

HLPK AND SHERMAN-TYPE THEOREMS FOR SCHUR COMPLEMENT OF MATRICES*

MAREK NIEZGODA[†]

Abstract. In this paper, by using majorization relations for tuples of matrices and of linear operations, some operator versions of Hardy–Littlewood–Pólya–Karamata and Sherman theorems for Schur complements of positive definite matrices are established. The stochasticity of the used operation matrices is partially reduced, and the commutativity of the involved matrices is engaged.

Key words. Stochastic matrix, Schur complement, Positive definite matrix, Fiedler–Markham inequality, HLPK theorem, Sherman theorem.

AMS subject classifications. 15A45, 15B48, 15B51, 15A27.

1. Introduction. As usual, we denote by \mathbb{R} the set of all reals, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

A real p -tuple $\mathbf{y} = (y_1, \dots, y_p)$ is said to be *majorized* by a real p -tuple $\mathbf{x} = (x_1, \dots, x_p)$ (written as, $\mathbf{y} \prec \mathbf{x}$), if $\sum_{i=1}^l y_{[i]} \leq \sum_{i=1}^l x_{[i]}$ for $l = 1, \dots, p$, with equality for $l = p$, where $x_{[i]}$ and $y_{[i]}$ stand for the i th largest entries of \mathbf{x} and \mathbf{y} , respectively [11].

A real $p \times q$ matrix $\mathbf{S} = (s_{ij})$ is said to be

- (i) *row stochastic* if $s_{ij} \geq 0$ for $i = 1, \dots, p$, $j = 1, \dots, q$, and $\sum_{j=1}^q s_{ij} = 1$ for $i = 1, \dots, p$,
- (ii) *column stochastic* if $s_{ij} \geq 0$ for $i = 1, \dots, p$, $j = 1, \dots, q$, and $\sum_{i=1}^p s_{ij} = 1$ for $j = 1, \dots, q$.

A real $p \times p$ matrix $\mathbf{S} = (s_{ij})$ is said to be *doubly stochastic* if \mathbf{S} is both row stochastic and column stochastic.

For given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, it holds that $\mathbf{y} \prec \mathbf{x}$ if and only if there exists a $p \times p$ doubly stochastic matrix \mathbf{S} satisfying $\mathbf{y} = \mathbf{x}\mathbf{S}$ [11].

Theorem A (Hardy–Littlewood–Pólya [11] and Karamata [10] (HLPK)). *Let $f : I \rightarrow \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, and $\mathbf{x} = (x_1, \dots, x_p) \in I^p$, $\mathbf{y} = (y_1, \dots, y_p) \in I^p$.*

If $\mathbf{y} \prec \mathbf{x}$, then

$$\sum_{i=1}^p f(y_i) \leq \sum_{i=1}^p f(x_i).$$

For applications and extensions of HLPK theorem, see e.g., [6, 10, 11, 18].

*Received by the editors on March 22, 2025. Accepted for publication on November 9, 2025. Handling Editor: Fuad Kittaneh. Corresponding Author: Marek Niezgoda.

[†]Retired from Institute of Mathematics, Pedagogical University of Cracow (Emeritus), Poland (mniezgoda247@gmail.com).

A more general result is as follows. (The symbol \mathbf{S}^T represents the transpose of a matrix \mathbf{S} .)

Theorem B (Sherman [20]). *Let $f : I \rightarrow \mathbb{R}$ be convex on an interval $I \subset \mathbb{R}$, and $\mathbf{x} = (x_1, \dots, x_p) \in I^p$, $\mathbf{y} = (y_1, \dots, y_q) \in I^q$, $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{R}_+^p$, $\mathbf{b} = (b_1, \dots, b_q) \in \mathbb{R}_+^q$.*

If

$$(1.1) \quad \mathbf{y} = \mathbf{xS} \quad \text{and} \quad \mathbf{a} = \mathbf{bS}^T,$$

for some $p \times q$ column stochastic matrix \mathbf{S} , then

$$(1.2) \quad \sum_{j=1}^q b_j f(y_j) \leq \sum_{i=1}^p a_i f(x_i).$$

Statements (1.1) are called *Sherman conditions*, and (1.2) is *Sherman inequality*.

Sherman Theorem B contains the HLPK Theorem A as a special case for $\mathbf{a} = \mathbf{b} = (1, 1, \dots, 1) \in \mathbb{R}_{++}^n$, with \mathbf{S} being doubly stochastic.

Recently, Sherman theorem has been generalized in various directions (see e.g., [1, 2, 8, 9, 13, 15]).

We now introduce some portion of needed matrix notations.

Throughout we denote by \mathbb{M}_n and \mathbb{H}_n the sets of all $n \times n$ complex matrices and of Hermitian matrices, respectively. The symbols I_n and 0 represent the identity matrix and the null matrix in \mathbb{M}_n . For $A, B \in \mathbb{H}_n$, we write $A \leq B$ if $B - A$ is positive semidefinite, and $A < B$ if $B - A$ is positive definite. We use the notation \mathbb{P}_n^+ (resp. \mathbb{P}_n^{++}) to stand for the set of all $n \times n$ positive semidefinite matrices (resp. the set of all $n \times n$ positive definite matrices).

Let A be a matrix of order n partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with A_{11} being of order k .

The matrix

$$A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$$

is called the *Schur complement* of A_{11} in A provided that A_{11} is invertible.

The matrix

$$A/A_{22} := A_{11} - A_{12}A_{22}^{-1}A_{21},$$

is called the *Schur complement* of A_{22} in A provided that A_{22} is invertible.

Theorem C (Haynsworth [7, Theorem 2], Fiedler–Markham [5, Theorem 1]). *Suppose A and B are $n \times n$ positive definite matrices partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} have order k .

(i) If $A_{11} > 0$ and $B_{11} > 0$, then

$$(1.3) \quad A/A_{11} + B/B_{11} \leq (A + B)/(A_{11} + B_{11}).$$

(ii) If $A_{22} > 0$ and $B_{22} > 0$, then

$$(1.4) \quad A/A_{22} + B/B_{22} \leq (A + B)/(A_{22} + B_{22}).$$

Equality is attained in (1.4) if and only if $A_{12}A_{22}^{-1} = B_{12}B_{22}^{-1}$.

In what follows, we consider the matrix map

$$(1.5) \quad \varphi(A) := A/A_{22} \quad \text{for } A \in \mathbb{P}_n^{++}.$$

By virtue of (1.4) one sees that φ is superadditive on \mathbb{P}_n^{++} with respect to the Loewner order \leq on \mathbb{H}_k .

It is a consequence of (1.4) that the following Fiedler–Markham inequality holds

$$(1.6) \quad \sum_{i=1}^p A_i/(A_i)_{22} \leq \left(\sum_{i=1}^p A_i \right) / \left(\sum_{i=1}^p A_i \right)_{22} \quad \text{for } A_i > 0, i = 1, 2, \dots, p.$$

To give a motivation for our further studies, suppose for a moment that $n = 2k$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} > 0$ and $A_{21} = A_{12} > 0$ with commuting A_{12} and A_{22} .

In this special situation, the Schur complement of A_{22} in A is

$$\begin{aligned} A/A_{22} &= A_{11} - A_{12}A_{22}^{-1}A_{21} = A_{11} - A_{12}A_{22}^{-1}A_{12} = A_{11} - A_{12}(A_{12}^{-1}A_{22})^{-1} \\ &= A_{11} - A_{12}^{1/2}(A_{12}^{-1/2}A_{22}A_{12}^{-1/2})^{-1}A_{12}^{1/2}. \end{aligned}$$

That is, A/A_{22} is the difference of the linear map $A \rightarrow A_{11}$ and the *perspective function* $(A_{12}, A_{22}) \mapsto A_{12}^{1/2}(A_{12}^{-1/2}A_{22}A_{12}^{-1/2})^{-1}A_{12}^{1/2}$ (see [4, 19]) of the order-reversing operator convex function $f = (\cdot)^{-1}$ (see [3, pp. 114 and 117]). Perspective functions of operator convex functions on commuting matrices were introduced by Effros [4]. The main idea is the equivalence of the convexity of a function and the joint convexity of its perspective. Moreover, the advantage of perspective functions is their scalar positive homogeneity. Therefore, the Schur complement map $A \mapsto \varphi(A) = A/A_{22}$ is (operator) concave. So, the theory of majorization can be used for studying properties of φ .

The purpose of the present paper is to extend Theorems **A** and **B** for φ from the scalar case to their operator versions, when the scalar homogeneity $\varphi(tA) = t\varphi(A)$, $0 < t \in \mathbb{R}$, is replaced with $\varphi(V^*AV) = V_1^*\varphi(A)V_1$ for $V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ (see Lemma 1, item (i), for details). Likewise, the usual meaning of convexity based on multiplication of matrices by positive scalars is replaced by the one based on operator multiplication $A \mapsto V^*AV$. Such approach is implemented in the next section, where operator majorization and operator stochasticity are introduced (cf. [12, 14, 16, 17]). Sections 3 and 4 are devoted to establishing and proving corresponding theorems of HLPK and Sherman types for tuples of positive definite matrices and the Schur complement map φ . In our considerations, *we do not restrict ourselves* to the above-mentioned case when $A_{21} = A_{12} > 0$ and A_{12} commutes with A_{22} .

2. Majorization on tuples of matrices. In this section, we continue introducing some algebraic notation and terminology needed in the sequel.

Given a $p \times 1$ matrix row $(A_1, A_2, \dots, A_p) \in \mathbb{M}_n^p$ and $p \times q$ operation matrix

$$\mathcal{S} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1q} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{p1} & \Phi_{p2} & \dots & \Phi_{pq} \end{pmatrix},$$

where $\Phi_{ij} : \mathbb{M}_n \rightarrow \mathbb{M}_n$ are linear operations, we define the *product*

$$(A_1, A_2, \dots, A_p)\mathcal{S} := (B_1, B_2, \dots, B_q),$$

by

$$(2.7) \quad B_j = \sum_{i=1}^p \Phi_{ij}(A_i) \quad \text{for } j = 1, 2, \dots, q.$$

A $p \times q$ operation matrix $\mathcal{S} = (\Phi_{ij})$ is called

(i) *positive*, if

$$A \geq 0 \quad \text{implies} \quad \Phi_{ij}(A) \geq 0 \quad \text{for } i = 1, 2, \dots, p, j = 1, 2, \dots, q,$$

(ii) *row stochastic*, if \mathcal{S} is positive and

$$(2.8) \quad \sum_{j=1}^q \Phi_{ij}(I_n) = I_n \quad \text{for } i = 1, 2, \dots, p,$$

(iii) *column stochastic*, if \mathcal{S} is positive and

$$(2.9) \quad \sum_{i=1}^p \Phi_{ij}(I_n) = I_n \quad \text{for } j = 1, 2, \dots, q.$$

A $p \times p$ operation matrix \mathcal{S} is called *doubly stochastic* if \mathcal{S} is both row stochastic and column stochastic.

Observe that if $S = (s_{ij})$ is a $p \times p$ real doubly stochastic matrix and all $n \times n$ matrices U_{ij} are unitary, then the operation matrix

$$\mathcal{S} = \begin{pmatrix} s_{11}U_{11}^*(\cdot)U_{11} & s_{12}U_{12}^*(\cdot)U_{12} & \dots & s_{1p}U_{1p}^*(\cdot)U_{1p} \\ s_{21}U_{21}^*(\cdot)U_{21} & s_{22}U_{22}^*(\cdot)U_{22} & \dots & s_{2p}U_{2p}^*(\cdot)U_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p1}U_{p1}^*(\cdot)U_{p1} & s_{p2}U_{p2}^*(\cdot)U_{p2} & \dots & s_{pp}U_{pp}^*(\cdot)U_{pp} \end{pmatrix},$$

is doubly stochastic.

Given a $1 \times p$ row $(\Psi_1, \Psi_2, \dots, \Psi_p)$ and a $p \times q$ matrix

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1q} \\ A_{21} & A_{22} & \dots & A_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ A_{p1} & A_{p2} & \dots & A_{pq} \end{pmatrix},$$

where $A_{ij} \in \mathbb{M}_n$ and $\Psi_j : \mathbb{M}_n \rightarrow \mathbb{M}_n$ are linear operations, we introduce the *product*

$$(\Psi_1, \Psi_2, \dots, \Psi_p)\mathcal{A} := (C_1, C_2, \dots, C_q),$$

by

$$(2.10) \quad C_j = \sum_{i=1}^p \Psi_i(A_{ij}) \quad \text{for } j = 1, 2, \dots, q.$$

We say that $(B_1, B_2, \dots, B_p) \in \mathbb{M}_n^p$ is *majorized* by $(A_1, A_2, \dots, A_p) \in \mathbb{M}_n^p$ with respect to a class \mathcal{K} of linear operations from \mathbb{M}_n to \mathbb{M}_n , written as

$$(B_1, B_2, \dots, B_p) \prec_{\mathcal{K}} (A_1, A_2, \dots, A_p),$$

if there exists a $p \times p$ doubly stochastic matrix $\mathcal{S} = (\Phi_{ij})$ with linear operations $\Phi_{ij} \in \mathcal{K}$ such that

$$(B_1, B_2, \dots, B_p) = (A_1, A_2, \dots, A_p)\mathcal{S}.$$

From now on, we employ the following class of positive operations:

$$\mathcal{K} = \{\Phi = V^*(\cdot)V : V \in \mathbb{M}_n\}.$$

In this situation, condition (2.8) reduces to

$$\sum_{j=1}^p V_{ij}^* V_{ij} = I_n \quad \text{for all } i = 1, 2, \dots, p,$$

while condition (2.9) becomes

$$\sum_{i=1}^p V_{ij}^* V_{ij} = I_n \quad \text{for all } j = 1, 2, \dots, p.$$

3. HLPK-type theorem for Schur complement. We begin with a technical lemma.

LEMMA 1. *Let*

$$(3.11) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix},$$

be matrices of order n , where A_{11} and V_1 are of order k , and A_{22} and V_2 are nonsingular.

(i) *Then,*

$$(3.12) \quad (V^*AV)/(V_2^*A_{22}V_2) = V_1^*(A/A_{22})V_1.$$

(ii) *If, in addition,*

$$(3.13) \quad AV = VA,$$

then

$$(3.14) \quad (A/A_{22})V_1 = V_1(A/A_{22}).$$

Proof. (i). It follows from (3.11) that

$$(3.15) \quad V^*AV = \begin{pmatrix} V_1^*A_{11}V_1 & V_1^*A_{12}V_2 \\ V_2^*A_{21}V_1 & V_2^*A_{22}V_2 \end{pmatrix}.$$

Hence, by the nonsingularity of $(V^*AV)_{22} = V_2^*A_{22}V_2$, we get

$$\begin{aligned} (V^*AV)/(V_2^*A_{22}V_2) &= V_1^*A_{11}V_1 - (V_1^*A_{12}V_2)(V_2^*A_{22}V_2)^{-1}(V_2^*A_{21}V_1) \\ &= V_1^*A_{11}V_1 - V_1^*A_{12}V_2V_2^{-1}A_{22}^{-1}V_2^{*-1}V_2^*A_{21}V_1 \\ &= V_1^*A_{11}V_1 - V_1^*A_{12}A_{22}^{-1}A_{21}V_1 = V_1^*(A_{11} - A_{12}A_{22}^{-1}A_{21})V_1 = V_1^*(A/A_{22})V_1, \end{aligned}$$

which completes the proof of (3.12).

(ii). By (3.11), we have

$$(3.16) \quad AV = \begin{pmatrix} A_{11}V_1 & A_{12}V_2 \\ A_{21}V_1 & A_{22}V_2 \end{pmatrix} \quad \text{and} \quad VA = \begin{pmatrix} V_1A_{11} & V_1A_{12} \\ V_2A_{21} & V_2A_{22} \end{pmatrix}.$$

On account of (3.13), statement (3.16) implies that

$$(3.17) \quad \begin{aligned} A_{11}V_1 &= V_1A_{11} \\ A_{12}V_2 &= V_1A_{12} \\ A_{21}V_1 &= V_2A_{21} \\ A_{22}V_2 &= V_2A_{22} \end{aligned}.$$

The last equality gives

$$(3.18) \quad A_{22}^{-1}V_2 = V_2A_{22}^{-1}. \quad \square$$

By using equalities (3.17) and (3.18), we infer that

$$\begin{aligned} (A/A_{22})V_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})V_1 = A_{11}V_1 - A_{12}A_{22}^{-1}A_{21}V_1 \\ &= V_1A_{11} - A_{12}A_{22}^{-1}V_2A_{21} = V_1A_{11} - A_{12}V_2A_{22}^{-1}A_{21} \\ &= V_1A_{11} - V_1A_{12}A_{22}^{-1}A_{21} = V_1(A_{11} - A_{12}A_{22}^{-1}A_{21}) = V_1(A/A_{22}), \end{aligned}$$

as required.

We are now in a position to demonstrate and prove a HLPK-type theorem for tuples of matrices and the Schur complement map φ .

THEOREM 2. For $j = 1, 2, \dots, p$, let $A_j, B_j \in \mathbb{P}_n^{++}$ be partitioned as

$$A_j = \begin{pmatrix} A_{j,11} & A_{j,12} \\ A_{j,21} & A_{j,22} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} B_{j,11} & B_{j,12} \\ B_{j,21} & B_{j,22} \end{pmatrix},$$

with invertible blocks $A_{j,22}$ and $B_{j,22}$ of order $n - k$.

Let

$$(3.19) \quad (B_1, B_2, \dots, B_p) = (A_1, A_2, \dots, A_p)\mathcal{S},$$

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for some $p \times p$ operation matrix $\mathcal{S} = (\Phi_{ij})$ with

$$\Phi_{ij} = V_{ij}^*(\cdot)V_{ij} \quad \text{and} \quad V_{ij} = \begin{pmatrix} V_{ij,1} & 0 \\ 0 & V_{ij,2} \end{pmatrix} \in \mathbb{M}_n \quad \text{for } i, j = 1, 2, \dots, p,$$

with invertible block $V_{ij,2}$ of order $n - k$.

Assume that A_i commutes with V_{ij} , i.e.,

$$(3.20) \quad A_i V_{ij} = V_{ij} A_i \quad \text{for } i, j = 1, 2, \dots, p,$$

and that the $p \times p$ operation matrix $\mathcal{S}_1 := (V_{ij,1}^*(\cdot)V_{ij,1})$ is row stochastic, i.e.,

$$(3.21) \quad \sum_{j=1}^p V_{ij,1}^* V_{ij,1} = I_k \quad \text{for } i = 1, 2, \dots, p.$$

Then,

$$(3.22) \quad \sum_{j=1}^p A_j / A_{j,22} \leq \sum_{j=1}^p B_j / B_{j,22}.$$

Proof. We deduce from (3.19) via (2.7) that

$$B_j = \sum_{i=1}^p \Phi_{ij}(A_i) = \sum_{i=1}^p V_{ij}^* A_i V_{ij} \quad \text{for } j = 1, 2, \dots, p.$$

By Lemma 1, eq. (3.12), we have

$$(V_{ij}^* A_i V_{ij}) / (V_{ij}^* A_i V_{ij})_{22} = (V_{ij}^* A_i V_{ij}) / (V_{ij,2}^* A_i V_{ij,2}) = V_{ij,1}^* (A_i / A_{i,22}) V_{ij,1}.$$

Again, by Lemma 1, eq. (3.14),

$$(A_i / A_{i,22}) V_{ij,1} = V_{ij,1} (A_i / A_{i,22}).$$

Hence, by the superadditivity of φ (see (1.5)-(1.6)) and by the row stochasticity of \mathcal{S}_1 (see (3.21)), we find that

$$\begin{aligned} \sum_{j=1}^p B_j / B_{j,22} &= \sum_{j=1}^p \varphi(B_j) = \sum_{j=1}^p \varphi \left(\sum_{i=1}^p V_{ij}^* A_i V_{ij} \right) \\ &\geq \sum_{j=1}^p \sum_{i=1}^p \varphi(V_{ij}^* A_i V_{ij}) = \sum_{j=1}^p \sum_{i=1}^p (V_{ij}^* A_i V_{ij}) / (V_{ij}^* A_i V_{ij})_{22} \\ &= \sum_{j=1}^p \sum_{i=1}^p V_{ij,1}^* (A_i / A_{i,22}) V_{ij,1} = \sum_{j=1}^p \sum_{i=1}^p V_{ij,1}^* V_{ij,1} (A_i / A_{i,22}) \\ &= \sum_{i=1}^p \sum_{j=1}^p V_{ij,1}^* V_{ij,1} (A_i / A_{i,22}) = \sum_{i=1}^p \left(\sum_{j=1}^p V_{ij,1}^* V_{ij,1} \right) (A_i / A_{i,22}) \\ &= \sum_{i=1}^p I_k (A_i / A_{i,22}) = \sum_{i=1}^p A_i / A_{i,22}. \end{aligned}$$

Thus, (3.22) is shown.

REMARK 3. The usage of the commutativity in the above proof of Theorem 2 allows to utilize the row stochasticity of the matrix \mathcal{S}_1 . In the case of real matrix \mathcal{S} , the required commutativity holds automatically, as in Corollary 4 and Theorem A.

COROLLARY 4. For $j = 1, 2, \dots, p$, let $A_j, B_j \in \mathbb{P}_n^{++}$ be partitioned as

$$A_j = \begin{pmatrix} A_{j,11} & A_{j,12} \\ A_{j,21} & A_{j,22} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} B_{j,11} & B_{j,12} \\ B_{j,21} & B_{j,22} \end{pmatrix},$$

with invertible blocks $A_{j,22}$ and $B_{j,22}$ of order $n - k$.

Let

$$(B_1, B_2, \dots, B_p) = (A_1, A_2, \dots, A_p)\mathbf{S},$$

for some $p \times p$ real row stochastic $\mathbf{S} = (s_{ij})$ with $s_{ij} > 0$.

Then,

$$(3.23) \quad \sum_{j=1}^p A_j/A_{j,22} \leq \sum_{j=1}^p B_j/B_{j,22}.$$

Proof. For $i, j = 1, 2, \dots, p$, we set $v_{ij} := s_{ij}^{1/2} > 0$ and $\mathcal{S} = (\Phi_{ij})$, where

$$V_{ij} := v_{ij}I_n, \quad V_{ij,1} := v_{ij}I_k, \quad V_{ij,2} := v_{ij}I_{n-k},$$

$$\Phi_{ij} = V_{ij}^*(\cdot)V_{ij} \quad \text{and} \quad V_{ij} = \begin{pmatrix} V_{ij,1} & 0 \\ 0 & V_{ij,2} \end{pmatrix} \in \mathbb{M}_n.$$

Then conditions (3.20) and (3.21) are fulfilled automatically.

In consequence, by Theorem 2, inequality (3.23) holds valid.

4. Sherman-type theorem for Schur complement.

We start this section with some initial material.

We consider operation matrices

$$\mathcal{S} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1q} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{p1} & \Phi_{p2} & \dots & \Phi_{pq} \end{pmatrix} \quad \text{and} \quad \mathcal{S}^T = \begin{pmatrix} \Phi_{11} & \Phi_{21} & \dots & \Phi_{p1} \\ \Phi_{12} & \Phi_{22} & \dots & \Phi_{p2} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{1q} & \Phi_{2q} & \dots & \Phi_{pq} \end{pmatrix},$$

where $\Phi_{ij} : \mathbb{M}_n \rightarrow \mathbb{M}_n$ are linear operations.

When $\Phi_{ij} = V_{ij}^*(\cdot)V_{ij}$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, we obtain

$$\mathcal{S}^T(I_n) = \begin{pmatrix} V_{11}^*V_{11} & V_{21}^*V_{21} & \dots & V_{p1}^*V_{p1} \\ V_{12}^*V_{12} & V_{22}^*V_{22} & \dots & V_{p2}^*V_{p2} \\ \vdots & \vdots & \vdots & \vdots \\ V_{1q}^*V_{1q} & V_{2q}^*V_{2q} & \dots & V_{pq}^*V_{pq} \end{pmatrix}.$$

Therefore, in light of (2.10), if $\Psi_j = K_j^*(\cdot)K_j$ for some $K_j \in \mathbb{M}_n$, $j = 1, 2, \dots, q$, then

$$(\Psi_1, \Psi_2, \dots, \Psi_q)\mathcal{S}^T(I_n)$$

$$\begin{aligned}
 &= (K_1^*(\cdot)K_1, K_2^*(\cdot)K_2, \dots, K_q^*(\cdot)K_q) \begin{pmatrix} V_{11}^*V_{11} & V_{21}^*V_{21} & \cdots & V_{p1}^*V_{p1} \\ V_{12}^*V_{12} & V_{22}^*V_{22} & \cdots & V_{p2}^*V_{p2} \\ \vdots & \vdots & \vdots & \vdots \\ V_{1q}^*V_{1q} & V_{2q}^*V_{2q} & \cdots & V_{pq}^*V_{pq} \end{pmatrix} \\
 &= (B_1, B_2, \dots, B_p),
 \end{aligned}$$

where

$$B_i = \sum_{j=1}^q K_j^* V_{ij}^* V_{ij} K_j \geq 0 \quad \text{for } i = 1, 2, \dots, p.$$

Each B_i can be represented as a Gramm matrix $L_i^* L_i$ for some $L_i \in \mathbb{M}_n$. (For example, $L_i = B_i^{1/2}$.)

Finally,

$$(L_1^* L_1, L_2^* L_2, \dots, L_p^* L_p) = (K_1^*(\cdot)K_1, K_2^*(\cdot)K_2, \dots, K_q^*(\cdot)K_q) S^T (I_n).$$

This is why a Sherman-type condition for our matrices is of the form (4.27).

THEOREM 5. For $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, let $A_i, B_j \in \mathbb{P}_n^{++}$ and $L_i, K_j \in \mathbb{M}_n$ be partitioned as

$$A_i = \begin{pmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} B_{j,11} & B_{j,12} \\ B_{j,21} & B_{j,22} \end{pmatrix},$$

$$(4.24) \quad L_i = \begin{pmatrix} L_{i,1} & 0 \\ 0 & L_{i,2} \end{pmatrix} \quad \text{and} \quad K_j = \begin{pmatrix} K_{j,1} & 0 \\ 0 & K_{j,2} \end{pmatrix},$$

with invertible blocks $A_{i,22}, B_{j,22}, L_{i,2}, K_{j,2}$ of order $n - k$.

Suppose $\mathcal{S} = (\Phi_{ij})$ is a $p \times q$ operation matrix with

$$(4.25) \quad \Phi_{ij} = V_{ij}^*(\cdot)V_{ij} \quad \text{and} \quad V_{ij} = \begin{pmatrix} V_{ij,1} & 0 \\ 0 & V_{ij,2} \end{pmatrix} \in \mathbb{M}_n,$$

with invertible block $V_{ij,2}$ of order $n - k$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$.

Let the following Sherman-type conditions be fulfilled:

$$(4.26) \quad (B_1, B_2, \dots, B_q) = (A_1, A_2, \dots, A_p) \mathcal{S},$$

$$(4.27) \quad (L_1^* L_1, L_2^* L_2, \dots, L_p^* L_p) = (K_1^*(\cdot)K_1, K_2^*(\cdot)K_2, \dots, K_q^*(\cdot)K_q) S^T (I_n).$$

Assume that A_i commutes with $V_{ij} K_j$, i.e.,

$$(4.28) \quad A_i V_{ij} K_j = V_{ij} K_j A_i \quad \text{for } i = 1, 2, \dots, p \text{ and } j = 1, 2, \dots, q.$$

Then,

$$(4.29) \quad \sum_{j=1}^p |L_{i,1}| (A_i / A_{i,22}) |L_{i,1}| \leq \sum_{j=1}^q K_{j,1}^* (B_j / B_{j,22}) K_{j,1}.$$

Proof. By making use (4.26) and (2.7), we obtain

$$(4.30) \quad B_j = \sum_{i=1}^p \Phi_{ij}(A_i) = \sum_{i=1}^p V_{ij}^* A_i V_{ij} \quad \text{for } j = 1, 2, \dots, p.$$

Next, from Lemma 1, (3.12), we deduce for $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ that

$$(4.31) \quad (V_{ij}^* A_i V_{ij}) / (V_{ij}^* A_i V_{ij})_{22} = (V_{ij}^* A_i V_{ij}) / (V_{ij,2}^* A_i V_{ij,2}) = V_{ij,1}^* (A_i / A_{i,22}) V_{ij,1},$$

and by (3.13)-(3.14) and (4.28), that

$$(4.32) \quad (A_i / A_{i,22}) V_{ij,1} K_{j,1} = V_{ij,1} K_{j,1} (A_i / A_{i,22}),$$

because $(V_{ij} K_j)_1 = V_{ij,1} K_{j,1}$ by (4.24) and (4.25).

Furthermore, by (4.27) and (2.10),

$$(4.33) \quad \sum_{j=1}^q K_j^* V_{ij}^* V_{ij} K_j = L_i^* L_i \quad \text{for } i = 1, 2, \dots, p,$$

which, by (4.24) and (4.25), leads to

$$(4.34) \quad \sum_{j=1}^q K_{j,1}^* V_{ij,1}^* V_{ij,1} K_{j,1} = L_{i,1}^* L_{i,1} \quad \text{for } i = 1, 2, \dots, p. \quad \square$$

Hence, by (4.30), (1.5)-(1.6), (4.31), (4.32), (4.34), we proceed as follows

$$\begin{aligned} & \sum_{j=1}^q K_{j,1}^* (B_j / B_{j,22}) K_{j,1} = \sum_{j=1}^q K_{j,1}^* \varphi(B_j) K_{j,1} = \sum_{j=1}^q K_{j,1}^* \varphi \left(\sum_{i=1}^p V_{ij}^* A_i V_{ij} \right) K_{j,1} \\ & \geq \sum_{j=1}^q K_{j,1}^* \left(\sum_{i=1}^p \varphi(V_{ij}^* A_i V_{ij}) \right) K_{j,1} = \sum_{j=1}^q \sum_{i=1}^p K_{j,1}^* ((V_{ij}^* A_i V_{ij}) / (V_{ij}^* A_i V_{ij})_{22}) K_{j,1} \\ & = \sum_{j=1}^q \sum_{i=1}^p K_{j,1}^* V_{ij,1}^* (A_i / A_{i,22}) V_{ij,1} K_{j,1} = \sum_{j=1}^q \sum_{i=1}^p K_{j,1}^* V_{ij,1}^* V_{ij,1} K_{j,1} (A_i / A_{i,22}) \\ & = \sum_{i=1}^p \sum_{j=1}^q K_{j,1}^* V_{ij,1}^* V_{ij,1} K_{j,1} (A_i / A_{i,22}) = \sum_{i=1}^p \left(\sum_{j=1}^q K_{j,1}^* V_{ij,1}^* V_{ij,1} K_{j,1} \right) (A_i / A_{i,22}) \\ & = \sum_{i=1}^p L_{i,1}^* L_{i,1} (A_i / A_{i,22}) = \sum_{i=1}^p |L_{i,1}| (A_i / A_{i,22}) |L_{i,1}|. \end{aligned}$$

The last equality holds true because A_i commutes with $\sum_{j=1}^q K_j^* V_{ij}^* V_{ij} K_j$ by (4.28) and with $L_i^* L_i = |L_i|^2$ by (4.33). Hence, A_i commutes with $|L_i|$. Next, by Lemma 1, (ii), $A_i / A_{i,22}$ commutes with $|L_{i,1}| = |L_{i,1}|$. This completes the proof of (4.29).

Acknowledgment: The author would like to thank an anonymous referee for giving valuable comments for improving the previous version of the manuscript.

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