Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

ELA

ON CARDINALITY OF PARETO SPECTRA*

ALBERTO SEEGER† AND JOSÉ VICENTE-PÉREZ‡

Abstract. How many Pareto eigenvalues are there in a matrix of a prescribed order? This note provides the best lower bound that it is known up to now for the maximal number of Pareto eigenvalues in a matrix of order n.

Key words. Cone-constrained eigenvalue problem, Pareto eigenvalue, Linear complementarity system.

AMS subject classifications. 15A18, 15A39, 65H17.

1. Introduction. Throughout this work, the space $\mathbb{R}^n$ is equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and componentwise ordering. So, the notation $x \geq \mathbf{0}$ indicates that each component of x is nonnegative. The dimension of the zero vector $\mathbf{0}$ is clear from the context. Let $\mathbb{M}_n$ denote the linear space of real matrices of order n. A real number λ is a *Pareto eigenvalue* of $A \in \mathbb{M}_n$ if the linear complementarity system

$$x \geq \mathbf{0}, \quad Ax - \lambda x \geq \mathbf{0}, \quad \langle x, Ax - \lambda x \rangle = 0$$

has a nonzero solution $x \in \mathbb{R}^n$. Such nonzero vector x is called a *Pareto eigenvector* of A. The set of all Pareto eigenvalues, denoted by $\Pi(A)$, is called the *Pareto spectrum* of A. This terminology has been introduced in [8] and further employed in [1, 2, 3, 7]. Numerical methods for computing Pareto eigenvalues can be found, for example, in [1, 5, 6, 7].

The concept of Pareto eigenvalue arises in various fields of applied mathematics.

EXAMPLE 1.1. Consider an autonomous differential system of the form

$$z(t) \geq \mathbf{0}, \quad Az(t) - \dot{z}(t) \geq \mathbf{0}, \quad \langle z(t), Az(t) - \dot{z}(t) \rangle = 0.$$

A nonzero solution $z(\cdot)$ to the above system is obtained by setting $z(t) = e^{\lambda t}x$, where λ is a Pareto eigenvalue of A and x is an associated Pareto eigenvector.

*Received by the editors on January 31, 2011. Accepted for publication on July 10, 2011. Handling Editor: Shmuel Friedland.

†Department of Mathematics, University of Avignon, 33 rue Louis Pasteur, 84000 Avignon, France (alberto.seeger@univ-avignon.fr).

‡Departamento de Estadística e Investigación Operativa, Universidad de Alicante, 03080 Alicante, Spain (jose.vicente@ua.es). This author was supported by FPI Program of MICINN of Spain under grant BES-2006-14041.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

The theory of Pareto spectra differs substantially from the classical spectral theory, in particular, when it comes to deal with cardinality issues. For instance, a matrix of order 10 has at most 10 eigenvalues, but the matrix can have more than 1500 Pareto eigenvalues! It remains a difficult open question to determine the exact value of

$$\pi_n := \max_{A \in \mathbb{M}_n} |\Pi(A)|,$$

where $|S|$ denotes the cardinality of a set S. The number π_n is called the *Pareto capacity* of $\mathbb{M}_n$. It can be verified that $\pi_1 = 1$ and $\pi_2 = 3$, but π_n for $n \geq 3$ needs to be determined.

Example 1.2. A computation shows that

$$A = \begin{bmatrix} 1 & 3 & 9 \\ -3 & 9 & 27 \\ -9 & 27 & 81 \end{bmatrix}$$

has 9 Pareto eigenvalues. Hence, $\pi_3 \geq 9$. One knows that a matrix of order 3 cannot have 11 Pareto eigenvalues or more (see [7]), but it is not clear to us whether 10 Pareto eigenvalues is possible.

There is neither an exact nor an asymptotic formula that helps to determine π_n. As shown in [7], the sequence $\{\pi_n\}_{n \geq 1}$ is increasing and sandwiched as follows:

$$2^n - 1 \leq \pi_n \leq n(2^{n-1} - 1) + 1.$$

The above bounds are crude in general, and they are getting worse for larger values of n. The purpose of this note is to improve the lower bound on Pareto capacity π_n.

2. Main result.

For the sake of convenience we recall a useful lemma taken from [8] in the following. The notation $A^J = [a_{i,j}]_{i,j \in J}$ denotes the principal submatrix of A which takes entries in the rows and columns of A indexed by J.

Lemma 2.1. *The scalar $\lambda \in \mathbb{R}$ is a Pareto eigenvalue of $A \in \mathbb{M}_n$ if and only if there exist a nonempty set $J \subset \{1, \ldots, n\}$ and a vector $\xi \in \mathbb{R}^{|J|}$ such that*

$$A^J \xi = \lambda \xi, \tag{2.1}$$

$$\xi_j > 0 \quad \textit{for all } \; j \in J, \tag{2.2}$$

$$\sum_{j \in J} a_{i,j} \, \xi_j \geq 0 \quad \textit{for all } \; i \notin J. \tag{2.3}$$

Furthermore, a Pareto eigenvector x associated to λ is constructed by setting

$$x_j = \begin{cases} \xi_j & \textit{if } \; j \in J, \\ 0 & \textit{if } \; j \notin J. \end{cases}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

ELA

Thus, in order to compute the whole Pareto spectrum of a matrix $A \in \mathbb{M}_n$, one must consider the $2^n - 1$ possible ways of selecting the index set J. For each choice of J one first solves the classical eigenvalue problem (2.1), and then checks whether or not the positivity constraint (2.2) and the binding constraint (2.3) are satisfied. The following theorem is our main result.

THEOREM 2.2. *For all $n \geq 2$,*

$$3(2^{n-1} - 1) \;\leq\; \pi_n. \tag{2.4}$$

Proof. We construct a matrix of order n whose number of Pareto eigenvalues is equal to the left-hand side of (2.4). Consider the matrix

$$A = \begin{bmatrix} s^2 & s^3 & s^4 & s^5 & \cdots \\ -s^3 & s^4 & s^5 & s^6 & \cdots \\ -s^4 & s^5 & s^6 & s^7 & \cdots \\ -s^5 & s^6 & s^7 & s^8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{2.5}$$

whose (i,j)-entry is

$$a_{i,j} = \begin{cases} -s^{i+1} & \text{if } i \geq 2, j = 1, \\ s^{i+j} & \text{otherwise.} \end{cases}$$

For a reason that will be clear in a moment, we force the parameter $s \in \mathbb{R}$ to satisfy the following two properties:

$$s > 1 + \sqrt{2}, \tag{2.6}$$

$$s \text{ is transcendental.} \tag{2.7}$$

In order to compute the Pareto spectrum of (2.5), we work out all the choices for the index set $J = \{j_1, \ldots, j_q\}$. For convenience' sake, we split the job into several steps:

Step 1. We examine the case $1 \notin J$. This case is easier because the condition (2.3) is automatically satisfied. By proceeding as in [7, Proposition 3] we get

$$\lambda^J := \operatorname{tr}(A^J) \;=\; \sum_{j \in J} s^{2j},$$

$$\xi^J := \left(s^{j_1}, \ldots, s^{j_q}\right)^T$$

as unique solution to (2.1)–(2.2). Strictly speaking, the eigenvector ξ^J is unique only up to normalization. We deduce that λ^J is a Pareto eigenvalue of A.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

ELA

Step 2. We find the eigenvalues of A^J when $1 \in J$. If $J = \{1\}$, then A^J has s^2 as unique eigenvalue. This subcase is of no interest because (2.2)–(2.3) becomes infeasible. Consider then $1 = j_1 < \cdots < j_q$ with $q \geq 2$. We claim that A^J has exactly two real nonzero eigenvalues. For proving this claim we expand the polynomial $p_J(\lambda) = \det(A^J - \lambda E)$, where E denotes an identity matrix of appropriate size. The general theory of characteristic polynomials (cf. [4, Chapter 1]) asserts that

$$\det(A - \lambda E) = (-1)^n \lambda^n + \sum_{r=1}^{n} (-1)^{n-r} c_r(A) \lambda^{n-r}$$

with $c_r(A) = \sum_{|I|=r} \det(A^I)$. Applied to A^J, the above determinantal expansion becomes

$$\det(A^J - \lambda E) = (-1)^q \lambda^q + \sum_{r=1}^{q} (-1)^{q-r} c_r(A^J) \lambda^{q-r}$$

with

$$c_r(A^J) = \sum_{I \subset J,\, |I|=r} \det(A^I). \tag{2.8}$$

While writing (2.8) we use the fact that $(A^J)^I = A^I$ for $I \subset J$. Given the structure of (2.5), we see that $\det(A^I) = 0$ if I contains at least two elements from $\{2, \ldots, n\}$. Thus,

$$\begin{aligned} c_1(A^J) &= \operatorname{tr}(A^J), \\ c_2(A^J) &= \sum_{j \in J \setminus \{1\}} \det\left(A^{\{1,j\}}\right), \\ c_r(A^J) &= 0 \quad \text{for all } r \geq 3 \end{aligned}$$

with

$$\det(A^{\{1,j\}}) = \det \begin{bmatrix} a_{1,1} & a_{1,j} \\ a_{j,1} & a_{j,j} \end{bmatrix} = 2s^{2(j+1)}.$$

We now have

$$p_J(\lambda) = (-1)^q \lambda^{q-2} \Psi_J(\lambda),$$

where

$$\Psi_J(\lambda) := \lambda^2 - \operatorname{tr}(A^J)\lambda + w^J, \tag{2.9}$$

$$w^J := \sum_{j \in J \setminus \{1\}} 2s^{2(j+1)} = 2s^2\left(\operatorname{tr}(A^J) - s^2\right). \tag{2.10}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

ELA

The root $\lambda = 0$ is thrown away because (2.1)–(2.2) becomes infeasible. The polynomial p_J admits two other real roots, namely,

$$\lambda_{\pm}^J \;=\; \frac{\operatorname{tr}(A^J) \,\pm\, \sqrt{\Delta^J}}{2}\,.$$

The condition (2.6) yields $\operatorname{tr}(A^J) > 2(2+\sqrt{2})s^2$, which in turn implies the positivity of the discriminant

$$\begin{aligned}\Delta^J &= \left[\operatorname{tr}(A^J)\right]^2 - 4w^J \\ &= \left[\operatorname{tr}(A^J)\right]^2 - 8s^2\operatorname{tr}(A^J) + 8s^4.\end{aligned}$$

Step 3. We establish some useful bounds for λ_-^J and λ_+^J. We have

$$\Delta^J = \left(s^{2j_2} + \cdots + s^{2j_q} + s^2\right)^2 - 8s^2\left(s^{2j_2} + \cdots + s^{2j_q}\right).$$

By expanding the square and rearranging terms, we get

$$\Delta^J = \left(s^{2j_2} + \cdots + s^{2j_q} - s^2\right)^2 - 4s^2\left(s^{2j_2} + \cdots + s^{2j_q}\right),$$

and therefore,

$$\sqrt{\Delta^J} \;<\; s^{2j_2} + \cdots + s^{2j_q} - s^2 \;=\; \operatorname{tr}(A^J) - 2s^2.$$

This leads to the following chain of strict inequalities:

$$s^2 \;<\; \lambda_-^J \;<\; \lambda_+^J \;<\; \operatorname{tr}(A^J) - s^2. \tag{2.11}$$

Step 4. We prove that λ_+^J and λ_-^J are Pareto eigenvalues of A. Consider the vectors ξ_-^J and ξ_+^J given by

$$\xi_{\pm}^J = \left(u_{\pm}^J, s^{j_2-n}, s^{j_3-n}, \ldots, s^{j_q-n}\right)^T$$

with

$$u_{\pm}^J = \left(\frac{\operatorname{tr}(A^J) - s^2}{\lambda_{\pm}^J - s^2}\right) s^{1-n}\,.$$

Due to (2.11), the term $u_{\pm}^J$ is well defined and positive. The remaining components of $\xi_{\pm}^J$ are also positive. Note that

$$\begin{aligned}&(s^2 - \lambda_{\pm}^J)u_{\pm}^J \;+\; s^{1+j_2}s^{j_2-n} + \cdots + s^{1+j_q}s^{j_q-n} \\ &=\; -s^{1-n}\left(\operatorname{tr}(A^J) - s^2\right) + s^{1-n}\left(s^{2j_2} + \cdots + s^{2j_q}\right) \;=\; 0.\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

On the other hand, for all $i \in \{j_2, \ldots, j_q\}$, we have

$$\begin{aligned}
\sum_{j\in J} a_{i,j}(\xi^J_\pm)_j &= -s^{1+i}u^J_\pm \ + \ s^{i+j_2}s^{j_2-n} + \cdots + s^{i+j_q}s^{j_q-n} \\
&= -s^{1+i}u^J_\pm \ + \ s^{i-n}\left(s^{2j_2} + \cdots + s^{2j_q}\right) \\
&= -s^{1+i}\left(\frac{\operatorname{tr}(A^J) - s^2}{\lambda^J_\pm - s^2}\right)s^{1-n} \ + \ s^{i-n}\left(\operatorname{tr}(A^J) - s^2\right) \\
&= s^{i-n}\left(\operatorname{tr}(A^J) - s^2\right)\left(1 - \frac{s^2}{\lambda^J_\pm - s^2}\right) \\
&= \lambda^J_\pm s^{i-n} \\
&= \lambda^J_\pm (\xi^J_\pm)_i.
\end{aligned}$$

The equality previous to the last one is due to the fact that

$$\left(\lambda^J_\pm\right)^2 - \operatorname{tr}(A^J)\lambda^J_\pm \ + \ 2s^2\left(\operatorname{tr}(A^J) - s^2\right) = 0.$$

We have shown in this way that $A^J\xi^J_\pm = \lambda^J_\pm \xi^J_\pm$. The formula

$$\sum_{j\in J} a_{i,j}(\xi^J_\pm)_j = \lambda^J_\pm s^{i-n} \tag{2.12}$$

applies also for $i \notin J$. Since the right-hand side of (2.12) is positive, the vector $\xi^J_\pm$ satisfies the inequality (2.3). This proves that λ^J_+ and λ^J_- are Pareto eigenvalues of A.

Step 5. Finally, we determine the cardinality of $\Pi(A)$. Both sets

$$\begin{aligned}
\mathcal{J}_1 &= \{J \subset \{1, \ldots, n\} : 1 \notin J,\, J \neq \emptyset\}, \\
\mathcal{J}_2 &= \{J \subset \{1, \ldots, n\} : 1 \in J,\, |J| \geq 2\}
\end{aligned}$$

have the same cardinality, namely, $2^{n-1}-1$. Each $J \in \mathcal{J}_1$ produces exactly one Pareto eigenvalue and each $J \in \mathcal{J}_2$ produces exactly two Pareto eigenvalues. More precisely,

$$\Pi(A) = \{\lambda^J : J \in \mathcal{J}_1\} \ \cup \ \{\lambda^J_- : J \in \mathcal{J}_2\} \ \cup \ \{\lambda^J_+ : J \in \mathcal{J}_2\}. \tag{2.13}$$

If all the elements on the right-hand side of (2.13) are distinct, then

$$|\Pi(A)| = 3(2^{n-1} - 1).$$

Let us examine whether there is a repetition in (2.13) or not. Take two distinct index sets $I, J \in \mathcal{J}_1 \cup \mathcal{J}_2$. There are three cases for consideration. The first case occurs when I and J are both in $\mathcal{J}_1$. The situation is then as in [7, Proposition 3] and we know that $\lambda^I \neq \lambda^J$. The second case occurs when I and J are both in $\mathcal{J}_2$. We claim that

$$\{\lambda^I_-,\ \lambda^I_+\} \cap \{\lambda^J_-,\ \lambda^J_+\} = \emptyset. \tag{2.14}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

We prove that the polynomials Ψ_I and Ψ_J do not have a common root. Recall that Ψ_J is given by (2.9)–(2.10). The polynomial Ψ_I is defined, of course, in a similar way. Ab absurdo, let $\Psi_I(\lambda) = \Psi_J(\lambda) = 0$ for some $\lambda \in \mathbb{R}$. Then

$$\lambda^2 - \mathrm{tr}(A^I)\lambda + 2s^2\left(\mathrm{tr}(A^I) - s^2\right) = 0, \tag{2.15}$$

$$\lambda^2 - \mathrm{tr}(A^J)\lambda + 2s^2\left(\mathrm{tr}(A^J) - s^2\right) = 0. \tag{2.16}$$

The combination of (2.15) and (2.16) leads to

$$\lambda\left(\mathrm{tr}(A^J) - \mathrm{tr}(A^I)\right) - 2s^2\left(\mathrm{tr}(A^J) - \mathrm{tr}(A^I)\right) = 0.$$

But $\mathrm{tr}(A^J) - \mathrm{tr}(A^I) \neq 0$ because $I \neq J$. Hence, $\lambda = 2s^2$. Substituting this value in (2.15) we get

$$4s^4 - 2s^2\mathrm{tr}(A^I) + 2s^2\left(\mathrm{tr}(A^I) - s^2\right) = 0.$$

This leads to $s = 0$, which contradicts (2.6). The proof of (2.14) is complete. Consider now the third and last case: $I \in \mathcal{J}_1$ and $J \in \mathcal{J}_2$. We claim that

$$\lambda^I \notin \{\lambda_-^J, \lambda_+^J\}. \tag{2.17}$$

The proof of this claim requires taking a look at the Pareto eigenvalues

$$\lambda^I(s) = \sum_{i\in I} s^{2i},$$

$$\lambda_\pm^J(s) = \frac{1}{2}\sum_{j\in J} s^{2j} \;\pm\; \frac{1}{2}\left[\left(\sum_{j\in J} s^{2j}\right)^2 - 8s^2 \sum_{j\in J\setminus\{1\}} s^{2j}\right]^{1/2}$$

as functions of the parameter s. The term inside the square root is positive when s ranges on the open interval $]1+\sqrt{2}, \infty[$. On this interval the function $\lambda_-^J(\cdot)$ is infinitely often differentiable, but it is not a polynomial. Hence, $\lambda^I(\cdot)$ and $\lambda_-^J(\cdot)$ are not the same function. For the same reason, the functions $\lambda^I(\cdot)$ and $\lambda_+^J(\cdot)$ are different. Let us put ourselves in the worst situation and suppose that

$$\lambda^I(s_*) \in \{\lambda_-^J(s_*), \lambda_+^J(s_*)\} \tag{2.18}$$

for a certain $s_* \in]1+\sqrt{2}, \infty[$. The condition (2.18) says that $\Psi_J(\lambda^I(s_*)) = 0$. Let us examine the term $\mathcal{P}_{I,J}(s) = \Psi_J(\lambda^I(s))$ as function of $s \in]1+\sqrt{2}, \infty[$. Clearly,

$$\mathcal{P}_{I,J}(s) = \left(\sum_{i\in I} s^{2i}\right)^2 - \left(\sum_{j\in J} s^{2j}\right)\left(\sum_{i\in I} s^{2i}\right) + 2s^2 \sum_{j\in J\setminus\{1\}} s^{2j}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

is a polynomial with integer coefficients. We shall prove that $\mathcal{P}_{I,J}$ is non-constant. If $\mathcal{P}_{I,J}$ were constant, then it should be identically zero because $\mathcal{P}_{I,J}(s_*) = 0$. Now, if $\mathcal{P}_{I,J}$ is the zero function, then

$$\lambda^I(s) \in \{\lambda_-^J(s), \lambda_+^J(s)\}$$

for all $s \in]1+\sqrt{2}, \infty[$. By a continuity argument, either $\lambda^I(\cdot) = \lambda_-^J(\cdot)$ or $\lambda^I(\cdot) = \lambda_+^J(\cdot)$. But we have seen already that $\lambda^I(\cdot) \neq \lambda_-^J(\cdot)$ and $\lambda^I(\cdot) \neq \lambda_+^J(\cdot)$. Summarizing, s_* is a root of a non-constant polynomial with integer coefficients. Hence, s_* is algebraic. Therefore, the condition (2.17) holds true under the transcendentality assumption (2.7). □

REMARK 2.3. If A_s is the matrix given by (2.5), then $|\Pi(A_s)| = 3(2^{n-1} - 1)$ whenever $s \in]1 + \sqrt{2}, \infty[$ is a transcendental number. This is enough for taking care of Theorem 2.2. The same proof shows that

$$S = \left\{ s \in]1 + \sqrt{2}, \infty[\, : |\Pi(A_s)| \neq 3(2^{n-1} - 1) \right\}$$

is a finite set containing only algebraic numbers. Numerical experiments suggest that S is empty, but this remains as a conjecture.

As one can see in Table 2.1, there is a large gap between the known lower and upper bounds for π_n. Some work needs to be done to make this gap smaller, hoping to determine the exact value of π_n.

n	3	4	5	6	7	8	9	10
$3(2^{n-1} - 1)$	9	21	45	93	189	381	765	1533
$n(2^{n-1} - 1) + 1$	10	29	76	187	442	1017	2296	5111

TABLE 2.1
Lower and upper bounds for π_n.

REFERENCES

[1] S. Adly and A. Seeger. A nonsmooth algorithm for cone-constrained eigenvalue problems. *Comp. Optim. Appl.*, 49:299–318, 2011.

[2] X.-D. Chen and Y.-C. Qi. The Pareto eigenvalue problem of some matrices (Chinese). *J. Shanghai Second Polytechnic Univ.*, 27:54–64, 2010.

[3] J.B. Hiriart-Urruty and A. Seeger. A variational approach to copositive matrices. *SIAM Review*, 52:593–629, 2010.

[4] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[5] J.J. Júdice, I.M. Ribeiro, S.S. Rosa, and H.D. Sherali. On the asymmetric eigenvalue complementarity problem. *Optim. Methods Softw.*, 24:549–568, 2009.

[6] J.J. Júdice, I.M. Ribeiro, and H.D. Sherali. The eigenvalue complementarity problem. *Comput. Optim. Appl.*, 37:139–156, 2007.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 758-766, August 2011

ELA

[7] A. Pinto da Costa and A. Seeger. Cone-constrained eigenvalue problems: theory and algorithms. *Comput. Optim. Appl.*, 45:25–57, 2010.

[8] A. Seeger. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. *Linear Algebra Appl.*, 292:1–14, 1999.