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LINEAR MAPS PRESERVING THE IDEMPOTENCY OF JORDAN PRODUCTS OF OPERATORS*

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Abstract. Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$ and let $\mathcal{I}^*(\mathcal{X})$ be the set of non-zero idempotent operators in $\mathcal{B}(\mathcal{X})$. A surjective map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ preserves nonzero idempotency of the Jordan products of two operators if for every pair $A, B \in \mathcal{B}(\mathcal{X})$, the relation $AB + BA \in \mathcal{I}^*(\mathcal{X})$ implies $\varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$. In this paper, the structures of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan products of two operators are given.

Key words. Banach space, Preserver, Idempotent, Jordan product.

AMS subject classifications. 47B49.

1. Introduction. This paper is a continuation of our recent work on preserver problems concerning certain properties of products or triple Jordan products of operators [3, 4, 13], and the related works in [2] and [7].

Let $\mathcal{X}$ be a complex Banach space, and let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. The dual of $\mathcal{X}$ is denoted by $\mathcal{X}'$ and the adjoint of $T \in \mathcal{B}(\mathcal{X})$ by T'. Let $\mathcal{I}^*(\mathcal{X})$, $\mathcal{I}_1(\mathcal{X})$ and $\mathcal{N}_1(\mathcal{X})$ be the set of nonzero idempotent operators, the set of rank-one idempotent operators and the set of rank-one nilpotent operators in $\mathcal{B}(\mathcal{X})$, respectively. If $\mathcal{X}$ has dimension n with $2 \leq n < \infty$, then $\mathcal{B}(\mathcal{X})$ is identified with the algebra $\mathcal{M}_n$ of $n \times n$ complex matrices and $\mathcal{I}_n(\mathcal{X})$ refers to the set of idempotent matrices in $\mathcal{M}_n$. For an operator $T \in \mathcal{B}(\mathcal{X})$, the range, the kernel and the rank of T are denoted by $R(T)$, $N(T)$ and rank T, respectively. Let $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}_1(\mathcal{X})$ denote the set of finite rank operators and the set of rank-one operators in $\mathcal{B}(\mathcal{X})$, respectively. For a non-zero vector $x \in \mathcal{X}$ and a non-zero $f \in \mathcal{X}'$, we denote by $x \otimes f$ the rank one operator defined by $(x \otimes f)y = f(y)x$, $y \in \mathcal{X}$. Note that every bounded linear rank one operator on $\mathcal{X}$ can be written in this form. The rank-one operator $x \otimes f$ is an idempotent operator if and only if $f(x) = 1$, and $x \otimes f$ is a nilpotent operator if and only if $f(x) = 0$. Given $P, Q \in \mathcal{I}(\mathcal{X})$, $P \leq Q$ if $PQ = QP = P$ and $P < Q$ if $P \leq Q$ with $P \neq Q$.

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In this paper, we are interested in determining the structure of linear surjective maps $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ for every pair $A, B \in \mathcal{B}(\mathcal{X})$ having the property that

$$AB + BA \in \mathcal{I}^*(\mathcal{X}) \Rightarrow \varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in \mathcal{I}^*(\mathcal{X}).$$

We will derive the following two theorems regarding the structure.

THEOREM 1.1. *Let φ be a linear map on $\mathcal{M}_n$ with $n \geq 3$. Then φ preserves the nonzero idempotency of Jordan products of two operators if and only if there exist an invertible matrix $A \in \mathcal{M}_n$ and a constant $\lambda \in \{1, -1\}$ such that one of the following holds.*

(1) $\varphi(X) = \lambda AXA^{-1}$ for all $X \in \mathcal{M}_n$;

(2) $\varphi(X) = \lambda AX^tA^{-1}$ for all $X \in \mathcal{M}_n$, where X^t is the transpose of X.

THEOREM 1.2. *Let $\mathcal{X}$ be a complex infinite dimensional Banach space and let φ be a linear surjective map on $\mathcal{B}(\mathcal{X})$. Then φ preserves the nonzero idempotency of Jordan products of two operators if and only if there exist a bounded invertible linear or conjugate-linear operator $A : \mathcal{X} \to \mathcal{X}$ and a constant $\lambda \in \{1, -1\}$ such that*

$$\varphi(X) = \lambda AXA^{-1} \text{ for all } X \in \mathcal{B}(\mathcal{X}),$$

or, only if $\mathcal{X}$ is reflexive, there exist a bounded invertible linear or conjugate-linear operator $A : \mathcal{X}' \to \mathcal{X}$ and a constant $\lambda \in \{1, -1\}$ such that

$$\varphi(X) = \lambda AX'A^{-1} \text{ for all } X \in \mathcal{B}(\mathcal{X}).$$

2. Preliminary results. Assume that $\mathcal{X}$ is a complex Banach space with dimension at least 3. In this section, we introduce some elementary results that will be used in the proofs of main theorems.

DEFINITION 2.1. [11] Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a filed $\mathbb{F}$. Linear operators $T_1, \ldots, T_n : \mathcal{U} \to \mathcal{V}$ are locally linearly dependent if $T_1u, \ldots, T_nu$ are linearly dependent for every $u \in \mathcal{U}$.

LEMMA 2.2. [11] *Assume that the operators A, B, I are locally linearly dependent. Then there exist scalars λ and μ such that $(A-\lambda)(B-\mu) = 0$ and either $(A-\lambda)^2 = 0$ or $(B-\mu)^2 = 0$.*

LEMMA 2.3. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be two idempotent operators. Then $zP+(1-z)Q \in \mathcal{I}^*(\mathcal{X})$ for any $z \in \mathbf{C}\backslash\{0, 1\}$ if and only if $P + Q = PQ + QP$.*

Proof. We obtain the desired result by direct calculation. □

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LEMMA 2.4. *Let $A \in \mathcal{B}(\mathcal{X})$ and $x \otimes f \in \mathcal{B}(\mathcal{X})$ be with $f(x) \neq 0$. Then $Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X})$ if and only if $Ax = \frac{1}{2f(x)}x$ or $A'f = \frac{1}{2f(x)}f$.*

Proof. Note that $Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X})$ implies that $Ax \otimes [f(Ax)f + f(x)A'f - f] + x \otimes [f(A^2x)f + f(Ax)A'f - A'f] = 0$. If Ax and x as well as $A'f$ and f are linearly independent, then $f(Ax) = 1,\ f(x) = 0$ and $f(A^2x) = 0$. But this contradicts the hypothesis $f(x) \neq 0$. Hence, either Ax and x are linearly dependent or $A'f$ and f are linearly dependent. Then we can compute that $Ax = \frac{1}{2f(x)}x$ or $A'f = \frac{1}{2f(x)}f$. ∎

The idea of the following lemma comes from [10].

LEMMA 2.5. *Let $A \in \mathcal{B}(\mathcal{X})$ be any non-scalar operator and let α be any fixed complex number. Then there exists an idempotent operator $P \in \mathcal{B}(\mathcal{X})$ of rank one such that α is an eigenvalue of $A + 2P$.*

Proof. As A is a non-scalar operator, we can find $x \in \mathcal{X}$ such that x and Ax are linearly independent. Define $P \in \mathcal{B}(\mathcal{X})$ by

$$Px = \tfrac{\alpha}{2}x - \tfrac{1}{2}Ax, \qquad PAx = \alpha(\tfrac{\alpha}{2} - 1)x - (1 - \tfrac{\alpha}{2})Ax \qquad \text{and} \qquad Pz = 0$$

for every $z \in \mathcal{X} \ominus [x, Ax]$. Clearly, P is an idempotent operator of rank one and $(A + 2P)x = \alpha x$. ∎

3. Main results. Assume that $\mathcal{X}$ is a complex Banach space with dimension at least 3 and we consider a linear surjective map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan products of operators, that is, $\varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$ whenever $AB + BA \in \mathcal{I}^*(\mathcal{X})$ for every pair $A, B \in \mathcal{B}(\mathcal{X})$.

LEMMA 3.1. *Let φ be as above. Then φ is injective.*

Proof. Assume that $\varphi(A) = 0$ for some non-zero operator $A \in \mathcal{B}(\mathcal{X})$. Suppose that there exists an $x \in \mathcal{X}$ such that x, Ax and A^2x are linearly independent. Then there is an $f \in \mathcal{X}'$ such that $f(x) = f(A^2x) = 0$ and $f(Ax) = 1$. It follows that $Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X})$. But, $\varphi(A)\varphi(x \otimes f) + \varphi(x \otimes f)\varphi(A) = 0 \notin \mathcal{I}^*(\mathcal{X})$. This contradiction implies that x, Ax and A^2x are linearly dependent for every $x \in \mathcal{X}$. By Lemma 2.2, there exist scalars λ and μ such that $(A - \lambda)(A^2 - \mu) = 0$ and either $(A - \lambda)^2 = 0$ or $(A^2 - \mu)^2 = 0$.

If $(A - \lambda)(A^2 - \mu) = 0$ and $(A - \lambda)^2 = 0$, then $(\mu - \lambda^2)A = \lambda(\mu - \lambda^2)$. When $\mu - \lambda^2 \neq 0$, we get that $A = \lambda I$. For any pair $x_1 \in \mathcal{X}$ and $f_1 \in \mathcal{X}'$ with $f_1(x_1) = \frac{1}{2\lambda}$, we have $Ax_1 \otimes f_1 + x_1 \otimes f_1 A \in \mathcal{I}^*(\mathcal{X})$ and then $\varphi(A)\varphi(x_1 \otimes f_1) + \varphi(x_1 \otimes f_1)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$. However, this contradicts $\varphi(A)\varphi(x_1 \otimes f_1) + \varphi(x_1 \otimes f_1)\varphi(A) = 0$. When $\mu - \lambda^2 = 0$ and $\lambda \neq 0$, we know that $A - \lambda I$ is a nilpotent operator and there is a non-zero vector $x_2 \in \mathcal{X}$ such that $Ax_2 = \lambda x_2$. Selecting $f_2 \in \mathcal{X}'$ with $f_2(x_2) = \frac{1}{2\lambda}$, we get $Ax_2 \otimes f_2 + x_2 \otimes f_2 A \in \mathcal{I}^*(\mathcal{X})$, which implies $\varphi(A)\varphi(x_2 \otimes f_2) + \varphi(x_2 \otimes f_2)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$.

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But $\varphi(A)\varphi(x_2\otimes f_2)+\varphi(x_2\otimes f_2)\varphi(A) = 0$ and we get a contradiction. When $\mu-\lambda^2 = 0$ and $\lambda = 0$, we know that $A^2 = 0$ and there exists a non-zero vector $x_3 \in \mathcal{X}$ such that x_3 and Ax_3 are linearly independent. Then there is an $f_3 \in \mathcal{X}'$ with $f_3(x_3) = 0$ and $f_3(Ax_3) = 1$. Hence, $Ax_3 \otimes f_3 + x_3 \otimes f_3A \in \mathcal{I}^*(\mathcal{X})$ and so $\varphi(A)\varphi(x_3 \otimes f_3) + \varphi(x_3 \otimes f_3)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$. However, this contradicts $\varphi(A)\varphi(x_3 \otimes f_3) + \varphi(x_3 \otimes f_3)\varphi(A) = 0$.

If $(A - \lambda)(A^2 - \mu) = 0$ and $(A^2 - \mu)^2 = 0$, then $(\mu - \lambda^2)A^2 = \lambda(\mu - \lambda^2)$. When $\mu - \lambda^2 \neq 0$, we know that $A^2 = \mu I$. Suppose $\mu \neq 0$. First, we assume that A is a non-scalar operator. Then there is an $x_4 \in \mathcal{X}$ such that x_4 and Ax_4 are linearly independent. We can find $f_4 \in \mathcal{X}'$ with $f_4(x_4) = \frac{1}{2\mu}$ and $f_4(Ax_4) = 0$. It follows $A(Ax_4) \otimes f_4 + (Ax_4) \otimes f_4A \in \mathcal{I}^*(\mathcal{X})$, which implies $\varphi(A)\varphi(Ax_4 \otimes f_4) + \varphi(Ax_4 \otimes f_4)\varphi(A) = 0 \in \mathcal{I}^*(\mathcal{X})$. This gives a contradiction. Second, we assume $A = \beta I$ for some non-zero number β. For any pair $x_5 \in \mathcal{X}$ and $f_5 \in \mathcal{X}'$ with $f_5(x_5) = \frac{1}{2\beta}$, we have $Ax_5\otimes f_5+x_5\otimes f_5A \in \mathcal{I}^*(\mathcal{X})$ and so $\varphi(A)\varphi(x_5\otimes f_5)+\varphi(x_5\otimes f_5)\varphi(A) = 0 \in \mathcal{I}^*(\mathcal{X})$. This also gives a contradiction. Hence, $\mu = 0$ and then $A^2 = 0$. Thus, there exists a non-zero vector $x_6 \in \mathcal{X}$ such that x_6 and Ax_6 are linearly independent. So, there is $f_6 \in \mathcal{X}'$ with $f_6(x_6) = 0$ and $f_6(Ax_6) = 0$. Obviously, $Ax_6 \otimes f_6 + x_6 \otimes f_6A \in \mathcal{I}^*(\mathcal{X})$ and then $\varphi(A)\varphi(x_6\otimes f_6)+\varphi(x_6\otimes f_6)\varphi(A) \in \mathcal{I}^*(\mathcal{X})$. However, $\varphi(A)\varphi(x_6\otimes f_6)+\varphi(x_6\otimes f_6)\varphi(A) = 0$ and this is also a contradiction. When $\mu - \lambda^2 = 0$, we know that $A^3 - \lambda A^2 - \lambda^2 A + \lambda^3 = 0$. Suppose $\lambda \neq 0$, we get $\frac{-1}{2\lambda^3}((A^2-\lambda A-\lambda^2 I)A+A(A^2-\lambda A-\lambda^2 I)) = I \in \mathcal{I}^*(\mathcal{X})$ implies $\frac{-1}{2\lambda^3}(\varphi(A^2 - \lambda A - \lambda^2 I)\varphi(A) + \varphi(A)\varphi(A^2 - \lambda A - \lambda^2 I)) = 0 \in \mathcal{I}^*(\mathcal{X})$. This is a contradiction. Suppose $\lambda = 0$, we know $A^3 = 0$. If $A^2 \neq 0$, then there is an $x_7 \in \mathcal{X}$ such that Ax_7 and A^2x_7 are linearly independent. So, there exists an $f_7 \in \mathcal{X}'$ such that $f_7(Ax_7) = 0$ and $f_7(A^2x_7) = 1$. Hence, $AAx_7 \otimes f_7 + Ax_7 \otimes f_7A \in \mathcal{I}^*(\mathcal{X})$ and then $\varphi(A)\varphi(Ax_7 \otimes f_7) + \varphi(x_7 \otimes f_7)\varphi(A) = 0 \in \mathcal{I}^*(\mathcal{X})$. This is also a contradiction. If $A^2 = 0$, then we can derive another contradiction by using a routine argument demonstrated above.

Thus, $A = 0$. Therefore, φ is injective. $\square$

LEMMA 3.2. *Let $N \in \mathcal{B}(\mathcal{X})$ be of finite rank and $N^2 = 0$. Then $\varphi(N)$ is a nilpotent operator.*

Proof. Note that $R(N)$ is finite dimensional. It is known that finite dimensional subspaces of a Banach space are complemented, and so $\mathcal{X} = R(N) + \mathcal{M}$ for some closed subspace $\mathcal{M}$ of $\mathcal{X}$ where $\mathcal{M}$ is a complementary subspace of $R(N)$. Then N has the following operator matrix

$$N = \begin{pmatrix} 0 & N_1 \\ 0 & 0 \end{pmatrix}.$$

Putting $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, we have that $(\frac{1}{2}(P+zN))(P+zN)+(P+zN)(\frac{1}{2}(P+zN)) = P + zN \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$. Thus, $(\frac{1}{2}\varphi(P + zN))\varphi(P + zN) + \varphi(P + zN)(\frac{1}{2}\varphi(P +$

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$zN)) = (\varphi(P) + z\varphi(N))^2 \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$. Hence, $(\varphi(P) + z\varphi(N))^2 = (\varphi(P) + z\varphi(N))^4$ for all $z \in \mathbb{C}$. That is,

$$(\varphi(P))^2 + z(\varphi(P)\varphi(N) + \varphi(N)\varphi(P)) + z^2(\varphi(N))^2 = (\varphi(P))^4 + \cdots + z^4(\varphi(N))^4$$

for all $z \in \mathbb{C}$. So, $(\varphi(N))^4 = 0$. $\square$

LEMMA 3.3. *If φ is surjective, then $\varphi(I) = \lambda I$ for some constant $\lambda \in \{1, -1\}$.*

Proof. Since φ is a surjective map, there exists a non-zero operator $A \in \mathcal{B}(\mathcal{X})$ such that $\varphi(A) = I$. Assume that A is a non-scalar operator. If there exists an $x \in \mathcal{X}$ such that x, Ax and A^2x are linearly independent, then there is an $f \in \mathcal{X}'$ such that $f(x) = f(A^2x) = 0$ and $f(Ax) = 1$. So, $Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X})$, which implies that $\varphi(A)\varphi(x \otimes f) + \varphi(x \otimes f)\varphi(A) = 2\varphi(x \otimes f) \in \mathcal{I}^*(\mathcal{X})$. But $2\varphi(x \otimes f)$ is a nilpotent operator by Lemma 3.2 and this contradiction implies that x, Ax and A^2x are linearly dependent for every $x \in \mathcal{X}$. Then there exist scalars λ and μ such that $(A - \lambda)(A^2 - \mu) = 0$ and either $(A - \lambda)^2 = 0$ or $(A^2 - \mu)^2 = 0$ by Lemma 2.2.

If $(A - \lambda)(A^2 - \mu) = 0$ and $(A - \lambda)^2 = 0$, then $(\mu - \lambda^2)A = \lambda(\mu - \lambda^2)$. Since A is a non-scalar operator, we know that $\mu = \lambda^2$. When $\lambda = 0$, we get that $A^2 = 0$ and there exist $x_1 \in \mathcal{X}$ and $f_1 \in \mathcal{X}'$ such that $f_1(x_1) = 0$ and $f_1(Ax_1) = 1$. So, $A(x_1 \otimes f_1) + (x_1 \otimes f_1)A \in \mathcal{I}^*(\mathcal{X})$ implies $\varphi(A)\varphi(x_1 \otimes f_1) + \varphi(x_1 \otimes f_1)\varphi(A) = 2\varphi(x_1 \otimes f_1) \in \mathcal{I}^*(\mathcal{X})$. However, $2\varphi(x_1 \otimes f_1)$ is a nilpotent operator by Lemma 3.2. This contradiction implies that $\lambda \neq 0$ and $A - \lambda$ is a nilpotent operator. Then there is a non-zero vector $x_2 \in \mathcal{X}$ such that $(A - \lambda)x_2 = 0$. For every $f_2 \in \mathcal{X}'$ with $f_2(x_2) = \frac{1}{2\lambda}$, we know that $Ax_2 \otimes f_2 + x_2 \otimes f_2A \in \mathcal{I}^*(\mathcal{X})$, which implies that $\varphi(A)\varphi(x_2 \otimes f_2) + \varphi(x_2 \otimes f_2)\varphi(A) = 2\varphi(x_2 \otimes f_2) \in \mathcal{I}^*(\mathcal{X})$. Moreover, $2\lambda\varphi(x_2 \otimes f_2) \in \mathcal{I}^*(\mathcal{X})$ follows from $2\lambda x_2 \otimes f_2 \in \mathcal{I}^*(\mathcal{X})$. Hence, $\lambda^2 = 1$. When $\lambda = 1$, we know that $A^2 - 2A + I = 0$ and so $(2I - A)\frac{A}{2} + \frac{A}{2}(2I - A) = I \in \mathcal{I}^*(\mathcal{X})$. Then, $\varphi(2I-A)\varphi(\frac{A}{2})+\varphi(\frac{A}{2})\varphi(2I-A) = 2\varphi(I)-I \in \mathcal{I}^*(\mathcal{X})$. Setting $2\varphi(I)-I = R \in \mathcal{I}^*(\mathcal{X})$, we get that $\varphi(I) = \frac{I+R}{2}$. However, $I\frac{I}{2} + \frac{I}{2}I = I \in \mathcal{I}^*(\mathcal{X})$ implies $\varphi(I)^2 \in \mathcal{I}^*(\mathcal{X})$. Thus, $\varphi(I)^2 = \frac{I+3R}{4}$ and so $R = I$. It follows that $\varphi(I) = I = \varphi(A)$ and then $A = I$ by the injection of φ. This is a contradiction. Similarly we can also obtain a contradiction when $\lambda = -1$.

If $(A - \lambda)(A^2 - \mu) = 0$ and $(A^2 - \mu)^2 = 0$, then $(\mu - \lambda^2)A^2 = \lambda(\mu - \lambda^2)$. When $\mu - \lambda^2 \neq 0$, we get $A^2 = \lambda I$. When $\lambda \neq 0$, there is a vector $x_4 \in \mathcal{X}$ such that x_4 and Ax_4 are linearly independent. Selecting $f_4 \in \mathcal{X}'$ with $f_4(x_4) = \frac{1}{\mu}$ and $f_4(Ax_4) = 0$, we know that $AAx_4 \otimes f_4 + Ax_4 \otimes f_4A \in \mathcal{I}^*(\mathcal{X})$ and so $2\varphi(Ax_4 \otimes f_4) \in \mathcal{I}^*(\mathcal{X})$. However, $\varphi(Ax_4 \otimes f_4)$ is a nilpotent operator. This contradiction shows that $\mu = 0$ and then $A^2 = 0$. Since x_4 and Ax_4 are linearly independent, there is an $f_5 \in \mathcal{X}'$ with $f_5(x_4) = 0$ and $f_5(Ax_4) = 1$. So, $Ax_4 \otimes f_5 + x_4 \otimes f_5A \in \mathcal{I}^*(\mathcal{X})$ and then $2\varphi(x_4 \otimes f_5) \in \mathcal{I}^*(\mathcal{X})$. But this contradicts with that $\varphi(x_4 \otimes f_5)$ is a nilpotent operator. When $\mu - \lambda^2 = 0$ and $\lambda = 0$, we know $A^3 = 0$. For the case that $A^2 = 0$, we also get

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a contradiction using a similar argument above. For the case that $A^2 \neq 0$, there is an $x_6 \in \mathcal{X}$ such that Ax_6 and A^2x_6 are linearly independent. Selecting $f_6 \in \mathcal{X}'$ such that $f_6(Ax_6) = 0$ and $f_6(A^2x_6) = 1$, we know $AAx_6 \otimes f_6 + Ax_6 \otimes f_6A \in \mathcal{I}^*(\mathcal{X})$ and then $2\varphi(Ax_6 \otimes f_6) \in \mathcal{I}^*(\mathcal{X})$. This contradicts the fact that $\varphi(Ax_6 \otimes f_6)$ is a nilpotent operator. So, $\lambda \neq 0$ and $A^2 - \lambda^2$ is a nilpotent operator. Hence, there is an $x_7 \in \mathcal{X}$ such that $A^2x_7 = \lambda^2x_7$. If x_7 and Ax_7 are linearly independent, then there is an $f_7 \in \mathcal{X}'$ such that $f_7(x_7) = \frac{1}{\lambda^2}$ and $f_7(Ax_7) = 0$. It follows that $AAx_7 \otimes f_7 + Ax_7 \otimes f_7A \in \mathcal{I}^*(\mathcal{X})$ which gives that $2\varphi(Ax_7 \otimes f_7) \in \mathcal{I}^*(\mathcal{X})$. While $\varphi(Ax_7 \otimes f_7)$ is a nilpotent operator, we get a contradiction. If $Ax_7 = \alpha x_7$ for some $\alpha \in \mathbb{C}$, then $\alpha^2 = \lambda^2$. Selecting $f_8 \in \mathcal{X}'$ such that $f_8(x_7) = \frac{1}{2\lambda^2}$, we get that $f_8(Ax_7) = \frac{\alpha}{2\lambda^2}$. By direct calculation, we know that $(2\lambda^2(Ax_7 \otimes f_8))(Ax_7 \otimes f_8) + (Ax_7 \otimes f_8)(2\lambda^2(Ax_7 \otimes f_8)) \in \mathcal{I}^*(\mathcal{X})$ and so $4\lambda^2\varphi(Ax_7 \otimes f_8)^2 \in \mathcal{I}^*(\mathcal{X})$. Moreover, $AAx_7 \otimes f_8 + Ax_7 \otimes f_8A \in \mathcal{I}^*(\mathcal{X})$ implies that $2\varphi(Ax_7 \otimes f_8) \in \mathcal{I}^*(\mathcal{X})$. It follows that $\lambda^2 = 1$. Since $I = 2A^2 - A^4 = (2I - A^2)(\frac{A^2}{2}) + (\frac{A^2}{2})(2I - A^2) = (2A - A^3)(\frac{A}{2}) + (\frac{A}{2})(2A - A^3)$, we have that $\varphi(2I - A^2)\varphi(\frac{A^2}{2}) + \varphi(\frac{A^2}{2})\varphi(2I - A^2) \in \mathcal{I}^*(\mathcal{X})$ and $2I - \varphi(A^3) \in \mathcal{I}^*(\mathcal{X})$. On the one hand, $\varphi(2I - A^2)\varphi(\frac{A^2}{2}) + \varphi(\frac{A^2}{2})\varphi(2I - A^2) = \varphi(I)\varphi(A^2) + \varphi(A^2)\varphi(I) - \varphi(A^2)^2 \in \mathcal{I}^*(\mathcal{X})$. On the other hand, $A^3 - A - \lambda A^2 + \lambda = 0$ implies $I = \frac{1}{\lambda}A + A^2 - \frac{1}{\lambda}A^3 = (\frac{1}{\lambda}I + A - \frac{1}{\lambda}A^2)(\frac{A}{2}) + (\frac{A}{2})(\frac{1}{\lambda}I + A - \frac{1}{\lambda}A^2)$. Hence, $\frac{1}{\lambda}[\varphi(A^2) - \varphi(I)] = I - E$ for some $E \in \mathcal{I}^*(\mathcal{X})$. It gives $\varphi(A^2)^2 - [\varphi(I)\varphi(A^2) + \varphi(A^2)\varphi(I)] + \varphi(I)^2 = I - E$. Furthermore, $\varphi(A^3) - I = \lambda[\varphi(I) - \varphi(A^2)]$ implies $(\varphi(A^3) - I)^2 = \varphi(I)^2 - [\varphi(I)\varphi(A^2) + \varphi(A^2)\varphi(I)] + \varphi(A^2)^2$. It follows $\varphi(A^3)^2 - 2\varphi(A^3) + I = I - E$ and $\varphi(A^3)^2 - 2\varphi(A^3) = -E$. However, $(2I - \varphi(A^3))^2 = 4I - 4\varphi(A^3) + \varphi(A^3)^2 = 2I - \varphi(A^3)$ implies $\varphi(A^3) = 2I - E$. So, $\frac{1}{\lambda}[\varphi(A^2) - \varphi(I)] = I - E = I + \varphi(A^3)$ and then $\varphi(\lambda A^3 + \lambda A - A^2 + I) = 0$. Using the fact that φ is a bijection, we get $\lambda A^3 + \lambda A - A^2 + I = 0$. Noting that $A^3 - A - \lambda A^2 + \lambda = 0$, we know $A = 0$ and this is a contradiction.

Therefore, $A = \mu I$ for some non-zero complex number μ. We thus have that $\varphi(I) = \lambda I$ for some constant $\lambda \in \mathbb{C}$ and $\lambda^2 = 1$. ∎

Next we assume that φ is surjective and $\varphi(I) = I$. We may replace φ by $-\varphi$ if $\varphi(I) = -I$.

Lemma 3.4. *φ has the following properties:*

(1) *$\varphi(\mathcal{I}(\mathcal{X})) \subseteq \mathcal{I}(\mathcal{X})$;*
(2) *φ preserves the orthogonality of idempotents;*
(3) *φ preserves the order of idempotents.*

Proof. (1) It follows directly from $\varphi(I) = I$, $\varphi(0) = 0$ and the fact that $(\frac{1}{2}P)I + I(\frac{1}{2}P) = P$ for any $P \in \mathcal{I}^*(\mathcal{X})$.

(2) If $P, Q \in \mathcal{I}^*(\mathcal{X})$ and $P \perp Q$, then $P + Q \in \mathcal{I}^*(\mathcal{X})$. So, $\varphi(P + Q) = \varphi(P) + \varphi(Q) \in \mathcal{I}^*(\mathcal{X})$ by (1). Since $\varphi(P), \varphi(Q) \in \mathcal{I}^*(\mathcal{X})$, we know that $\varphi(P) \perp \varphi(Q)$.

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(3) Let $P, Q \in \mathcal{I}^*(\mathcal{X})$ and $P < Q$. Then $PQ = QP = P$ and $Q = P + (Q - P)$. Clearly, $P \in \mathcal{I}^*(\mathcal{X})$ and $Q - P \in \mathcal{I}^*(\mathcal{X})$. Thus, $\varphi(Q) - \varphi(P) \in \mathcal{I}^*(\mathcal{X})$, and we get $2\varphi(P) = \varphi(Q)\varphi(P) + \varphi(P)\varphi(Q)$. This implies that $\varphi(Q)\varphi(P) = \varphi(P)\varphi(Q) = \varphi(P)$. $\square$

LEMMA 3.5. $\varphi(\mathcal{I}_1(\mathcal{X})) \subseteq \mathcal{I}_1(\mathcal{X})$.

Proof. Let $P = x \otimes f \in \mathcal{I}_1(\mathcal{X})$ for some $x \in \mathcal{X}$ and $f \in \mathcal{X}'$ with $f(x) = 1$. Then $\varphi(P) \in \mathcal{I}^*(\mathcal{X})$ by Lemma 3.4. Assume that $\operatorname{rank} \varphi(P) \geq 2$. Then there exists a $R \in \mathcal{I}_1(\mathcal{X})$ such that $R < \varphi(P)$ and so $\varphi(P) - R \in \mathcal{I}^*(\mathcal{X})$. Since φ is bijective, there is a non-zero operator $B \in \mathcal{B}(\mathcal{X})$ such that $R = \varphi(B)$.

If x, Bx and B^2x are linearly independent, there is a $g \in \mathcal{X}'$ such that $g(Bx) = 1 - f(Bx)$, $g(B^2x) = 1 - f(B^2x)$ and $g(x) = -1$. Then $Bx \otimes (f+g) + x \otimes (f+g)B \in \mathcal{I}^*(\mathcal{X})$ implies $\varphi(B)\varphi(x \otimes (f+g)) + \varphi(x \otimes (f+g))\varphi(B) = 2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R \in \mathcal{I}^*(\mathcal{X})$. On the one hand, $-x \otimes g \in \mathcal{I}^*(\mathcal{X})$ implies $-\varphi(x \otimes g) \in \mathcal{I}^*(\mathcal{X})$, and $zx \otimes f \cdot (-x) \otimes g + (1 - z)(-x) \otimes g \cdot x \otimes f \in \mathcal{I}^*(\mathcal{X})$ implies $z\varphi(x \otimes f) \cdot \varphi((-x) \otimes g) + (1-z)\varphi((-x) \otimes g)\varphi(x \otimes f) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$. On the other hand, $zx \otimes f \cdot (-x) \otimes g + (1 - z)(-x) \otimes g \cdot x \otimes f = z(-x) \otimes g + (1 - z)x \otimes f \in \mathcal{I}^*(\mathcal{X})$ implies $z\varphi((-x) \otimes g) + (1 - z)\varphi(x \otimes f) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$. Hence, $\varphi((-x) \otimes g)\varphi(x \otimes f)$, $\varphi(x \otimes f)\varphi((-x) \otimes g) \in \mathcal{I}^*(\mathcal{X})$ and

$$\begin{aligned}
&\varphi(x \otimes f)\varphi((-x) \otimes g)\varphi(x \otimes f) + \varphi((-x) \otimes g)\varphi(x \otimes f)\varphi((-x) \otimes g) \\
&= \varphi(x \otimes f)\varphi((-x) \otimes g) + \varphi((-x) \otimes g)\varphi(x \otimes f) \\
&= \varphi(x \otimes f) + \varphi((-x) \otimes g).
\end{aligned}$$

Under the decomposition $\mathcal{X} = R(R) + (R(P) \ominus N(R)) + N(P)$, R, P and $-\varphi(x \otimes g)$ have the following operator matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \varphi(-x \otimes g) = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

respectively. By direct calculation, we know

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & 0 \end{pmatrix} = \begin{pmatrix} 1+Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & 1+Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}.$$

So, $Q_{11} = 1,\ Q_{12} = 0,\ Q_{21} = 0,\ Q_{33} = 0$ and $Q_{22} = 1$. It follows that

$$2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R = \begin{pmatrix} 0 & 0 & -Q_{13} \\ 0 & 0 & 0 \\ -Q_{31} & 0 & 0 \end{pmatrix}.$$

Hence, we get a contradiction since $2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R \in \mathcal{I}^*(\mathcal{X})$.

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If x, Bx and B^2x are linearly dependent and x and Bx are linearly independent, then $B^2x = \lambda x + \mu Bx$ for some $\lambda,\ \mu \in \mathbb{C}$. When $\mu = 0$, there is $h \in \mathcal{X}'$ such that $h(Bx) = 1 - f(Bx)$ and $h(x) = -1$. Then we know that $Bx \otimes (f+h) + x \otimes (f+h)B \in \mathcal{I}^*(\mathcal{X})$ implies $\varphi(B)\varphi(x\otimes(f+h)) + \varphi(x\otimes(f+h))\varphi(B) = 2R + R\varphi(x\otimes h) + \varphi(x\otimes h)R \in \mathcal{I}^*(\mathcal{X})$. Note that $-x \otimes h \in \mathcal{I}^*(\mathcal{X})$ and $zx \otimes f \cdot (-x) \otimes h + (1-z)(-x) \otimes h \cdot x \otimes f = z(-x) \otimes h + (1-z)x \otimes f \in \mathcal{I}^*(\mathcal{X})$. So, we can get another contradiction using a similar argument demonstrated above. Now, we assume that $\mu \neq 0$. If $\lambda = 0$, then $B^2x = \mu Bx$. Thus, there is $h_1 \in \mathcal{X}'$ such that $h_1(Bx) = \frac{\mu}{2} - f(Bx)$ and $h_1(x) = -1$. It follows that $(I - \frac{2}{\mu}B)x \otimes (f + h_1) + x \otimes (f + h_1)(I - \frac{2}{\mu}B) \in \mathcal{I}^*(\mathcal{X})$ and then

$$\begin{aligned}&\varphi(I - \tfrac{2}{\mu}B)\varphi(x \otimes (f + h_1)) + \varphi(x \otimes (f + h_1))\varphi(I - \tfrac{2}{\mu}B)\\ &= 2P - \tfrac{4}{\mu}R - 2\varphi(-x \otimes h_1) + \tfrac{2}{\mu}[\varphi(-x \otimes h_1)R + R\varphi(-x \otimes h_1)] \in \mathcal{I}^*(\mathcal{X}).\end{aligned}$$

Note that $-x \otimes h_1 \in \mathcal{I}^*(\mathcal{X})$ and $zx \otimes f \cdot (-x) \otimes h_1 + (1-z)(-x) \otimes h_1 \cdot x \otimes f = z(-x) \otimes h_1 + (1-z)x \otimes f \in \mathcal{I}^*(\mathcal{X})$. We get that $-\varphi(x\otimes h_1) \in \mathcal{I}^*(\mathcal{X})$, $z\varphi(x\otimes f)\cdot\varphi(-x)\otimes h_1) + (1-z)\varphi((-x)\otimes h_1)\varphi(x\otimes f) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$ and $z\varphi(-x)\otimes h_1) + (1-z)\varphi(x\otimes f) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}$. Under the decomposition $\mathcal{X} = R(R) + (R(P) \ominus N(R)) + N(P)$, R, P and $-\varphi(x \otimes h_1)$ have the following operator matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(-x\otimes h_1) = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

respectively. By direct calculation, we know that

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & 0 \end{pmatrix} = \begin{pmatrix} 1 + Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & 1 + Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}.$$

So, $Q_{11} = 1,\ Q_{12} = 0,\ Q_{21} = 0,\ Q_{33} = 0$ and $Q_{22} = 1$. It follows that

$$\begin{aligned}&\varphi(I - \tfrac{2}{\mu}B)\varphi(x \otimes (f + h_1)) + \varphi(x \otimes (f + h_1))\varphi(I - \tfrac{2}{\mu}B)\\ &= 2P - \tfrac{4}{\mu}R - 2\varphi(-x \otimes h_1) + \tfrac{2}{\mu}[\varphi(-x \otimes h_1)R + R\varphi(-x \otimes h_1)] \in \mathcal{I}^*(\mathcal{X})\\ &= \begin{pmatrix} 0 & 0 & (-2 + \frac{2}{\mu})Q_{13} \\ 0 & 0 & -2Q_{23} \\ (-2 + \frac{2}{\mu})Q_{31} & -2Q_{32} & 0 \end{pmatrix} \in \mathcal{I}^*(\mathcal{X}).\end{aligned}$$

Clearly, this is a contradiction. If $\lambda \neq 0$, then there is $h_2 \in \mathcal{X}'$ such that $h_2(Bx) = \frac{-\mu}{2} - f(Bx)$ and $h_2(x) = -f(x)$. Hence, $(I - \frac{2}{\mu}B)x \otimes (f + h_2) + x \otimes (f + h_2)(I - \frac{2}{\mu}B) \in \mathcal{I}^*(\mathcal{X})$, and thus,

$$\begin{aligned}&\varphi(I - \tfrac{2}{\mu}B)\varphi(x \otimes (f + h_2)) + \varphi(x \otimes (f + h_2))\varphi(I - \tfrac{2}{\mu}B)\\ &= 2P - \tfrac{4}{\mu}R - 2\varphi(-x \otimes h_2) + \tfrac{2}{\mu}[\varphi(-x \otimes h_2)R + R\varphi(-x \otimes h_2)] \in \mathcal{I}^*(\mathcal{X}).\end{aligned}$$

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By an argument similar to that above, we also get a contradiction.

If x, Bx and B^2x are linearly dependent and x and Bx are linearly dependent, then $Bx = \beta x$ for some $\beta \in \mathbb{C}$. When $\beta = 0$, we know that $\frac{1}{2}((I + zB)x \otimes f + x \otimes f(I + zB)) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}\backslash\{0\}$ implies that $\frac{1}{2}((I + zR)P + P(I + zR)) = P + zR \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}\backslash\{0\}$. Obviously, this is a contradiction. When $\beta \neq 0$, we know that $\frac{1}{2(1+\beta z)}((I + zB)x \otimes f + x \otimes f(I + zB)) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}\backslash\{0\}$ with $1 + \beta z \neq 0$ implies that $\frac{1}{2(1+\beta z)}((I + zR)P + P(I + zR)) = \frac{1}{(1+\beta z)}(P + zR) \in \mathcal{I}^*(\mathcal{X})$ for all $z \in \mathbb{C}\backslash\{0\}$ with $1 + \beta z \neq 0$. This is also a contradiction.

Therefore, $\varphi(P) \in \mathcal{I}_1(\mathcal{X})$. □

The proof of the following lemma is similar to that of [13, Lemma 2.7].

Lemma 3.6. *$\varphi(\mathcal{N}_1(\mathcal{X})) \subseteq \mathcal{N}_1(\mathcal{X})$.*

Proof. Let $N = x \otimes f \in \mathcal{N}_1(\mathcal{X})$ for some non-zero $x \in \mathcal{X}$ and non-zero $f \in \mathcal{X}'$ such that $f(x) = 0$. Then $\varphi(N) \in \mathcal{N}(\mathcal{X})$ by Lemma 3.2. Taking an $f_1 \in \mathcal{X}'$ such that $f_1(x) = 1$ and setting $Q = x \otimes f_1$, we know that both Q and $Q+N$ are in $\mathcal{I}_1(\mathcal{X})$. So are both $\varphi(Q)$ and $\varphi(Q+N)$ by Lemma 3.5. Then there exist $y_1,\ y_2 \in \mathcal{X}$ and $g_1,\ g_2 \in \mathcal{X}'$ such that $g_1(y_1) = g_2(y_2) = 1$, $\varphi(Q) = y_1 \otimes g_1$ and $\varphi(Q + N) = y_2 \otimes g_2$. Putting $P = \frac{1}{2}((Q + N) + Q) = \frac{1}{2}N + Q$, we get that $P \in \mathcal{I}_1(\mathcal{X})$ and then $\varphi(P) \in \mathcal{I}_1(\mathcal{X})$. However, $\varphi(P) = \frac{1}{2}(\varphi(Q + N) + \varphi(Q)) = \frac{1}{2}(y_1 \otimes g_1 + y_2 \otimes g_2)$. It follows that either y_1 and y_2 or g_1 and g_2 are linearly dependent. If y_1 and y_2 are linearly dependent, then we may assume that $y_1 = y_2$. Thus, $\varphi(P) = \frac{1}{2}y_1 \otimes (g_1 + g_2) \in \mathcal{I}_1(\mathcal{X})$ and then $g_1(y_1) + g_2(y_1) = 2$. Since $g_1(y_1) = 1$, we have $g_2(y_1) = 1$. Thus, $\varphi(N) = \varphi(N + Q) - \varphi(Q) = y_1 \otimes (g_1 - g_2)$ and $(g_1 - g_2)(y_1) = 0$. Hence, $\varphi(N) \in \mathcal{N}_1(\mathcal{X})$. We can get that $\varphi(N) \in \mathcal{N}_1(\mathcal{X})$ by similar discussion if g_1 and g_2 are linearly dependent. □

Corollary 3.7. *$\varphi(\mathcal{F}_1(\mathcal{X})) \subseteq \mathcal{F}_1(\mathcal{X})$ and $\varphi(\mathcal{F}(\mathcal{X})) \subseteq \mathcal{F}(\mathcal{X})$.*

Proof. Since every non-nilpotent rank-one operator is a non-zero scalar multiple of rank-one idempotent operator, we know that $\varphi(\mathcal{F}_1(\mathcal{X})) \subseteq \mathcal{F}_1(\mathcal{X})$ by Lemma 3.5 and Lemma 3.6 and the linearity of φ. Moreover, every finite-rank operator can be written as a linear combination of finitely many rank-one operators. It follows from the linearity of φ that $\varphi(\mathcal{F}(\mathcal{X})) \subseteq \mathcal{F}(\mathcal{X})$. □

Note that a linear map φ on $\mathcal{F}(\mathcal{X})$ is rank non-increasing if $\operatorname{rank}\varphi(X) \leq \operatorname{rank} X$ for any $X \in \mathcal{F}(\mathcal{X})$. We start this section with [5, Corollary 2.1.5], which is restated in the following.

Lemma 3.8. *Let φ be a linear map on $\mathcal{F}(\mathcal{X})$ which is rank non-increasing such that* $\operatorname{rank}\varphi(T_0) > 1$ *for some $T_0 \in \mathcal{F}(\mathcal{X})$. Then one of the following holds.*

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(1) *There exist linear injective maps* $A : \mathcal{X} \to \mathcal{X}$ *and* $C : \mathcal{X}' \to \mathcal{X}'$ *such that* $\varphi(x \otimes f) = Ax \otimes Cf$ *for all* $x \in \mathcal{X}$ *and* $f \in \mathcal{X}'$.
(2) *There exist linear injective maps* $A : \mathcal{X}' \to \mathcal{X}$ *and* $C : \mathcal{X} \to \mathcal{X}'$ *such that* $\varphi(x \otimes f) = Af \otimes Cx$ *for all* $x \in \mathcal{X}$ *and* $f \in \mathcal{X}'$.

PROPOSITION 3.9. *Let* φ *be a surjective linear map on* $\mathcal{B}(\mathcal{X})$ *preserving the nonzero idempotency of Jordan products of two operators such that* $\varphi(I) = I$. *Then one of the statements in Lemma 3.8 holds.*

Proof. From Corollary 3.7, we know that the restriction $\varphi|_{\mathcal{F}(\mathcal{X})}$ of φ on $\mathcal{F}(\mathcal{X})$ is a linear map preserving rank non-increasing. Let Q be a rank-2 idempotent operator. Then there exists a $P \in \mathcal{I}_1(\mathcal{X})$ such that $P < Q$. By Lemma 3.4, we know that $\varphi(P) < \varphi(Q)$ and $\varphi(P) \in \mathcal{I}_1(\mathcal{X})$. So, $\operatorname{rank} \varphi(Q) \geq 2$. Then we get the desired results from Lemma 3.8. ∎

Proof of Theorem 1.1. The sufficiency is clear. Let φ be a linear map on $\mathcal{M}_n$ preserving the nonzero idempotency of Jordan product of two operators. Then φ is injective by Lemma 3.1 and thus bijective. We now have $\varphi(I) = \lambda I$ for some constant $\lambda \in \{1, -1\}$. we may assume $\varphi(I) = I$. Then one of two statements in Proposition 3.1 holds.

If (1) holds, then we easily have that $\varphi(X) = AXB$ for all $X \in \mathcal{M}_n$. It is clear that $B = A^{-1}$. We have that $\varphi(X) = AXA^{-1}$ for all $X \in \mathcal{M}_n$.

If (2) holds, then we similarly have that $\varphi(X) = AX^tA^{-1}$ for all $X \in \mathcal{M}_n$. ∎

We next consider the infinite dimensional case.

LEMMA 3.10. *Let* φ *be a surjective linear map on* $\mathcal{B}(\mathcal{X})$ *preserving the nonzero idempotency of Jordan products of two operators such that* $\varphi(I) = I$. *Then* $\varphi(\mathcal{F}_1(\mathcal{X})) = \mathcal{F}_1(\mathcal{X})$.

Proof. By Corollary 3.7, it is sufficient to prove that $\mathcal{F}_1(\mathcal{X}) \subseteq \varphi(\mathcal{F}_1(\mathcal{X}))$. Let $T \in \mathcal{B}(\mathcal{H})$ such that $\varphi(T) = z \otimes h$ is of rank-one for some $z \in \mathcal{X}$ and $h \in \mathcal{X}'$. Clearly, T is a non-scalar operator. So, there is $x \in \mathcal{X}$ such that x and Tx are linearly independent. If $\operatorname{rank} T > 1$, then there exists $y \in \mathcal{X}$ such that Tx and Ty are linearly independent. It follows that x and y are linearly independent. Applying [9, Lemma 2.1], we know that $x + \delta y$ and $T(x + \delta y)$ are linearly independent for all but finite number $\delta \in \mathbb{C}$. For any fixed $\alpha \in \mathbb{C}$, there is a rank-one idempotent $P(\delta, \alpha)$ such that $(T + P(\delta, \alpha))(x + \delta y) = \alpha(x + \delta y)$ by Lemma 2.4. Finding $f_{(\delta,\alpha)} \in \mathcal{X}'$ to satisfy $f_{(\delta,\alpha)}(x + \delta y) = \frac{1}{2\alpha}$, we get that $\frac{1}{2\alpha}((T + P(\delta, \alpha))(x + \delta y) \otimes f_{(\delta,\alpha)} + (x + \delta y) \otimes f_{(\delta,\alpha)}(T + P(\delta, \alpha))) \in \mathcal{I}^*(\mathcal{X})$, which implies that $\frac{1}{2\alpha}(\varphi(T + P(\delta, \alpha))\varphi((x + \delta y) \otimes f_{(\delta,\alpha)}) + \varphi((x + \delta y) \otimes f_{(\delta,\alpha)})\varphi(T + P(\delta, \alpha))) \in \mathcal{I}^*(\mathcal{X})$. Let $\varphi(P(\delta, \alpha)) = e_{(\delta,\alpha)} \otimes g_{(\delta,\alpha)}$. If Proposition 3.9 (1) holds, then $\frac{1}{2\alpha}(\varphi(T + P(\delta, \alpha))\varphi((x + \delta y) \otimes f_{(\delta,\alpha)}) + \varphi((x + \delta y) \otimes f_{(\delta,\alpha)})\varphi(T + P(\delta, \alpha))) =$

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$\frac{1}{2\alpha}(\varphi(T+P(\delta,\alpha))A(x+\delta y)\otimes Cf_{(\delta,\alpha)}+A(x+\delta y)\otimes Cf_{(\delta,\alpha)}\varphi(T+P(\delta,\alpha)))\in\mathcal{I}^*(\mathcal{X})$. This yields that $\varphi(T+P(\delta,\alpha))A(x+\delta y)=\alpha A(x+\delta y)$ or $\varphi(T+P(\delta,\alpha))'Cf_{(\delta,\alpha)}=\alpha Cf_{(\delta,\alpha)}$. When $\varphi(T+P(\delta,\alpha))A(x+\delta y)=\alpha A(x+\delta y)$, we have that $h(A(x+\delta y))z+g_{(\delta,\alpha)}(A(x+\delta y))e_{(\delta,\alpha)}=\alpha A(x+\delta y)$ for fixed $\delta\in\mathbb{C}\backslash\{0\}$ and any $\alpha\in\mathbb{C}\backslash\{0\}$. It follows that

$$g_{(\delta,\alpha_1)}(A(x+\delta y))e_{(\delta,\alpha_1)}-g_{(\delta,\alpha_2)}(A(x+\delta y))e_{(\delta,\alpha_2)}=(\alpha_1-\alpha_2)A(x+\delta y)$$

for fixed $\delta\in\mathbb{C}$ and any $\alpha_1,\alpha_2\in\mathbb{C}\backslash\{0\}$. Obviously,

$$(1+\alpha_2-\alpha_1)g_{(\delta,\alpha_1)}(A(x+\delta y))=g_{(\delta,\alpha_2)}(A(x+\delta y))g_{(\delta,\alpha_1)}(e_{(\delta,\alpha_2)})$$

for fixed $\delta\in\mathbb{C}$ and any $\alpha_1,\alpha_2\in\mathbb{C}\backslash\{0\}$. Hence, $g_{(\delta,\alpha_2)}(A(x+\delta y))g_{(\delta,\alpha_1)}(e_{(\delta,\alpha_2)})=0$ for fixed $\delta\in\mathbb{C}$ and any $\alpha_1,\alpha_2\in\mathbb{C}\backslash\{0\}$ with $1+\alpha_2=\alpha_1$. So, $g_{(\delta,\alpha_2)}(A(x+\delta y))=0$ or $g_{(\delta,\alpha_1)}(e_{(\delta,\alpha_2)})=0$ for fixed $\delta\in\mathbb{C}$ and any $\alpha_1,\alpha_2\in\mathbb{C}\backslash\{0\}$ with $1+\alpha_2=\alpha_1$. According to Lemma 3.8, we know that $\varphi(P(\delta,\alpha)):\mathcal{X}\to[A(x+\delta y),AT(x+\delta y)]$ and so $g_{(\delta,\alpha_1)}(e_{(\delta,\alpha_2)})\neq 0$. Hence, $g_{(\delta,\alpha_2)}(A(x+\delta y))=0$ for fixed $\delta\in\mathbb{C}$ and any $\alpha_2\in\mathbb{C}\backslash\{0\}$. With this observation, we know that $h(A(x+\delta y))z=\alpha_2A(x+\delta y)$ for fixed $\delta\in\mathbb{C}$ and any $\alpha_2\in\mathbb{C}\backslash\{0\}$. This is a contradiction. When $\varphi(T+P(\delta,\alpha))'f_{(\delta,\alpha)}=\alpha f_{(\delta,\alpha)}$, we get that $\alpha\in\sigma_p(z\otimes h+e_{(\delta,\alpha)}\otimes g_{(\delta,\alpha)})$ for fixed $\delta\in\mathbb{C}$ and any $\alpha\in\mathbb{C}\backslash\{0\}$, where $\sigma_p(z\otimes h+e_{(\delta,\alpha)}\otimes g_{(\delta,\alpha)})$ denotes that point spectrum of $z\otimes h+e_{(\delta,\alpha)}\otimes g_{(\delta,\alpha)}$. Note that $\mathcal{R}(z\otimes h+e_{(\delta,\alpha)}\otimes g_{(\delta,\alpha)})\subseteq[z]+[A(x+\delta y),AT(x+\delta y)]$ for any $\alpha\in\mathbb{C}$. Here is a contradiction. If Proposition 3.9 (2) holds, then we get a contradiction again. Thus, T is of rank-one. □

The idea of the following proof comes from [9] and [13].

Proof of Theorem 1.2. The sufficiency is clear. Now, we prove the necessity and assume that $\varphi(I)=I$ by Lemma 3.3.

By Lemma 3.10, φ maps the set of rank-one operators onto itself. This implies that the injective linear maps A and C mentioned in Proposition 3.9 are bijective. Suppose Proposition 3.9 (1) holds. By Lemma 3.4 and the linearity of φ, it can be shown that $Cf(Ax)=f(x)$ for all $x\in\mathcal{X}$ and $f\in\mathcal{X}'$, which implies that C is the adjoint of A^{-1}, and hence C is bounded. Thus, A^{-1} and A are bounded too. Furthermore, for any $y\in\mathcal{X}$,

$$\varphi(x\otimes f)y=(Ax\otimes Cf)y=(Cf)(y)Ax=f(A^{-1}y)Ax=A(x\otimes f)A^{-1}y.$$

Thus, $\varphi(T)=ATA^{-1}$ for any rank-one operator T. Moreover, every finite-rank operator can be written as a linear combination of finitely many rank-one operators. With the linearity of φ, we know that $\varphi(T)=ATA^{-1}$ for any finite-rank operator T. Replacing φ by $A^{-1}\varphi A$, we may assume that $\varphi(T)=T$ for every finite-rank operator T. We next prove that $\varphi(T)=T$ for every non-scalar and infinite-rank operator $T\in\mathcal{B}(\mathcal{X})$.

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Let $T \in \mathcal{B}(\mathcal{X})$ be any non-scalar and infinite-rank operator. By the property of φ, we know that

$$TR + RT \in \mathcal{I}(\mathcal{X})\backslash\{0\} \Rightarrow \varphi(T)\varphi(R) + \varphi(R)\varphi(T) = \varphi(T)R + R\varphi(T) \in \mathcal{I}(\mathcal{X})\backslash\{0\}$$

for every $R \in \mathcal{F}_1(\mathcal{X})$. Suppose there is $x \in \mathcal{X}$ such that $(\varphi(A)+2P_0)x$ and $(A+2P_0)x$ are linearly independent for some rank-one idempotent $P_0 \in \mathcal{I}^*(\mathcal{X})$ If x and $(A+2P_0)x$ are linearly independent, then there exists a rank-one idempotent $Q \in \mathcal{I}(\mathcal{X})$ such that $(A+2P_0+2Q)x = x$ by Lemma 2.4. So, $(\varphi(A)+2P_0+2Q)x \neq (A+2P_0+2Q)x$ and there is $f \in \mathcal{X}'$ such that $f((\varphi(A)+2P_0+2Q)x) = 0$ and $f((A+2P_0+2Q)x) = 1$. It follows that $\frac{1}{2}((A+2P_0+2Q)x \otimes f + x \otimes f(A+2P_0+2Q)) \in \mathcal{I}^*(\mathcal{X})$ and then $\frac{1}{2}(\varphi(A+2P_0+2Q)x \otimes f + x \otimes f\varphi(A+2P_0+2Q)) \in \mathcal{I}^*(\mathcal{X})$. Hence, $f((\varphi(A)+2P_0+2Q)^2x)x = 2(\varphi(A)+2P_0+2Q)x$. Thus, $f((\varphi(A)+2P_0+2Q)^2x) = 0$ and so $(\varphi(A)+2P_0+2Q)x = 0$. This is a contradiction. If $(A+2P_0)x = \gamma x$ for some non-zero $\gamma \in \mathbb{C}$, then there is $g \in \mathcal{X}'$ such that $g((\varphi(A)+2P_0)x) = 0$ and $g((A+2P_0)x) = \gamma$. Hence, $\frac{1}{2\gamma}((A+2P_0)x \otimes g + x \otimes g(A+2P_0)) \in \mathcal{I}^*(\mathcal{X})$, which implies that $\frac{1}{2\gamma}(\varphi(A+2P_0)x \otimes g + x \otimes g\varphi(A+2P_0)) \in \mathcal{I}^*(\mathcal{X})$. By direct calculation, we get that $g((\varphi(A)+2P_0)^2x)x = 2\gamma(\varphi(A)+2P_0)x$ and so $f((\varphi(A)+2P_0+2Q)^2x) = 0$. Thus, $(\varphi(A)+2P_0+2Q)x = 0$ and this is a contradiction. Thus, $\varphi(A)+2P$ and $A+2P$ are locally linearly dependent for any rank-one idempotent $P \in \mathcal{I}^*(\mathcal{X})$. By [10, Theorem 2.4], we know that there is $\eta(P) \in \mathbb{C}$ such that $\varphi(A)+2P = \eta(P)(A+2P)$ for any rank-one idempotent $P \in \mathcal{I}^*(\mathcal{X})$. Since $A+2P$ is a non-scalar operator, we can find $x \in \mathcal{X}$ such that x and $(A+2P)x$ are linearly independent. Then there exists an idempotent operator $Q_1 \in \mathcal{B}(\mathcal{X})$ of rank one such that $(A+2P+2Q_1)x = x$. There is $h \in \mathcal{X}'$ such that $h(x) = 1$. So, $\frac{1}{2}((A+2P+2Q_1)x \otimes h + x \otimes h(A+2P+2Q_1)) \in \mathcal{I}^*(\mathcal{X})$. It follows that $\frac{1}{2}(\varphi(A+2P+2Q_1)x \otimes h + x \otimes h\varphi(A+2P+2Q_1)) \in \mathcal{I}^*(\mathcal{X})$. By direct calculation, we have that $\varphi(A+2P+2Q_1)x = x$ or $\varphi(A+2P+2Q_1)'h = h$. Therefore, $\eta(P+Q_1) = 1$ and then $\varphi(A) = A$.

Suppose Proposition 3.9 (2) holds. Then we have $(Cx)(Af) = f(x)$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}'$ by a similar argument. So, $C' = A^{-1}K^{-1}$, where K is the natural embedding of X into X''. Thus, A^{-1} is bounded and so as $(A^{-1})'$ and $C = (A^{-1})'K$. As C and $(A^{-1})'$ are bijective, as so K and hence X is reflective. Now by a similar argument, $\varphi(T) = AT'A^{-1}$ for every finite-rank operator T and hence $\varphi(T) = AT'A^{-1}$ for all every $T \in \mathcal{B}(\mathcal{X})$. $\square$

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