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EXTREME SPECTRA REALIZATION BY REAL SYMMETRIC TRIDIAGONAL AND REAL SYMMETRIC ARROW MATRICES*

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Abstract. We consider the following two problems: to construct a real symmetric arrow matrix A and to construct a real symmetric tridiagonal matrix A, from a special kind of spectral information: one eigenvalue $\lambda^{(j)}$ of the $j \times j$ leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$; and one eigenpair $(\lambda^{(n)}, \mathbf{x})$ of A. Here we give a solution to the first problem, introduced in [J. Peng, X.Y. Hu, and L. Zhang. Two inverse eigenvalue problems for a special kind of matrices. *Linear Algebra Appl.*, 416:336-347, 2006.]. In particular, for both problems to have a solution, we give a necessary and sufficient condition in the first case, and a sufficient condition in the second one. In both cases, we also give sufficient conditions in order that the constructed matrices be nonnegative. Our results are constructive and they generate algorithmic procedures to construct such matrices.

Key words. Real symmetric tridiagonal matrices, Real symmetric arrow matrices, Eigenproblem.

AMS subject classifications. 65F15, 65F18, 15A18.

1. Introduction.

A discrete undamped system, consisting of n masses $m_i > 0$ connected by n springs of stiffness $k_i > 0$, is characterized by the generalized eigenvalue problem:

$$(K_n - \lambda M_n)\,\mathbf{u} = 0, \tag{1.1}$$

where $\lambda = \omega^2$, with ω a natural frequency and u the respective vibration mode, while $M_n = diag\,\{m_1, m_2, \ldots, m_n\}$ and

$$K_n = \begin{bmatrix} k_1 - k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}$$

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are the mass matrix and the stiffness matrix, respectively.

The generalized eigenvalue problem (1.1) can be expressed as the standard eigenvalue problem

$$(J_n - \lambda I_n)\,\mathbf{x} = 0,$$

where $J_n = M_n^{-\frac{1}{2}} K_n M_n^{-\frac{1}{2}}$ and $x = M_n^{\frac{1}{2}}\mathbf{u}$. It is well known that the system (M_n, K_n) can be reconstructed from some spectral information of the matrix J_n. There are several procedures to reconstruct J_n. One of the most popular procedures considers the n eigenvalues of J_n, and the $n-1$ eigenvalues of the submatrix J_{n-1}, obtained by deleting the last (first) row and column of J (see [5]).

Recently, new spectral information to reconstruct the system (M_n, K_n) have been considered by Huang et al. in [7]:

i. λ_i and λ_i^*, associate to the maximal and minimal natural frequencies of (M_i, K_i), $i = 1, \ldots, n$, respectively.
ii. $\lambda_i \in \mathbb{R}^+$, associate to the maximal (or minimal) frequency of (M_i, K_i), $i = 1, \ldots, n$, and $x \in \mathbb{R}^n$, the mode corresponding to λ_n,

where (M_i, K_i) is the subsystem of (M_n, K_n), that is, obtained by fixing the mass m_{i+1}. Note that the subsystems (M_i, K_i) correspond to the leading principal submatrices of J_n.

Peng et al. [8] propose two inverse eigenvalue problems to reconstruct an arrow matrix from similar spectral data. Unfortunately, the proposed solutions in [8] are not correct. In [9], a correct solution is given for this problem, while in [10], a Jacobi matrix is reconstructed from the same spectral data.

In this paper, we study two inverse eigenproblems. One of them related with the construction of an $n \times n$ symmetric arrow matrix of the form

$$\begin{bmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & & & \\ b_2 & & a_3 & & \\ \vdots & & & \ddots & \\ b_{n-1} & & & & a_n \end{bmatrix}, \quad a_j, b_j \in \mathbb{R}, \tag{1.2}$$

and the other one related with the construction of an $n \times n$ symmetric tridiagonal

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matrix of the form

$$\begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}, \quad a_j, b_j \in \mathbb{R}. \tag{1.3}$$

The following eigenproblem was proposed and discussed in [8, Problem II]:

Problem 1.1. Given n real numbers $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$, and a real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, find an $n \times n$ symmetric arrow matrix A of the form (1.2), where the a_i are all distinct for $i = 2, 3, \ldots, n$ and the b_i are all positive, and such that $\lambda^{(j)}$ is an eigenvalue of the $j \times j$ leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A.

We observe that in this problem, the formulae to compute the a_i and b_i, given in [8, Theorem 2], may lead us to a matrix which does not satisfy the requirements. For instance, the numbers

$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
2	$\frac{(5-\sqrt{65})}{2}$	-2	$\frac{(5+\sqrt{85})}{2}$

and the vector

$$\mathbf{x} = \left(\frac{1}{2}(\sqrt{85} - 1), 4, 2, 1 \right)^T$$

satisfy the conditions in [8, Theorem 2]. However, the unique resultant matrix is

$$A = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix},$$

which has repeated diagonal entries.

In this work, we rediscuss the above eigenproblem to give a necessary and sufficient condition, which leads us to a correct solution. We also study the following eigenproblem for real symmetric tridiagonal matrices:

Problem 1.2. Given n real numbers $\lambda^{(j)}$, $j = 1, 2, \ldots, n$ and a real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, find necessary and sufficient conditions for the existence of an $n \times n$ symmetric tridiagonal matrix A of the form (1.3), such that $\lambda^{(j)}$ is an eigenvalue of the $j \times j$ leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A.

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This family of matrices appears in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems, which arise in many applications, including modern control theory and vibration analysis [1]–[10].

We shall denote as I_j the identity matrix of order j; as A_j the $j \times j$ leading principal submatrix of A; as $P_j(\lambda)$ the characteristic polynomial of A_j, and as $\lambda^{(j)}$ an eigenvalue of A_j. $\sigma(A_j)$ will denote the spectrum of A_j.

The paper is organized as follows: In Section 2, we consider Problem 1.1 with $\lambda^{(j)}$ as the maximal eigenvalue of A_j, and in this case, we give a correct solution to the eigenproblem in [8, Problem II]. We also give sufficient conditions in order that the symmetric arrow solution matrix is nonnegative. In Section 3, we consider Problem 1.2 with $\lambda^{(j)}$ being the maximal eigenvalue of A_j, and give sufficient conditions for the problem to have a solution with all its b_i entries positive and also a nonnegative solution. Finally, in Section 4, we show some examples to illustrate the results. Our results are constructive, in the sense that they generate an algorithmic procedure to construct the solution matrices.

2. Solution to Problem 1.1. We recall the following well known lemmas:

Lemma 2.1. *Let A be an $n \times n$ matrix of the form (1.2) and let $P_j(\lambda)$ the characteristic polynomial of the $j \times j$ leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$\begin{aligned} P_1(\lambda) &= (\lambda - a_1), \\ P_2(\lambda) &= (\lambda - a_2) P_1(\lambda) - b_1^2, \\ P_j(\lambda) &= (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda - a_i), \quad j = 3, 4, \ldots, n. \end{aligned}$$

This recurrence relation can be written as:

$$P_j(\lambda) = \prod_{i=1}^{j} (\lambda - a_i) - \sum_{k=1}^{j-1} \left(b_k^2 \prod_{\substack{i=2 \\ i \neq k+1}}^{j} (\lambda - a_i) \right), \quad j = 1, 2, \ldots, n. \tag{2.1}$$

Lemma 2.2. [8] *Let A be an $n \times n$ matrix of the form (1.2) with all its diagonal entries a_j distinct, $j = 2, 3, \ldots, n$, and all its entries b_i positive, $i = 1, 2, \ldots, n-1$. Then*
i) the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are all simple,
ii) the zeros of $P_{j-1}(\lambda)$ strictly separate those of $P_j(\lambda)$.

Lemma 2.3. [9] *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and maximal zero of $P(\lambda)$, then*

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1. If $\mu < \lambda_1$, we have $(-1)^n P(\mu) > 0$,
2. If $\mu > \lambda_n$, we have $P(\mu) > 0$.

LEMMA 2.4. [9] *Let A be a matrix of the form (1.2) with $b_i \neq 0$, $i = 1, 2, \ldots, n-1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then*

$$\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)}.$$

LEMMA 2.5. [8] *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a set of orthonormal eigenvectors associated to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A of the form (1.2), with all its diagonal entries a_j distinct, $j = 2, 3, \ldots, n$, and all its entries b_i positive, $i = 1, 2, \ldots, n-1$. Then $x_{\mu j} \neq 0$ for $\mu, j = 1, 2, \ldots, n$, where $x_{\mu j}$ denotes the μ-th entry of the vector $\mathbf{x}_j$.*

Now we propose a solution to Problem 1.1.

THEOREM 2.6. *Given n real numbers $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n)}$ and a real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, then there exists a matrix A of the form (1.2), with all its diagonal entries a_j distinct, $j = 2, 3, \ldots, n$, and all its entries b_i positive, $i = 1, 2, \ldots, n-1$, in such a way that $\lambda^{(j)}$ is the maximal eigenvalue of the $j \times j$ leading principal submatrix A_j, $j = 1, 2, \ldots, n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A if and only if the following conditions are satisfied:*

$$x_1 x_i > 0, \quad i = 1, \ldots, n, \tag{2.2}$$

$$\frac{x_1^2}{x_2^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(2)}\right)}{\lambda^{(2)} - \lambda^{(1)}}, \tag{2.3}$$

$$\frac{x_1^2}{x_j^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right)}{P_{j-1}\left(\lambda^{(j)}\right)}, \quad j = 3, 4, \ldots, n, \tag{2.4}$$

and

$$\frac{x_{k+1}}{x_{j+1}} \neq \frac{b_k}{b_j}, \quad j = 2, 3, \ldots, n-1 \text{ and } k = 1, 2, \ldots, j-1. \tag{2.5}$$

Proof. First, we observe that to prove the existence of a matrix A with the required spectral properties is equivalent to show that the system of equations

$$P_j\left(\lambda^{(j)}\right) = 0, \quad j = 1, 2, \ldots, n, \tag{2.6}$$

$$A\mathbf{x} = \lambda^{(n)}\mathbf{x}, \tag{2.7}$$

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has real solutions a_j and b_{j-1}, such that $a_j \neq a_k$, $j \neq k$ and $b_{j-1} > 0$.

From (2.6), it is clear that

$$a_1 = \lambda^{(1)}.$$

For $j = 2, 3, \ldots, n$, the entries a_j and b_{j-1} can be obtained as follows: We rewrite (2.7) as:

$$\left.\begin{aligned} a_1x_1 + b_1x_2 + \cdots + b_{n-1}x_n = \lambda^{(n)}x_1 \\ b_{j-1}x_1 + a_jx_j = \lambda^{(n)}x_j,\ j = 2, 3, \ldots, n \end{aligned}\right\}. \tag{2.8}$$

From (2.2) we have $x_i \neq 0$ for $i = 1, \ldots, n$, and then

$$a_j = \lambda^{(n)} - b_{j-1}\frac{x_1}{x_j}, \quad j = 2, 3, \ldots, n. \tag{2.9}$$

By substituting (2.9) in (2.6), for $j = 2$, we obtain the following quadratic equation in b_1:

$$b_1^2 - b_1\frac{x_1}{x_2}P_1\left(\lambda^{(2)}\right) + \left(\lambda^{(n)} - \lambda^{(2)}\right)P_1\left(\lambda^{(2)}\right) = 0.$$

This equation has real solutions

$$b_1 = \frac{1}{2}\left[\frac{x_1}{x_2}P_1\left(\lambda^{(2)}\right) \pm \sqrt{\triangle_1}\right]$$

if

$$\triangle_1 = \frac{x_1^2}{x_2^2}\left[P_1\left(\lambda^{(2)}\right)\right]^2 - 4\left(\lambda^{(n)} - \lambda^{(2)}\right)P_1\left(\lambda^{(2)}\right) \geq 0,$$

or, from Lemma 2.2, if

$$\frac{x_1^2}{x_2^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(2)}\right)}{P_1\left(\lambda^{(2)}\right)},$$

which is condition (2.3). Moreover, from Lemma 2.2 (ii) and Lemma 2.4,

$$4\left(\lambda^{(n)} - \lambda^{(2)}\right)P_1\left(\lambda^{(2)}\right) > 0$$

and

$$\frac{x_1}{x_2}P_1\left(\lambda^{(2)}\right) > \left|\sqrt{\triangle_1}\right|.$$

Then, it follows that $b_1 > 0$.

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From Lemma 2.1, for $j = 3, 4, \dots, n$, it follows that

$$P_j\left(\lambda^{(j)}\right) = \left(\lambda^{(j)} - a_j\right) P_{j-1}\left(\lambda^{(j)}\right) - b_{j-1}^2 \prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right) = 0. \tag{2.10}$$

By substituting (2.9) in (2.10) we obtain the quadratic equation

$$b_{j-1}^2 \prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right) - b_{j-1}\frac{x_1}{x_j} P_{j-1}\left(\lambda^{(j)}\right) + \left(\lambda^{(n)} - \lambda^{(j)}\right) P_{j-1}\left(\lambda^{(j)}\right) = 0, \tag{2.11}$$

whose real solutions are given by

$$b_{j-1} = \frac{\frac{x_1}{x_j} P_{j-1}\left(\lambda^{(j)}\right) \pm \sqrt{\triangle_{j-1}}}{2\prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right)}$$

if

$$\triangle_{j-1} = \frac{x_1^2}{x_j^2}\left[P_{j-1}\left(\lambda^{(j)}\right)\right]^2 - 4\left(\lambda^{(n)} - \lambda^{(j)}\right) P_{j-1}\left(\lambda^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right) \geq 0,$$

or, from Lemma 2.2, if

$$\frac{x_1^2}{x_j^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right)}{P_{j-1}\left(\lambda^{(j)}\right)},$$

which is condition (2.4). Moreover, from Lemma 2.2 and Lemma 4 in [9],

$$4\left(\lambda^{(n)} - \lambda^{(j)}\right) P_{j-1}\left(\lambda^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right) > 0$$

and

$$\frac{x_1}{x_j} P_{j-1}\left(\lambda^{(j)}\right) > \left|\sqrt{\triangle_{j-1}}\right|.$$

Then it follows that $b_{j-1} > 0$.

Moreover, from (2.11), (2.1) and (2.9), we have

$$a_1x_1 + b_1x_2 + \cdots + b_{n-1}x_n = a_1x_1 + \sum_{k=1}^{n-2} b_k x_{k+1} + b_{n-1}x_n$$

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$$
\begin{aligned}
&= a_1x_1 + \sum_{k=1}^{n-2} b_k x_{k+1} + \frac{x_1 P_{n-1}\left(\lambda^{(n)}\right)}{\prod\limits_{i=2}^{n-1}\left(\lambda^{(n)} - a_i\right)} \\
&= a_1x_1 + \sum_{k=1}^{n-2} b_k x_{k+1} + x_1 \frac{\prod\limits_{i=1}^{n-1}\left(\lambda^{(n)} - a_i\right) - \sum\limits_{k=1}^{n-2}\left(b_k^2 \prod\limits_{\substack{i=2\\ i\neq k+1}}^{n-1}\left(\lambda^{(n)} - a_i\right)\right)}{\prod\limits_{i=2}^{n-1}\left(\lambda^{(n)} - a_i\right)} \\
&= a_1x_1 + \sum_{k=1}^{n-2} b_k x_{k+1} + x_1\left[\left(\lambda^{(n)} - a_1\right) - \frac{1}{x_1}\sum_{k=1}^{n-2} b_k x_{k+1}\right] \\
&= \lambda^{(n)} x_1.
\end{aligned}
$$

Thus, the first equation in (2.8) is satisfied.

Finally, from (2.5), we have for $j = 2$ and $k = 1$,

$$\frac{x_2}{x_3} \neq \frac{b_1}{b_2}.$$

Then,

$$a_3 = \lambda^{(n)} - b_2\frac{x_1}{x_3} \neq \lambda^{(n)} - b_1\frac{x_1}{x_2} = a_2.$$

For $j = 3, 4, \ldots, n-1$,

$$\frac{x_{k+1}}{x_{j+1}} \neq \frac{b_k}{b_j}, \quad k = 1, 2, \ldots, j-1.$$

Then,

$$a_{j+1} = \lambda^{(n)} - b_j\frac{x_1}{x_{j+1}} \neq \lambda^{(n)} - b_k\frac{x_1}{x_{k+1}} = a_{k+1}, \quad k = 1, 2, \ldots, j-1,$$

and the entry a_{j+1}, $j = 2, 3, \ldots, n-1$ is distinct from each one of entries $a_2, a_3, \ldots, a_j$. Hence, all diagonal entries a_i, $i = 2, 3, \ldots, n$ are distinct. Thus, there exists $A = A_n$, of the form (1.2), such that $\lambda^{(j)}$, $j = 1, 2, \ldots, n$, is the maximal eigenvalue of the $j \times j$ leading principal submatrix A_j of A, with a_i distinct, $i = 2, 3, \ldots, n$, and b_i positive, $i = 1, 2, \ldots, n-1$. Besides, as (2.7) holds, $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A.

To show the necessity, observe that condition (2.2) is obtained from Lemma 2.5. The conditions (2.3) and (2.4) comes from the fact that $b_j > 0$ for $j = 1, 2, \ldots, n-1$. Finally, since a_i, $i = 2, 3, \ldots, n$, are all distinct and (2.9) holds, condition (2.5) is satisfied. $\square$

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REMARK 2.7. Theorem 2.6 also holds if we take $\lambda^{(j)}$ as the minimal eigenvalue of A_j. In this case we must have $\lambda^{(n)} < \cdots < \lambda^{(2)} < \lambda^{(1)}$ and condition (2.2) becomes $x_1 x_i < 0$, $i = 1, \ldots, n$. The proof is completely similar.

Now we give sufficient conditions in order that the solution matrix A in Problem 1.1 be nonnegative.

COROLLARY 2.8. *Given n real numbers $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n)}$ and a real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, there exists an $n \times n$ nonnegative matrix A of the form (1.2), such that $\lambda^{(j)}$ is the maximal eigenvalue of the $j \times j$ leading principal submatrix A_j, $j = 1, 2, \ldots, n$ and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A, if the following conditions are satisfied:*

$$x_1 x_i > 0, \quad i = 1, \ldots, n, \tag{2.12}$$

$$\frac{x_1^2}{x_2^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(2)}\right)}{\lambda^{(2)} - \lambda^{(1)}}, \tag{2.13}$$

$$\frac{x_1^2}{x_j^2} \geq \frac{4\left(\lambda^{(n)} - \lambda^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda^{(j)} - a_i\right)}{P_{j-1}\left(\lambda^{(j)}\right)}, \quad j = 3, \ldots, n, \tag{2.14}$$

$$\lambda^{(1)} \geq 0; \quad \lambda^{(n)} \geq b_{j-1}\frac{x_1}{x_j}, \quad j = 2, 3, \ldots, n.$$

Proof. From the proof of Theorem 2.6, conditions (2.12), (2.13) and (2.14) guarantee that the system of equations (2.6)-(2.7) has real solutions a_j and positive solutions b_{j-1}. It remains to show the nonnegativity of the diagonal entries a_j. Clearly, from (2.14) and (2.9), $a_1 = \lambda^{(1)} \geq 0$ and

$$a_j = \lambda^{(n)} - b_{j-1}\frac{x_1}{x_j} \geq 0, \quad j = 2, 3, \ldots, n. \quad \square$$

3. Solution to Problem 1.2. We start by recalling the following well known result:

LEMMA 3.1. *Let A be an $n \times n$ symmetric tridiagonal matrix of the form (1.3). Let A_j be the $j \times j$ leading principal submatrix of A, with characteristic polynomial $P_j(\lambda)$, $j = 1, 2, \ldots, n$. Then the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies*

$$P_j(\lambda) = (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda), \quad j = 1, 2, \ldots, n,$$

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where $P_0(\lambda) = P_{-1}(\lambda) = 1$ *and* $b_0 = 0$.

The next result gives sufficient conditions for a solution to Problem 1.2 with $b_i > 0$, $i = 1, 2, \ldots, n-1$.

Theorem 3.2. *Given* n *real numbers* $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n)}$, *and a real vector* $\mathbf{x} = (x_1, \ldots, x_n)^T$, *there exists an* $n \times n$ *matrix* A *of the form (1.3), with positive entries* b_i, *such that* $\lambda^{(j)}$ *is the maximal eigenvalue of the* $j \times j$ *leading principal submatrix* A_j *of* A, $j = 1, 2, \ldots, n$, *and* $(\lambda^{(n)}, \mathbf{x})$ *is an eigenpair of* A, *if the following conditions are satisfied:*

$$x_i x_{i+1} > 0, \quad i = 1, 2, \ldots, n \tag{3.1}$$

and

$$\lambda^{(n)} > a_j + b_{j-1}\frac{x_{j-1}}{x_j}, \quad j = 1, 2, \ldots, n-1. \tag{3.2}$$

Proof. To show the existence of a matrix A of the form (1.3), with the desired spectral properties, it is equivalent to show that the system of equations

$$P_j\left(\lambda^{(j)}\right) = 0, \quad j = 1, 2, \ldots, n, \tag{3.3}$$

$$A\mathbf{x} = \lambda^{(n)}\mathbf{x}, \tag{3.4}$$

has real solutions a_j and $b_{j-1} > 0$, where (3.4) can be written as

$$\left.\begin{array}{rl} a_1x_1 + b_1x_2 & = \lambda^{(n)}x_1 \\ b_1x_1 + a_2x_2 + b_2x_3 & = \lambda^{(n)}x_2 \\ b_2x_2 + a_3x_3 + b_3x_4 & = \lambda^{(n)}x_3 \\ \vdots & \quad\vdots \\ b_{j-1}x_{j-1} + a_jx_j + b_jx_{j+1} & = \lambda^{(n)}x_j \\ \vdots & \quad\vdots \\ b_{n-2}x_{n-2} + a_{n-1}x_{n-1} + b_{n-1}x_n & = \lambda^{(n)}x_{n-1} \\ b_{n-1}x_{n-1} + a_nx_n & = \lambda^{(n)}x_n \end{array}\right\}. \tag{3.5}$$

From (3.1), (3.3) and (3.5), it is clear that $a_1 = \lambda^{(1)}$,

$$b_1 = \left(\lambda^{(n)} - a_1\right)\frac{x_1}{x_2} > 0$$

and

$$b_j = \left(\lambda^{(n)} - a_j\right)\frac{x_j}{x_{j+1}} - b_{j-1}\frac{x_{j-1}}{x_{j+1}}, \tag{3.6}$$

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$j = 2, 3, \ldots, n-1$. From (3.2),

$$\left(\lambda^{(n)} - a_j\right) \frac{x_{j+1}}{x_{j+1}} > b_{j-1} \frac{x_{j-1}}{x_j}$$

and

$$\left(\lambda^{(n)} - a_j\right) \frac{x_j}{x_{j+1}} > b_{j-1} \frac{x_{j-1}}{x_{j+1}}.$$

Thus, $b_j > 0$, $j = 2, 3, \ldots, n-1$. Besides, from (3.3) it follows that

$$P_j\left(\lambda^{(j)}\right) = \left(\lambda^{(j)} - a_j\right) P_{j-1}\left(\lambda^{(j)}\right) - b_{j-1}^2 P_{j-2}\left(\lambda^{(j)}\right) = 0,$$

$j = 2, 3, \ldots, n$, and

$$a_j = \lambda^{(j)} - b_{j-1}^2 \frac{P_{j-2}\left(\lambda^{(j)}\right)}{P_{j-1}\left(\lambda^{(j)}\right)}, \ j = 2, 3, \ldots, n.$$

Observe that from Lemma 2.3 we have $P_{j-1}\left(\lambda^{(j)}\right) > 0$.

To show that the last equation in (3.5) is satisfied, we first show that

$$b_j = \frac{x_j}{x_{j+1}} \frac{P_j\left(\lambda^{(n)}\right)}{P_{j-1}\left(\lambda^{(n)}\right)}, \quad j = 1, 2, \ldots, n-1, \tag{3.7}$$

where $P_{j-1}\left(\lambda^{(n)}\right) > 0$ because of Lemma 2.3. For $j = 1$, we have

$$b_1 = \frac{x_1}{x_2}\left(\lambda^{(n)} - a_1\right) = \frac{x_1}{x_2} \frac{P_1\left(\lambda^{(n)}\right)}{P_0\left(\lambda^{(n)}\right)}.$$

Now, suppose that (3.7) is true for $j = k$:

$$b_k = \frac{x_k}{x_{k+1}} \frac{P_k\left(\lambda^{(n)}\right)}{P_{k-1}\left(\lambda^{(n)}\right)}.$$

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Then, from (3.6),

$$\begin{aligned}
b_{k+1} &= \left(\lambda^{(n)} - a_{k+1}\right)\frac{x_{k+1}}{x_{k+2}} - b_k\frac{x_k}{x_{k+2}} \\
&= \left(\lambda^{(n)} - a_{k+1}\right)\frac{x_{k+1}}{x_{k+2}} - b_k\frac{x_k}{x_{k+2}}\left[\frac{x_k}{x_{k+1}}\frac{P_k\left(\lambda^{(n)}\right)}{P_{k-1}\left(\lambda^{(n)}\right)}\frac{x_{k+1}}{x_k}\frac{P_{k-1}\left(\lambda^{(n)}\right)}{P_k\left(\lambda^{(n)}\right)}\right] \\
&= \left(\lambda^{(n)} - a_{k+1}\right)\frac{x_{k+1}}{x_{k+2}} - b_k\frac{x_k}{x_{k+2}}\left[b_k\frac{x_{k+1}}{x_k}\frac{P_{k-1}\left(\lambda^{(n)}\right)}{P_k\left(\lambda^{(n)}\right)}\right] \\
&= \frac{x_{k+1}}{x_{k+2}}\left[\left(\lambda^{(n)} - a_{k+1}\right) - b_k^2\frac{P_{k-1}\left(\lambda^{(n)}\right)}{P_k\left(\lambda^{(n)}\right)}\right] \\
&= \frac{x_{k+1}}{x_{k+2}}\left[\frac{\left(\lambda^{(n)} - a_{k+1}\right)P_k\left(\lambda^{(n)}\right) - b_k^2 P_{k-1}\left(\lambda^{(n)}\right)}{P_k\left(\lambda^{(n)}\right)}\right] \\
&= \frac{x_{k+1}}{x_{k+2}}\frac{P_{k+1}\left(\lambda^{(n)}\right)}{P_k\left(\lambda^{(n)}\right)}.
\end{aligned}$$

Thus, (3.7) holds and

$$x_{n-1} = b_{n-1}\frac{P_{n-2}\left(\lambda^{(n)}\right)}{P_{n-1}\left(\lambda^{(n)}\right)}.$$

Now,

$$\begin{aligned}
b_{n-1}x_{n-1} + a_n x_n &= b_{n-1}x_{n-1} + \left[\lambda^{(n)} - b_{n-1}^2\frac{P_{n-2}\left(\lambda^{(n)}\right)}{P_{n-1}\left(\lambda^{(n)}\right)}\right]x_n \\
&= b_{n-1}x_{n-1} + \lambda^{(n)}x_n - b_{n-1}\left[b_{n-1}\frac{P_{n-2}\left(\lambda^{(n)}\right)}{P_{n-1}\left(\lambda^{(n)}\right)}\right]x_n \\
&= b_{n-1}x_{n-1} + \lambda^{(n)}x_n - b_{n-1}x_{n-1} \\
&= \lambda^{(n)}x_n,
\end{aligned}$$

and the last equation in (3.5) is true.

Hence, there exists A of the form (1.3), with positive entries b_i, $\lambda^{(j)}$ being an eigenvalue of the leading principal matrix A_j, $j = 1, 2, \ldots, n$ and $(\lambda^{(n)}, \mathbf{x})$ an eigenpair of A. ☐

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Remark 3.3. Theorem 3.2 also holds if we take $\lambda^{(j)}$ as the minimal eigenvalue of A_j. In this case we must have $\lambda^{(n)} < \cdots < \lambda^{(2)} < \lambda^{(1)}$ and condition (3.1) become $x_i x_{i+1} < 0$, $i = 1, \ldots, n$. The proof is completely similar.

Now we give sufficient conditions in order that the solution matrix in Problem 1.2 (with $\lambda^{(j)}$ as the minimal eigenvalue of A_j) be nonnegative.

Corollary 3.4. *Let the n real numbers $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n)}$, and a real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ be given. If*

$$x_i x_{i+1} < 0, \quad i = 1, 2, \ldots, n-1, \tag{3.8}$$

$$\lambda^{(n)} > a_j + b_{j-1}\frac{x_{j-1}}{x_j}, \quad j = 2, 3, \ldots, n-1, \tag{3.9}$$

and

$$\lambda^{(1)} \geq 0 \quad \text{and} \quad \lambda^{(j)} \geq b_{j-1}^2 \frac{P_{j-2}\left(\lambda^{(j)}\right)}{P_{j-1}\left(\lambda^{(j)}\right)}, \quad j = 1, 2, \ldots, n, \tag{3.10}$$

then there exists an $n \times n$ nonnegative matrix A of the form (1.3), such that $\lambda^{(j)}$ is the maximal eigenvalue of the $j \times j$ leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$ and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A.

Proof. From Theorem 3.2, conditions (3.8) and (3.9) guarantee the existence of A of the form (1.3) with positive entries b_i, $\lambda^{(j)}$ being the maximal eigenvalue of A_j, and $(\lambda^{(n)}, \mathbf{x})$ an eigenpair of A. It only remains to show the nonnegativity of the diagonal entries a_j, $j = 1, 2, \ldots, n$. From (3.10), it follows that $a_1 = \lambda^{(1)} \geq 0$ and

$$a_j = \lambda^{(j)} - b_{j-1}^2 \frac{P_{j-2}\left(\lambda^{(j)}\right)}{P_{j-1}\left(\lambda^{(j)}\right)} \geq 0, \quad j = 2, 3, \ldots, n. \qquad \square$$

4. Examples.

Example 4.1. The real numbers

$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$
3.7542	4.4013	9.7618	10.2563	11.5778	11.5789

and the real vector

$$\mathbf{x} = \begin{bmatrix} -0.6942 & -0.0742 & -0.5200 & -0.2501 & -0.4237 & -0.0123 \end{bmatrix}^T$$

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satisfy conditions of Corollary 2.8. Then we may construct the nonnegative symmetric arrow matrix

$$A = \begin{bmatrix} 3.7542 & 0.9006 & 6.1056 & 1.9487 & 4.0160 & 0.1232 \\ 0.9006 & 3.1479 & & & & \\ 6.1056 & & 3.4273 & & & \\ 1.9487 & & & 6.1696 & & \\ 4.0160 & & & & 4.9990 & \\ 0.1232 & & & & & 4.6291 \end{bmatrix},$$

where the spectra of the matrices A_j are $\sigma(A_1) = \{\ \mathbf{3.7542}\ \}$,

$\sigma(A_2) = \{\ 2.5008 \quad \mathbf{4.4013}\ \}$,

$\sigma(A_3) = \{\ -2.5863 \quad 3.1539 \quad \mathbf{9.7618}\ \}$,

$\sigma(A_4) = \{\ -2.7960 \quad 3.1538 \quad 5.8848 \quad \mathbf{10.2563}\ \}$,

$\sigma(A_5) = \{\ -3.7669 \quad 3.1535 \quad 4.5001 \quad 6.0335 \quad \mathbf{11.5778}\ \}$,

$\sigma(A_6) = \{\ -3.7678 \quad 3.1535 \quad 4.4989 \quad 4.6301 \quad 6.0335 \quad \mathbf{11.5789}\ \}$,

and $A\mathbf{x} = (11.5789)\mathbf{x}$.

Example 4.2. Now we consider the following spectral information:

$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$
3.7542	2.5008	−2.5863	−2.7960	−3.7669	−3.7678

and

$$\mathbf{x} = \begin{bmatrix} -0.7097 & 0.0924 & 0.6022 & 0.1392 & 0.3251 & 0.0104 \end{bmatrix}^T.$$

Then, we obtain

$$A = \begin{bmatrix} 3.7542 & 0.9006 & 6.1056 & 1.9487 & 4.0160 & 0.1232 \\ 0.9006 & 3.1479 & & & & \\ 6.1056 & & 3.4273 & & & \\ 1.9487 & & & 6.1696 & & \\ 4.0160 & & & & 4.9990 & \\ 0.1232 & & & & & 4.6291 \end{bmatrix},$$

where

$\sigma(A_1) = \{\ \mathbf{3.7542}\ \}$,

$\sigma(A_2) = \{\ \mathbf{2.5008} \quad 4.4013\ \}$,

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$\sigma(A_3) = \{ \ \mathbf{-2.5863} \quad 3.1539 \quad 9.7618 \ \}$,

$\sigma(A_4) = \{ \ \mathbf{-2.7960} \quad 3.1538 \quad 5.8848 \quad 10.2563 \ \}$,

$\sigma(A_5) = \{ \ \mathbf{-3.7669} \quad 3.1535 \quad 4.5001 \quad 6.0335 \quad 11.5778 \ \}$,

$\sigma(A_6) = \{ \ \mathbf{-3.7678} \quad 3.1535 \quad 4.4989 \quad 4.6301 \quad 6.0335 \quad 11.5789 \ \}$,

and $A\mathbf{x} = (-3.7678)\mathbf{x}$.

Example 4.3. Given

$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$	$\lambda^{(7)}$
0.6829	3.4716	4.4172	4.4194	4.4195	4.4198	4.4238

and the vector

$$\mathbf{x} = \begin{bmatrix} 0.2275 & 0.5257 & 0.4566 & 0.0532 & 0.1407 & 0.1829 & 0.6383 \end{bmatrix}^T,$$

we construct the nonnegative symmetric tridiagonal matrix

$$A = \begin{bmatrix} 0.6829 & 1.6186 & & & & & \\ 1.6186 & 2.5321 & 1.3716 & & & & \\ & 1.3716 & 2.8277 & 0.1452 & & & \\ & & 0.1452 & 0.6665 & 0.9498 & & \\ & & & 0.9498 & 0.3571 & 2.8518 & \\ & & & & 2.8518 & 1.0901 & 0.3266 \\ & & & & & 0.3266 & 4.3302 \end{bmatrix},$$

with the required spectral properties, where

$\sigma(A_1) = \{ \ \mathbf{0.6829} \ \}$,

$\sigma(A_2) = \{ \ -0.2566 \quad \mathbf{3.4716} \ \}$,

$\sigma(A_3) = \{ \ -0.4208 \quad 2.0463 \quad \mathbf{4.4172} \ \}$,

$\sigma(A_4) = \{ \ -0.4219 \quad 0.6569 \quad 2.0548 \quad \mathbf{4.4194} \ \}$,

$\sigma(A_5) = \{ \ -0.4642 \quad -0.4101 \quad 1.4613 \quad 2.0599 \quad \mathbf{4.4195} \ \}$,

$\sigma(A_6) = \{ \ -2.3258 \quad -0.4217 \quad 0.6992 \quad 2.0540 \quad 3.7309 \quad \mathbf{4.4198} \ \}$,

$\sigma(A_7) = \{ \ -2.3320 \quad -0.4217 \quad 0.6964 \quad 2.0540 \quad 3.6512 \quad 4.4150 \quad \mathbf{4.4238} \ \}$,

and $A\mathbf{x} = (4.4238)\mathbf{x}$.

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