} *and* B_{11} *are positive (or negative), and* $a_{ij} = 0$ *for all* $j < i$ *,* $i, j \in S_2$*, then, for* $(\alpha, \beta) \in I_{(\alpha,\beta)}$ *given by (2.2), the weighted* k*-subdirect sum* $C = A \oplus_k^{\alpha,\beta} B$ *is a Nekrasov matrix.*

Proof. Since A is a Nekrasov matrix, it follows that $|a_{ii}| > h_i(A)$ for all $i \in S_1 \cup S_2$. We divide three cases to prove that $|c_{ii}| > h_i(C)$ for all $i \in S_1 \cup S_2 \cup S_3$.

Case 1: $i \in S_1$. Similarly to the proof of Case 1 in Theorem 3, it follows that $|c_{ii}| > h_i(C)$.

Case 2: $i \in S_2$. For $i = n_1 - k + 1$, by the assumptions, it follows that

$$\begin{aligned}
|c_{n_1-k+1,n_1-k+1}| &= \alpha|a_{t+1,t+1}| + \beta|b_{11}| \\
&> \alpha h_{t+1}(A) + \beta r_1(B) \\
&= \alpha \left(\sum_{j=1}^{t} |a_{t+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+2}^{n_1} |a_{t+1,j}| \right) + \beta \sum_{j=2}^{n_2} |b_{1j}| \\
&= \alpha \sum_{j=1}^{t} |a_{t+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+2}^{n_1} (\alpha|a_{t+1,j}| + \beta|b_{1,j-t}|) + \beta \sum_{j=k+1}^{n_2} |b_{1j}| \\
&\geq \alpha \sum_{j=1}^{t} |a_{t+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+2}^{n_1} |c_{t+1,j}| + \beta \sum_{j=k+1}^{n_2} |b_{1j}| \\
&\geq \sum_{j=1}^{t} |a_{t+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+2}^{n_1} |c_{t+1,j}| + \sum_{j=k+1}^{n_2} |b_{1j}| \ \text{(by (2.2))} \\
&= \sum_{j=1}^{t} |c_{t+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+2}^{n_1} |c_{t+1,j}| + \sum_{j=n_1+1}^{n} |c_{t+1,j}| \\
&= h_{t+1}(C).
\end{aligned}$$

Suppose that $|c_{ii}| > h_i(C)$ for all $i < t + m$, where m is a positive integer and $1 < m \leq k$. Since

$$\begin{aligned}
|c_{t+m,t+m}| &= \alpha|a_{t+m,t+m}| + \beta|b_{m,m}| \\
&> \alpha h_{t+m}(A) + \beta r_m(B) \\
&= \alpha \left(\sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+1}^{t+m-1} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| \right) + \beta r_m(B) \\
&= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| + \beta r_m(B)
\end{aligned}$$

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$$\begin{aligned}
&= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+m+1}^{n_1} (\alpha |a_{t+m,j}| + \beta |b_{m,j-t}|) + \beta \sum_{j=1}^{m-1} |b_{m,j}| + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| \\
&\geq \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+m+1}^{n_1} |\alpha a_{t+m,j} + \beta b_{m,j-t}| + \beta \sum_{j=1}^{m-1} |b_{m,j}| + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| \\
&\geq \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+m+1}^{n_1} |c_{t+m,j}| + \beta \sum_{j=1}^{m-1} |b_{m,j}| + \sum_{j=k+1}^{n_2} |b_{m,j}| \text{ (by (2.2))} \\
&= \sum_{j=1}^{t} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n_1} |c_{t+m,j}| + \sum_{j=t+1}^{t+m-1} |c_{t+m,j}| + \sum_{j=n_1+1}^{n} |c_{t+m,j}| \\
&\geq \sum_{j=1}^{t} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n} |c_{t+m,j}| \\
&= h_{t+m}(C),
\end{aligned}$$

it follows that $|c_{ii}| > h_i(C)$ for all $i \in S_2$.

Case 3: $i \in S_3$. When $i = n_1 + 1$, we have

$$\begin{aligned}
|c_{n_1+1,n_1+1}| = |b_{k+1,k+1}| &> r_{k+1}(B) \\
&= \sum_{j=1}^{k} |b_{k+1,j}| + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \\
&\geq \sum_{j=1}^{n_1} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\
&= h_{n_1+1}(C).
\end{aligned}$$

Suppose that $|c_{ii}| > h_i(C)$ for all $n_1 < i \leq n_1 + l - 1$, where l is a positive integer and $1 < l \leq n_2 - k$. Since

$$\begin{aligned}
|c_{n_1+l,n_1+l}| = |b_{k+l,k+l}| > r_{k+l}(B) &= \sum_{j=1}^{k+l-1} |b_{k+l,j}| + \sum_{j=k+l+1}^{n_2} |b_{k+l,j}| \\
&= \sum_{j=t+1}^{n_1} |c_{n_1+l,j}| + \sum_{j=n_1+1}^{n_1+l-1} |c_{n_1+l,j}| + \sum_{j=n_1+l+1}^{n} |c_{n_1+l,j}| \\
&\geq \sum_{j=1}^{n_1+l-1} |c_{n_1+l,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+l+1}^{n} |c_{n_1+l,j}| \\
&= h_{n_1+l}(C),
\end{aligned}$$

it follows that $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Cases 1, 2, and 3, it follows that for any $i \in S_1 \cup S_2 \cup S_3$, $|c_{ii}| > h_i(C)$. This completes the proof. ∎

EXAMPLE 3.3. *Consider the following matrices in [18]:*

$$A = \left[\begin{array}{c:ccc} 7.9 & -0.5 & -0.5 & -0.5 \\ \hdashline -9 & 12 & -5 & -5 \\ -4 & 0 & 9.6 & -3 \\ -4.9 & 0 & 0 & 6 \end{array}\right], \quad B = \left[\begin{array}{ccc:c} 1.51 & -0.4 & -0.5 & -0.6 \\ 0 & 1.63 & -0.8 & -0.8 \\ -0.5 & -0.1 & 1.6 & -0.9 \\ \hdashline -0.5 & -0.8 & -0.2 & 2.9 \end{array}\right],$$

where A is a Nekrasov matrix and B is an SDD matrix. For $k = 3$, if we take $\alpha = 1.2$, then from (2.2) we have

$$\beta \geq \max\left\{0, \max_{i \in S_2}\left\{1 + \frac{(1-\alpha)\sum_{j=1}^{n_1-k}|a_{ij}|\frac{h_j(A)}{|a_{jj}|}}{\sum_{j=n_1+1}^{n}|b_{i-t,j-t}|}\right\}\right\} = 0.8101.$$

Observe that $a_{32} = a_{42} = a_{43} = 0$, which satisfies the hypotheses of Theorem 4. Therefore, by Theorem 4, it follows that $A \oplus_3^{1.2,\beta} B$ is a Nekrasov matrix for any $\beta \geq 0.8101$. In fact, if we take $\beta = 0.9$, then

$$C = A \oplus_3^{1.2,0.9} B = \left[\begin{array}{c:ccc:c} 7.9 & -0.5 & -0.5 & -0.5 & 0 \\ \hdashline -9 & 15.759 & -6.36 & -6.45 & -0.6 \\ -4 & 0 & 12.987 & -4.32 & -0.8 \\ -4.9 & -0.45 & -0.09 & 8.64 & -0.9 \\ \hdashline 0 & -0.5 & -0.8 & -0.2 & 2.9 \end{array}\right].$$

By computation, $|c_{ii}| > h_i(C)$ holds for $i = 1, 2, \ldots, 5$, i.e., $C = A \oplus_3^{1.2,0.9} B$ is a Nekrasov matrix.

THEOREM 5. *Let $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ and $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ be two matrices partitioned as in (1.1). Let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$, and let S_1, S_2, and S_3 be sets defined by (2.1). Assume that A is a Nekrasov matrix and B is an SDD matrix. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), and for all $i \in S_2$, $\frac{h_i(A)}{|a_{ii}|} \geq \frac{r_{i-t}(B)}{|b_{i-t,i-t}|}$, where $t = n_1 - k$, then, for $(\alpha, \beta) \in I_{(\alpha,\beta)}$ given by (2.2), the k-weighted subdirect sum $C = A \oplus_k^{\alpha,\beta} B$ is a Nekrasov matrix.*

Proof. Since A is a Nekrasov matrix, it follows that $|a_{ii}| > h_i(A)$ for all $i \in S_1 \cup S_2$. We divide three cases to prove that $|c_{ii}| > h_i(C)$ for all $i \in S_1 \cup S_2 \cup S_3$.

Case 1: $i \in S_1$. Similarly to the proof of Case 1 in Theorem 3, it follows that $|c_{ii}| > h_i(C)$.

Case 2: $i \in S_2$. When $i = n_1 - k + 1$, similarly to the proof of Case 2 in Theorem 4, we have

$$|c_{n_1-k+1,n_1-k+1}| = \alpha|a_{t+1,t+1}| + \beta|b_{11}| > \alpha h_{t+1}(A) + \beta r_1(B) \geq h_{t+1}(C).$$

Suppose that $|c_{ii}| = \alpha|a_{ii}| + \beta|b_{i-t,i-t}| > \alpha h_i(A) + \beta r_{i-t}(B) \geq h_i(C)$ for all $i < t + m$, where m is a positive integer and $1 < m \leq k$. It follows from $\frac{h_i(A)}{|a_{ii}|} \geq \frac{r_{i-t}(B)}{|b_{i-t,i-t}|}$ that

$$\frac{\alpha h_i(A) + \beta r_{i-t}(B)}{\alpha|a_{ii}| + \beta|b_{i-t,i-t}|} \leq \frac{h_i(A)}{|a_{ii}|}, \ i < t + m.$$

Hence,

$$\begin{aligned}
h_{t+m}(C) &= \sum_{j=1}^{t} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n} |c_{t+m,j}| \\
&\leq \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+1}^{t+m-1} |\alpha a_{t+m,j} + \beta b_{m,j-t}| \frac{\alpha h_j(A) + \beta r_{j-t}(B)}{|\alpha a_{jj} + \beta b_{j-t,j-t}|} \\
&\quad + \sum_{j=t+m+1}^{n_1} |\alpha a_{t+m,j} + \beta b_{m,j-t}| + \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\
&\leq \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+1}^{t+m-1} |\alpha a_{t+m,j} + \beta b_{m,j-t}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| \\
&\quad + \beta \sum_{j=t+m+1}^{n_1} |b_{m,j-t}| + \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\
&\leq \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+1}^{t+m-1} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \beta \sum_{j=t+1}^{t+m-1} |b_{m,j-t}| + \alpha \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| \\
&\quad + \beta \sum_{j=t+m+1}^{n_1} |b_{m,j-t}| + \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\
&\leq \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+1}^{t+m-1} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| + \beta \sum_{j=t+1}^{t+m-1} |b_{m,j-t}| \\
&\quad + \beta \sum_{j=t+m+1}^{n_1} |b_{m,j-t}| + \beta \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\
&= \alpha \left(\sum_{j=1}^{t+m-1} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| \right) + \beta \left(\sum_{j=t+1}^{t+m-1} |b_{m,j-t}| + \sum_{j=t+m+1}^{n} |b_{m,j-t}| \right) \\
&= \alpha h_{t+m}(A) + \beta r_m(B).
\end{aligned}$$

This implies that

$$|c_{t+m,t+m}| = \alpha |a_{t+m,t+m}| + \beta |b_{t+m-t,t+m-t}| > \alpha h_{t+m}(A) + \beta r_{t+m-t}(B) \geq h_{t+m}(C).$$

Hence, we can conclude that $|c_{ii}| > h_i(C)$ for all $i \in S_2$.

Case 3: $i \in S_3$. Similarly to the proof of Case 3 in Theorem 4, it follows that $|c_{ii}| > h_i(C)$.

From Cases 1, 2, and 3, we have that for any $i \in S_1 \cup S_2 \cup S_3$, $|c_{ii}| > h_i(C)$. This completes the proof. ☐

EXAMPLE 3.4. *Consider the matrices* $A_1 = [a_{ij}]$ *and* $A_2 = [b_{ij}]$ *as defined in the introduction, where* A_1 *is a Nekrasov matrix and* A_2 *is an SDD matrix. For* $k = 4$, *the index set* S_2 *is given by* $S_2 = \{3, 4, 5, 6\}$. *By computations, we obtain the following inequalities:*

$$\frac{h_3(A_1)}{|a_{33}|} = 0.9524 > 0.8750 = \frac{r_1(A_2)}{|b_{11}|}, \frac{h_4(A_1)}{|a_{44}|} = 0.9841 > 0.8611 = \frac{r_2(A_2)}{|b_{22}|},$$

$$\frac{h_5(A_1)}{|a_{55}|} = 0.9868 > 0.7934 = \frac{r_3(A_2)}{|b_{33}|}, \text{ and } \frac{h_6(A_1)}{|a_{66}|} = 0.9290 > 0.6809 = \frac{r_4(A_2)}{|b_{44}|}.$$

These inequalities satisfy the hypotheses of Theorem 5. Consequently, by Theorem 5 and formula (2.2), the matrix $A_1 \oplus_4^{\alpha,\beta} A_2$ *is a Nekrasov matrix for all* $\alpha \geq 1$ *and* $\beta \geq 1$.

Example 3.5. *Consider the following matrices in [10]:*

$$A=\left[\begin{array}{c|ccc} 8 & -0.5 & -0.5 & -0.5\\ \hline -9 & 16 & 5 & -5\\ -6 & -4 & 15 & -3\\ -5 & -1 & -1 & 6 \end{array}\right],\quad B=\left[\begin{array}{ccc|c} 7 & 1 & -1.2 & 2\\ 7 & 88 & 2 & -3\\ 2 & 0.5 & 13 & -2\\ \hline 0.5 & 3 & 1 & 6 \end{array}\right],$$

where A is a Nekrasov matrix and B is an SDD matrix. For $k=3$, if we take $\alpha=2$, then by (2.2) it holds that

$$\beta \geq \max\left\{0, \max_{i\in S_2}\left\{1+\frac{(1-\alpha)\sum_{j=1}^{n_1-k}|a_{ij}|\frac{h_j(A)}{|a_{jj}|}}{\sum_{j=n_1+1}^{n}|b_{i-t,j-t}|}\right\}\right\}=0.6250.$$

Since $a_{ij}\neq 0$, for each $j<i$ with $i,j\in S_2=\{2,3,4\}$, it follows that the matrix A does not satisfy the hypotheses of Theorem 4, so we cannot use Theorem 4 to verify whether $C=A\oplus_3^{2,\beta}B$ is a Nekrasov matrix or not for those $\beta\geq 0.6250$. However, by computations,

$$\frac{h_2(A)}{|a_{22}|}>0.7304>0.6=\frac{r_1(B)}{|b_{11}|},\frac{h_3(A)}{|a_{33}|}>0.4697>0.1364>\frac{r_2(B)}{|b_{22}|},$$

and

$$\frac{h_4(A)}{|a_{44}|}>0.3562>0.3462>\frac{r_3(B)}{|b_{33}|},$$

which satisfy the hypotheses of Theorem 5. Therefore, from Theorem 5, $C=A\oplus_3^{2,\beta}B$ is a Nekrasov matrix for any $\beta\geq 0.6250$. In fact, without loss of generality, assume that $\beta=0.8$. Then, the weighted 3-subdirect sum $C=A\oplus_3^{2,0.8}B$ gives

$$C=\left[\begin{array}{c|ccc|c} 8 & -0.5 & -0.5 & -0.5 & 0\\ \hline -9 & 37.6 & 10.8 & -10.96 & 2\\ -6 & -2.4 & 100.4 & -4.4 & -3\\ -5 & -0.4 & -1.6 & 22.4 & -2\\ \hline 0 & 0.5 & 3 & 1 & 6 \end{array}\right].$$

In fact, since $h_1(C)=1.5, h_2(C)=25.4475, h_3(C)=10.1493, h_4(C)=3.37$, and $h_5(C)=0.7921$, it follows that $|c_{ii}|>h_i(C)$ holds for $i=1,2,\ldots,5$, i.e., $C=A\oplus_3^{2,0.8}B$ is a Nekrasov matrix.

Theorem 6. *Let $A=[a_{ij}]\in\mathbb{C}^{n_1\times n_1}$ and $B=[b_{ij}]\in\mathbb{C}^{n_2\times n_2}$ be two matrices partitioned as in* (1.1). *Let k be an integer such that $1\leq k\leq\min\{n_1,n_2\}$, and let S_1, S_2, and S_3 be sets defined by (2.1). Assume that A is a Nekrasov matrix and B is an SDD matrix. If all diagonal entries of A_{22} and B_{11} are positive (or negative), then, for $(\alpha,\beta)\in I_{(\alpha,\beta)}$ given by (2.2) such that*

$$\alpha|a_{ii}|+\beta|b_{i-t,i-t}|-\alpha h_i(A)-\beta r_{i-t}(B)>\sum_{j=1}^{i-t-1}\delta_{i-t,j},\ i\in S_2\backslash\{t+1\}, \tag{3.1}$$

the weighted k-subdirect sum $C=A\oplus_k^{\alpha,\beta}B$ is a Nekrasov matrix, where $t=n_1-k$ and

$$\delta_{i-t,j}=|\alpha a_{i,t+j}+\beta b_{i-t,j}|\frac{h_{t+j}(A,B)}{\alpha|a_{t+j,t+j}|+\beta|b_{j,j}|}-\alpha|a_{i,t+j}|\frac{h_{t+j}(A)}{|a_{t+j,t+j}|}-\beta|b_{i-t,j}|,$$

with

$$h_{t+1}(A,B):=\sum_{j=1}^{t}|a_{t+1,j}|\frac{h_j(A)}{|a_{jj}|}+\sum_{j=k+1}^{n_2}|b_{1j}|+\sum_{j=2}^{k}|\alpha a_{t+1,t+j}+\beta b_{1j}|,$$

and for $p = 2, \dots, i-t-1$,

$$\begin{aligned} h_{t+p}(A,B) :&= \sum_{j=1}^{t} |a_{t+p,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=k+1}^{n_2} |b_{pj}| \\ &+ \sum_{j=1}^{p-1} |\alpha a_{t+p,t+j} + \beta b_{pj}| \frac{h_{t+j}(A,B)}{\alpha|a_{t+j,t+j}| + \beta|b_{jj}|} + \sum_{j=1}^{p-1} |\alpha a_{t+j,t+j} + \beta b_{jj}|. \end{aligned}$$

Proof. Since A is a Nekrasov matrix, it follows that $|a_{ii}| > h_i(A)$ for all $i \in S_1 \cup S_2$. We divide three cases to prove that $|c_{ii}| > h_i(C)$ for all $i \in S_1 \cup S_2 \cup S_3$.

Case 1: $i \in S_1$. Similarly to the proof of Case 1 in Theorem 3, it follows that $|c_{ii}| > h_i(C)$.

Case 2: $i \in S_2$. When $i = n_1 - k + 1$, similarly to the proof of Case 2 in Theorem 4, we have

$$|c_{n_1-k+1,n_1-k+1}| = \alpha|a_{t+1,t+1}| + \beta|b_{11}| > \alpha h_{t+1}(A) + \beta r_1(B) \geq h_{t+1}(C).$$

Suppose that $|c_{ii}| > h_i(C)$ for all $i < t+m$, where m is a positive integer and $1 < m \leq k$. Next we prove that $|c_{t+m,t+m}| > h_{t+m}(C)$. By the assumptions, we have

$$\begin{aligned} |c_{t+m,t+m}| &= \alpha|a_{t+m,t+m}| + \beta|b_{m,m}| \\ &> \alpha h_{t+m}(A) + \beta r_m(B) + \sum_{j=1}^{m-1} \delta_{m,j} \\ &= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+1}^{t+m-1} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=t+m+1}^{n_1} |a_{t+m,j}| \\ &\quad + \beta \sum_{j=1}^{m-1} |b_{m,j}| + \beta \sum_{j=m+1}^{k} |b_{m,j}| + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| + \sum_{j=1}^{m-1} \delta_{m,j} \\ &= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \alpha \sum_{j=1}^{m-1} |a_{t+m,t+j}| \frac{h_{t+j}(A)}{|a_{t+j,t+j}|} + \alpha \sum_{j=m+1}^{k} |a_{t+m,t+j}| \\ &\quad + \beta \sum_{j=1}^{m-1} |b_{m,j}| + \beta \sum_{j=m+1}^{k} |b_{m,j}| + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| + \sum_{j=1}^{m-1} \delta_{m,j} \\ &= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} (\alpha |a_{t+m,t+j}| \frac{h_{t+j}(A)}{|a_{t+j,t+j}|} + \beta |b_{m,j}|) \\ &\quad + \sum_{j=m+1}^{k} (\alpha|a_{t+m,t+j}| + \beta|b_{m,j}|) + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| \\ &\quad + \sum_{j=1}^{m-1} (|\alpha a_{t+m,t+j} + \beta b_{m,j}| \frac{h_{t+j}(A,B)}{\alpha|a_{t+j,t+j}| + \beta|b_{jj}|} - \alpha|a_{t+m,t+j}| \frac{h_{t+j}(A)}{|a_{t+j,t+j}|} - \beta|b_{m,j}|) \end{aligned}$$

$$\begin{aligned}
&= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} |\alpha a_{t+m,t+j} + \beta b_{m,j}| \frac{h_{t+j}(A,B)}{\alpha |a_{t+j,t+j}| + \beta |b_{jj}|} \\
&\quad + \sum_{j=m+1}^{k} (\alpha |a_{t+m,t+j}| + \beta |b_{m,j}|) + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| \\
&= \alpha \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} |\alpha a_{t+m,t+j} + \beta b_{m,j}| \frac{h_{t+j}(C)}{\alpha |a_{t+j,t+j}| + \beta |b_{jj}|} \\
&\quad + \sum_{j=m+1}^{k} (\alpha |a_{t+m,t+j}| + \beta |b_{m,j}|) + \beta \sum_{j=k+1}^{n_2} |b_{m,j}| \\
&\geq \sum_{j=1}^{t} |a_{t+m,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} |\alpha a_{t+m,t+j} + \beta b_{m,j}| \frac{h_{t+j}(C)}{\alpha |a_{t+j,t+j}| + \beta |b_{jj}|} \\
&\quad + \sum_{j=m+1}^{k} (\alpha |a_{t+m,t+j}| + \beta |b_{m,j}|) + \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\
&\geq \sum_{j=1}^{t} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n_1} |c_{t+m,j}| + \sum_{j=n_1+1}^{n} |c_{t+m,j}| \\
&= \sum_{j=1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n} |c_{t+m,j}| \\
&= h_{t+m}(C).
\end{aligned}$$

Hence, by mathematical induction, we have $|c_{ii}| > h_i(C)$ for all $i \in S_2$.

Case 3: $i \in S_3$. Similarly to the proof of Case 3 in Theorem 4, we have $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Cases 1, 2, and 3, we know that $|c_{ii}| > h_i(C)$ for any $i \in S_1 \cup S_2 \cup S_3$. The proof is complete. □

EXAMPLE 3.6. *Consider the following matrices:*

$$A = \left[\begin{array}{cc:cc} 2 & 0.5 & 0.5 & 0.5 \\ -4 & 4 & 0.25 & 0.25 \\ \hdashline -4 & -4 & 10 & 0.5 \\ 2.5 & 1 & 1 & 4 \end{array}\right], \quad B = \left[\begin{array}{cc:cc} 4 & 1 & 1 & -1 \\ 1 & 8 & -2 & 3 \\ \hdashline 2 & 1 & 6 & 2 \\ -3 & 3 & -4 & 12 \end{array}\right],$$

where A is a Nekrasov matrix and B is an SDD matrix. For $k = 2$, we take $\alpha = 1.5$, then from (2.2) we have

$$\beta \geq \max\left\{0, \max_{i \in S_2}\left\{1 + \frac{(1-\alpha)\sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|}}{\sum_{j=n_1+1}^{n} |b_{i-t,j-t}|}\right\}\right\} = 0.7250.$$

Take $\beta = 0.8$. For $i \in S_2 \backslash \{t+1\} = \{4\}$, we have

$$\alpha |a_{ii}| + \beta |b_{i-t,i-t}| - \alpha h_i(A) - \beta r_{i-t}(B) = 2.4350,$$

and

$$\begin{aligned}\sum_{j=1}^{i-t-1}\delta_{i-t,j} &= |\alpha a_{i,t+j}+\beta b_{i-t,j}|\frac{h_{t+j}(A,B)}{\alpha|a_{t+j,t+j}|+\beta|b_{jj}|}-\alpha|a_{i,t+j}|\frac{h_{t+j}(A)}{|a_{t+j,t+j}|}-\beta|b_{i-t,j}|\\ &= |\alpha a_{43}+\beta b_{21}|\frac{h_{2+1}(A,B)}{\alpha|a_{33}|+\beta|b_{11}|}-\alpha|a_{43}|\frac{h_3(A)}{|a_{33}|}-\beta|b_{21}|\\ &< 0,\end{aligned}$$

which implies that

$$\alpha|a_{44}|+\beta|b_{22}|-\alpha h_4(A)-\beta r_2(B)>\delta_{21}.$$

These satisfy the hypotheses of Theorem 6, and thus by Theorem 6, it holds that $C = A\oplus_2^{1.5,0.8} B$ is a Nekrasov matrix. In contrast, both Theorem 4 and Theorem 5 become invalid because $a_{43}=1$ and $\frac{h_3(A)}{|a_{33}|}=\frac{7}{10}<\frac{3}{4}=\frac{r_1(B)}{|b_{11}|}$. In fact,

$$C = A\oplus_2^{1.5,0.8} B = \left[\begin{array}{cc:cc:cc} 2 & 0.5 & 0.5 & 0.5 & 0 & 0\\ -4 & 4 & 0.25 & 0.25 & 0 & 0\\ \hdashline -4 & -4 & 18.2 & 1.55 & 1 & -1\\ 2.5 & 1 & 2.3 & 12.4 & -2 & 3\\ \hdashline 0 & 0 & 2 & 1 & 6 & 2\\ 0 & 0 & -3 & 3 & -4 & 12 \end{array}\right].$$

By computations, $h_i(C)<|c_{ii}|$ for $i=1,2,\ldots,6$, which means that C is a Nekrasov matrix.

Note that in Theorem 6 if, for $i\in S_2\backslash\{t+1\}$,

$$\sum_{j=1}^{i-t-1}|\alpha a_{i,t+j}+\beta b_{i-t,j}|\frac{h_{t+j}(A,B)}{\alpha|a_{t+j,t+j}|+\beta|b_{jj}|}\le\sum_{j=1}^{i-t-1}\left(\alpha|a_{i,t+j}|\frac{h_{t+j}(A)}{|a_{t+j,t+j}|}+\beta|b_{i-t,j}|\right), \tag{3.2}$$

then

$$\sum_{j=1}^{i-t-1}\delta_{i-t,j}\le 0,$$

which together with (3.1) immediately yields the following easily checkable sufficient condition.

COROLLARY 7. *Let $A=[a_{ij}]\in\mathbb{C}^{n_1\times n_1}$ and $B=[b_{ij}]\in\mathbb{C}^{n_2\times n_2}$ be two matrices partitioned as in (1.1). Let k be an integer such that $1\le k\le\min\{n_1,\ n_2\}$, and let S_1, S_2, and S_3 be sets defined by (2.1). Assume that A is a Nekrasov matrix and B is an SDD matrix. If all diagonal entries of A_{22} and B_{11} are positive (or negative), then, for $(\alpha,\beta)\in I_{(\alpha,\beta)}$ such that (3.2) holds, the weighted k-subdirect sum $C=A\oplus_k^{\alpha,\beta}B$ is a Nekrasov matrix.*

EXAMPLE 3.7. *Consider the following matrices from [29]:*

$$A=\left[\begin{array}{ccc:cccc} 8 & 0.5 & 1 & 0.5 & 1 & -1 & -1\\ -1 & 6 & 1 & 1 & 1 & 1 & 1\\ 9 & -1 & 14 & 2 & 1 & -1 & -1\\ \hdashline 9 & 7 & 2 & 18 & -1 & 0.5 & 0.5\\ 8 & 6 & 4 & 0 & 18 & 1 & 1\\ 4 & 8 & 4 & 2 & 1 & 20 & 2\\ -8 & -12 & 6 & -3 & 2 & 7 & 36 \end{array}\right],\quad B=\left[\begin{array}{cccc:ccc} 4 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5\\ 1 & 8 & -2 & -1 & 0.5 & 0.5 & 1\\ 1.5 & -0.5 & 10 & 1 & 1 & -3 & -1\\ 3 & 3 & 2 & 12 & 0.5 & 1 & 1.5\\ \hdashline -1 & -1 & -1 & -1 & 9 & 1 & 1\\ 3 & -2 & 1 & 1 & 1 & 15 & 2\\ 5 & 3 & 1 & 1 & 3 & 2 & 20 \end{array}\right].$$

It is easy to verify that A is a Nekrasov matrix and B is an SDD matrix. For $k = 4, \alpha = 1.1$, by computations, we have

$$\beta \geq \max\left\{0, \max_{i\in S_2}\left\{1 + \frac{(1-\alpha)\sum_{j=1}^{n_1-k}|a_{ij}|\frac{h_j(A)}{|a_{jj}|}}{\sum_{j=n_1+1}^{n}|b_{i-t,j-t}|}\right\}\right\} = 0.7339.$$

Take $\beta = 0.8$, the weighted 4-subdirect sum of A and B is

$$C = A \oplus_4^{1.1,0.8} B = \left[\begin{array}{ccc|cccc|ccc} 8 & 0.5 & 1 & 0.5 & 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 6 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 9 & -1 & 14 & 2 & 1 & -1 & -1 & 0 & 0 & 0 \\ \hline 9 & 7 & 2 & 23 & -0.7 & 0.95 & 0.95 & 0.5 & 0.5 & 0.5 \\ 8 & 6 & 4 & 0.8 & 26.2 & -0.5 & 0.3 & 0.5 & 0.5 & 1 \\ 4 & 8 & 4 & 3.4 & 0.7 & 30 & 3 & 1 & -3 & -1 \\ -8 & -12 & 6 & -0.9 & 4.6 & 9.3 & 49.2 & 0.5 & 1 & 1.5 \\ \hline 0 & 0 & 0 & -1 & -1 & -1 & -1 & 9 & 1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 1 & 1 & 1 & 15 & 2 \\ 0 & 0 & 0 & 5 & 3 & 1 & 1 & 3 & 2 & 20 \end{array}\right].$$

It is easy to verify that (3.2) holds for all $i \in S_2 \backslash \{t+1\} = \{5, 6, 7\}$. Hence, by Corollary 7, it follows that $C = A \oplus_4^{1.1,0.8} B$ is a Nekrasov matrix.

Next, we provide some sufficient conditions to ensure that the weighted subdirect sum of two Nekrasov matrices belongs to the same class.

THEOREM 8. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be all Nekrasov matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let k, S_1, S_2, S_3, t be the same as Theorem 4. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $A_{21} = 0$, and for $i, j \in S_2, j \neq i$,*

$$|\alpha a_{ij} + \beta b_{i-t,j-t}| \leq |\beta b_{i-t,j-t}|,$$

where $\alpha \geq 0$ and $\beta \geq 1$, then the weighted k-subdirect sum $C = A \oplus_k^{\alpha,\beta} B$ is a Nekrasov matrix.

Proof. We divide the following three cases to finish the proof.

Case 1: $i \in S_1$. Similarly to Case 1 in the proof of Theorem 3, we have for each $i \in S_1$, $|c_{ii}| > h_i(C)$.

Case 2: $i \in S_2$. We first prove that $h_i(C) \leq h_{i-t}(B)$. For $i = n_1 - k + 1$, by the assumptions it holds that

$$\begin{aligned} h_{n_1-k+1}(C) &= \sum_{j=1}^{n_1-k}|c_{n_1-k+1,j}|\frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+2}^{n}|c_{n_1-k+1,j}| \\ &= \sum_{j=1}^{n_1-k}|a_{n_1-k+1,j}|\frac{h_j(A)}{|a_{jj}|} + \sum_{j=n_1-k+2}^{n}|c_{n_1-k+1,j}| \\ &= \sum_{j=n_1-k+2}^{n_1}|\alpha a_{n_1-k+1,j} + \beta b_{1,j-t}| + \sum_{j=n_1+1}^{n}|b_{1,j-t}| \\ &\leq \beta\sum_{j=n_1-k+2}^{n}|b_{1,j-t}| = \beta h_1(B). \end{aligned}$$

Assume that $h_i(C) \le h_{i-t}(B)$ holds for all $i < n_1 - k + m$, where m is a positive integer and $1 < m \le k$. Since

$$\begin{aligned} h_{t+m}(C) &= \sum_{j=1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n} |c_{t+m,j}| \\ &= \sum_{j=t+1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n} |c_{t+m,j}| \\ &\le \beta \sum_{j=t+1}^{t+m-1} |b_{m,j-t}| \frac{h_{j-t}(B)}{|b_{j-t,j-t}|} + \beta \sum_{j=t+m+1}^{n_1} |b_{m,j-t}| + \beta \sum_{j=n_1+1}^{n} |b_{m,j-t}| \\ &= h_m(B), \end{aligned}$$

it follows that $h_i(C) \le h_{i-t}(B)$. This immediately leads to

$$|c_{ii}| = |\alpha a_{ij} + \beta b_{i-t,j-t}| \ge |b_{i-t,i-t}| > h_{i-t}(B) \ge h_i(C).$$

Case 3: $i \in S_3$. For $i = n_1 + 1$, we have

$$\begin{aligned} |c_{n_1+1,n_1+1}| &= |b_{k+1,k+1}| \\ &> h_{k+1}(B) \\ &= \sum_{j=1}^{k} |b_{k+1,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \\ &= \sum_{j=1}^{n_1-k} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+1}^{n_1} |c_{n_1+1,j}| \frac{h_{j-t}(B)}{|b_{j-t,j-t}|} + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\ &\ge \sum_{j=1}^{n_1} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\ &= h_{n_1+1}(C). \end{aligned}$$

Suppose that $|c_{ii}| \ge h_{i-t}(B) > h_i(C)$ for all $n_1 < i \le n_1+l-1$, where l is a positive integer and $1 < l \le n_2-k$. Since

$$\begin{aligned} |c_{n_1+l,n_1+l}| &= |b_{k+l,k+l}| \\ &> h_{k+l}(B) \\ &= \sum_{j=1}^{k+l-1} |b_{k+l,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+l+1}^{n_2} |b_{k+l,j}| \\ &= \sum_{j=1}^{n_1-k} |c_{n_1+l,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+1}^{n_1+l-1} |c_{n_1+l,j}| \frac{h_{j-t}(B)}{|b_{j-t,j-t}|} + \sum_{j=n_1+l+1}^{n} |c_{n_1+l,j}| \\ &\ge \sum_{j=1}^{n_1+l-1} |c_{n_1+l,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+l+1}^{n} |c_{n_1+l,j}| \\ &= h_{n_1+l}(C), \end{aligned}$$

it follows that $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Case 1, Case 2, and Case 3, we have that $|c_{ii}| > h_i(C)$ for any $i \in S_1 \cup S_2 \cup S_3$. This implies that $C = A \oplus_k^{\alpha,\beta} B$ is a Nekrasov matrix. $\square$

Example 3.8. *Consider the following two Nekrasov matrices in [10]:*

$$A = \left[\begin{array}{c|ccc} 9 & -4 & -1 & -2 \\ \hline 0 & 10 & 1 & 2 \\ 0 & 2 & 4 & 2 \\ 0 & 1 & 3 & 7 \end{array}\right] \text{ and } B = \left[\begin{array}{ccc|c} 60 & -15 & -15 & -15 \\ -75 & 105 & -45 & 0 \\ -60 & -60 & 120 & -15 \\ \hline -15 & -15 & -15 & 45 \end{array}\right].$$

For $k = 3$, if we take $\alpha = 0.5, \beta = 1$, then it is easy to verify that $|\alpha a_{ij} + \beta b_{i-t,j-t}| \le |\beta b_{i-t,j-t}|$ holds for all $i, j \in S_2 = \{2, 3, 4\}, j \neq i$, which satisfies the hypotheses of Theorem 8. Therefore, by Theorem 8, we get that $C = A \oplus_3^{0.5,1} B$ is a Nekrasov matrix. In fact,

$$C = A \oplus_3^{0.5,1} B = \left[\begin{array}{c|ccc|c} 9 & -4 & -1 & -2 & 0 \\ \hline 0 & 65 & -14.5 & -14 & -15 \\ 0 & -74 & 107 & -44 & 0 \\ 0 & -59.5 & -58.5 & 123.5 & -15 \\ \hline 0 & -15 & -15 & -15 & 45 \end{array}\right].$$

By computation, $h_1(C) = 7, h_2(C) = 43.5, h_3(C) = 93.5231, h_4(C) = 105.951, h_5(C) = 36.0177$. Obviously, $|c_{ii}| > h_i(C)$ holds for $i = 1, 2, \ldots, 5$, i.e., $C = A \oplus_2^{0.5,1} B$ is a Nekrasov matrix.

Using the same technique, we can easily obtain the following result.

Theorem 9. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be all Nekrasov matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let k, S_1, S_2, S_3, t be the same as Theorem 4. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $B_{12} = 0$, and for $j \neq i, i, j \in S_2$,*

$$|\alpha a_{ij} + \beta b_{i-t,j-t}| \le |\alpha a_{ij}|,$$

where $\alpha \ge 1$ and $\beta \ge 0$, then the weighted k-subdirect sum $C = A \oplus_k^{\alpha,\beta} B$ is a Nekrasov matrix.

Example 3.9. *Consider the following two Nekrasov matrices:*

$$A = \left[\begin{array}{c|ccc} 9 & -4 & -1 & -2 \\ \hline 17 & 40 & 10 & 15 \\ 15 & 25 & 65 & 25 \\ 13 & 20 & 20 & 50 \end{array}\right] \text{ and } B = \left[\begin{array}{ccc|c} 35 & -15 & -15 & 0 \\ -15 & 58 & -45 & 0 \\ -30 & -35 & 64 & 0 \\ \hline -15 & -15 & -15 & 45 \end{array}\right].$$

For $k = 3$, if we take $\alpha = 1.2, \beta = 0.5$, then $|\alpha a_{ij} + \beta b_{i-t,j-t}| \le |\alpha a_{ij}|$ holds for all $i, j \in S_2 = \{2, 3, 4\}, j \neq i$, which satisfies the hypotheses of Theorem 9. Therefore, by Theorem 9, we get that $C = A \oplus_3^{1.2,0.5} B$ is a Nekrasov matrix. In fact,

$$C = A \oplus_3^{1.2,0.5} B = \left[\begin{array}{c|ccc|c} 9 & -4 & -1 & -2 & 0 \\ \hline 17 & 65.5 & 4.5 & 10.5 & 0 \\ 15 & 22.5 & 107 & 7.5 & 0 \\ 13 & 9 & 6.5 & 92 & 0 \\ \hline 0 & -15 & -15 & -15 & 45 \end{array}\right].$$

By computation, $h_1(C) = 7, h_2(C) = 28.2222, h_3(C) = 28.8613, h_4(C) = 15.7422, h_5(C) = 13.0758$. Hence, $|c_{ii}| > h_i(C)$ holds for $i = 1, 2, \ldots, 5$, which implies that $C = A \oplus_2^{1.2,0.5} B$ is a Nekrasov matrix.

4. Conclusions. In this paper, we give some sufficient conditions for the weighted k-subdirect sum $A \oplus_k^{\alpha,\beta} B$ to be a Nekrasov matrix when A is a Nekrasov matrix and B is an SDD matrix, or A and B are both Nekrasov matrices. These provide affirmative answers to questions Q-I and Q-III for Nekrasov matrices. Nevertheless, questions Q-II and Q-IV for Nekrasov matrices remain open and worth studying in the future. In addition, although $A \oplus_k^{\alpha,\beta} B$ is a Nekrasov matrix, $B \oplus_k^{\alpha,\beta} A$ is not a Nekrasov matrix in general. So finding some conditions such that $B \oplus_k^{\alpha,\beta} A$ is a Nekrasov matrix will be one of the major tasks in the future.

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