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EIGENVALUES AND COMPONENT FACTORS IN GRAPHS*

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Abstract. For a set $\mathcal{H}$ of connected graphs, an $\mathcal{H}$-factor of G is a spanning subgraph F of G if each component of F is isomorphic to an element of $\mathcal{H}$. Kano, Lu and Yu [Electron. J. Combin. 26 (2019) P4.33] provided a good characterization based on an isolated vertex condition for the existence of a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor in graphs. Motivated by the above elegant result, we in this paper focus on the existence of a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor in graphs from perspective of eigenvalues. By adopting a crucial technique due to Tait [J. Combin. Theory Ser. A 166 (2019) 42-58] and combining typical spectral methods and structural analysis, we present tight sufficient conditions in terms of the spectral radius and the distance spectral radius for a graph to contain a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, respectively.

Key words. Component factors, Spectral radius, Distance spectral radius.

AMS subject classifications. 05C50; 05C35.

1. Introduction. Let $G = (V(G), E(G))$ be a simple undirected graph with n vertices and e edges, where $n = |V(G)|$ and $e = |E(G)|$. For a vertex v of a graph G, the degree of v in G is denoted by $d_G(v)$. For a vertex subset S of G, we denote by $G-S$ and $G[S]$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges and the subgraph of G induced by S, respectively. For two vertex disjoint graphs G_1 and G_2, we denote by $G_1 \cup G_2$ the *disjoint union* of G_1 and G_2. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. We denote by $iso(G)$ the number of isolated vertices in G. For undefined terms and notions, one can refer to [2].

Given a graph G of order n, the *adjacency matrix* of G is the 0-1 matrix $A(G) = (a_{ij})_{n\times n}$ indexed by the vertex set $V(G)$ of G, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the *spectral radius* of G. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance* between v_i and v_j, denoted by $p_{ij}(G)$, is the length of a shortest path from v_i to v_j. The *distance matrix* of G, denoted by $D(G)$, is an $n \times n$ matrix with its rows and columns indexed by $V(G)$. For $i \neq j$, the (i, j)-entry of $D(G)$ is equal to $p_{ij}(G)$. Also, $p_{ii}(G) = 0$. Clearly, $D(G)$ is a real symmetric matrix with zeros on the diagonal. In this paper, we always use J to denote the all-one matrix, and I to denote the identity square matrix. We can order the eigenvalues of $D(G)$ as $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G))$. We call $\lambda_1(D(G))$ the *distance spectral radius* of G.

For any tree T, let $Leaf(T) = \{v \in T : d_T(v) = 1\}$ be the set of leaves of T. Kano et al. [7] defined a new class of trees $\mathcal{T}(2k+1)$. Let $k \geq 2$ be an integer and let T be a tree that satisfies the following two conditions: for every vertex $v \in V(T)$,

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(i) $d_{T-Leaf(T)}(v) \in \{1, 3, \ldots, 2k+1\}$, and
(ii) $2 \cdot$ (the number of leaves adjacent to v in T) $+ d_{T-Leaf(T)}(v) \leq 2k+1$ in T.

Define a new tree T^* from T as follows (see Fig. 1).

(iii) insert a new vertex of degree 2 into each edge of $T - Leaf(T)$, and
(iv) for each vertex v of $T - Leaf(T)$ with $d_{T-Leaf(T)}(v) = 2r+1 < 2k+1$ in T, add $k - r -$ (the number of leaves adjacent to v in T) pendant edges to v.

Then the resulting tree T^* has a fractional $[1, k+\frac{1}{2}]$-factor h such that every pendant edge e_1 of T^* has $h(e_1) = 1$, all other edges e_2 have $h(e_2) = \frac{1}{2}$.

The set of such trees T^* for all trees T is denoted by $\mathcal{T}(2k+1)$.

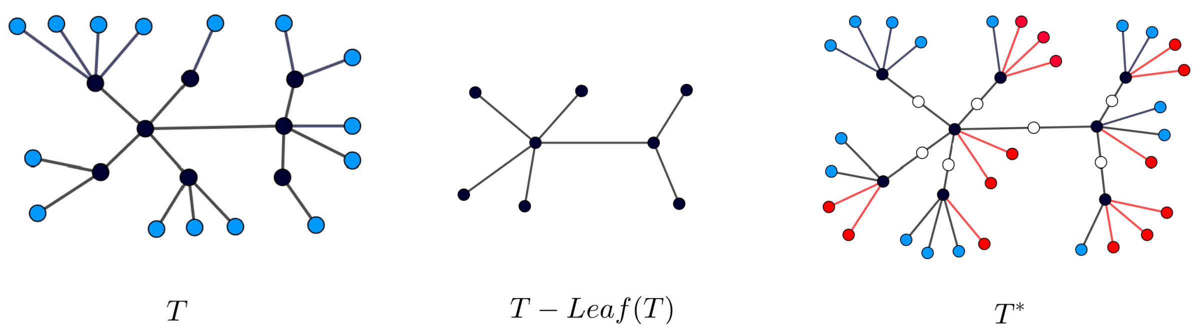

FIGURE 1. *A tree T which satisfies (i) and (ii) with $k = 4$, the tree $T - Leaf(T)$ and the tree T^* obtained from T.*

Let $\mathcal{H}$ be a set of connected graphs. A spanning subgraph F of G is called *an $\mathcal{H}$-factor* or *a component factor* if each component of F is isomorphic to an element of $\mathcal{H}$. Very recently, the relationship between the existence of component factors and the spectral radius is a natural and interesting topic, which has attracted much attention in spectral graph theory. Li and Miao [10] established a lower bound on the spectral radius of a graph G to guarantee that the graph G has a path factor. Moreover, Miao and Li [12] presented a sufficient condition based on the spectral radius of G to ensure that a graph G contains a star factor. Lou et al. [8] provided a tight sufficient condition in terms of the spectral radius for the existence of a $\{K_2, \{C_k\}\}$-factor in a graph with minimum degree δ, where $k \geq 3$ is an integer.

Kano et al. [7] proposed a necessary and sufficient condition for a graph to contain a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor.

THEOREM 1.1 (Kano et al. [7]). *Let $k \geq 2$ be an integer. Then a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor if and only if*

$$iso(G-S) \leq \left(k + \frac{1}{2}\right)|S|,$$

for any vertex subset $S \subseteq V(G)$.

Based on Theorem 1.1, we in this paper obtain a tight sufficient condition in terms of the spectral radius for a graph to contain a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor.

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Theorem 1.2. *Let G be a connected graph of order n and $k \geq 2$ be an integer. If*

$$\rho(G) \geq \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)),$$

then G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor unless $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.

Recently, researchers have increasingly focused on the relationship between structural properties and the distance spectral radius of graphs. Zhang and Lin[14] proposed sufficient conditions based on the distance spectral radius to guarantee the existence of a perfect matching in connected graphs and connected balanced bipartite graphs, respectively. Subsequently, Zhang et al. [15] generalized the result of [14] and provided a sufficient condition in terms of the distance spectral radius to ensure the existence of a perfect matching in connected balanced bipartite graphs with given minimum degree. Li et al. [11] proved an upper bound on the distance spectral radius for a connected graph to contain a fractional perfect matching. Later, Miao and Li [12] presented an upper bound on the distance spectral radius to guarantee the existence of a star factor in a connected graph. Zhang and van Dam [16] posed sufficient conditions in terms of the distance spectral radius to ensure k-extendability of connected graphs and connected balanced bipartite graphs, respectively. Lou et al. [9] proved sufficient conditions with respect to the distance spectral radius for a graph to be t-tough.

Motivated by the above results, we pose a tight sufficient condition based on the distance spectral radius to guarantee the existence of a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor in a connected graph.

Theorem 1.3. *Let G be a connected graph of order n and $k \geq 2$ be an integer. If*

$$\lambda_1(D(G)) \leq \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))),$$

then G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor unless $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.

2. Preliminaries. We in this section will present some auxiliary results that will be used in our arguments. A real symmetric matrix M is denoted by

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

where $M_{i,j}$ is the submatrix corresponding to rows in X_i and columns in X_j where $(X_1, X_2, \ldots, X_m)$ is a partition of $X = \{1, 2, \ldots, n\}$. The quotient matrix $R(M)$ of the matrix M (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The above partition is called *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.1 (Brouwer and Haemers [1], Godsil and Royle [3], Haemers [6]). *Let M be a real symmetric matrix and let $R(M)$ be its equitable quotient matrix. Then the eigenvalues of the quotient matrix $R(M)$ are eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then the spectral radius of the quotient matrix $R(M)$ equals to the spectral radius of M.*

Lemma 2.2 (Godsil and Royle [3]). *If H is a spanning subgraph of a connected graph G, then $\rho(H) \leq \rho(G)$ with equality if and only if $H \cong G$.*

Tait [13] proved the following elegant result, which plays an essential role in our proof of Theorem 1.2.

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LEMMA 2.3 (Tait [13]). *Let H_1 be a graph on n_0 vertices with maximum degree d and H_2 be a graph on $n-n_0$ vertices with maximum degree d'. H_1 and H_2 may have loops or multiple edges, where loops add 1 to the degree. Let H be the join of H_1 and H_2. Define*

$$B = \begin{bmatrix} d & n-n_0 \\ n_0 & d' \end{bmatrix},$$

then $\rho(H) \le \lambda_1(B)$.

LEMMA 2.4 (Godsil [4]). *Let e be an edge of a graph G such that $G-e$ is connected. Then*

$$\lambda_1(D(G)) < \lambda_1(D(G-e)).$$

Let $W(G) = \sum_{i<j} d_{ij}(G)$ be the *Wiener index* of a connected graph G of order n. By the Rayleigh quotient[5], one can easily have

$$\lambda_1(D(G)) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T D(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \ge \frac{\mathbf{1}^T D(G)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2W(G)}{n}, \tag{2.1}$$

where $\mathbf{1} = (1,1,\ldots,1)^T$.

3. Proof of Theorem 1.2. In this section, we give the proof of Theorem 1.2. Before presenting our proof, we first prove some critical lemmas.

LEMMA 3.1. *$\rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)) > n-k-2$ and $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ contains no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, where $k \ge 2$.*

Proof. Note that K_{n-k-1} is a proper subgraph of $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. By Lemma 2.2, we have $\rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)) > \rho(K_{n-k-1}) = n-k-2$. Now take $S = \{v\}$ with $d(v) = n-1$. Then

$$iso(K_1 \vee (K_{n-k-2} \cup (k+1)K_1) - S) \ge k+1 > \left(k+\frac{1}{2}\right)|S|.$$

By Theorem 1.1, we have $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ contains no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. □

Let $G' = K_s \vee (K_{n-s-\lfloor (k+\frac{1}{2})s\rfloor -1} \cup (\lfloor (k+\frac{1}{2})s\rfloor + 1)K_1)$, where $s \ge 1$, $k \ge 2$ and $n \ge s + \lfloor (k+\frac{1}{2})s\rfloor + 1$.

LEMMA 3.2. *If $s \ge 2$ and $k \ge 2$, then $\rho(G') \le n-k-2$.*

Proof. We distinguish the proof into two cases.

CASE 1. $n \ge s + \lfloor (k+\frac{1}{2})s\rfloor + 2$.

Let $T = \{v |\ v \in V(G')$ and $d_{G'}(v) = n-1\}$. Then $|T| = s$ and $G'[T] \cong K_s$. Define

$$B_1 = \begin{bmatrix} s-1 & n-s \\ s & n-s-\left\lfloor \left(k+\frac{1}{2}\right)s\right\rfloor - 2 \end{bmatrix}.$$

Note that $G'[T]$ has s vertices with the maximum degree $s-1$ and $G'[V(G')\setminus T]$ has $n-s$ vertices with the maximum degree $n-s-\lfloor (k+\frac{1}{2})s\rfloor - 2$. By Lemma 2.3, we have $\rho(G') \le \lambda_1(B_1)$. The characteristic polynomial of B_1 is

$$f(x) = x^2 - \left(n - \left\lfloor \left(k+\frac{1}{2}\right)s\right\rfloor - 3\right)x - (s-1)\left\lfloor \left(k+\frac{1}{2}\right)s\right\rfloor - n - s + 2.$$

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Note that the axis of symmetry of $f(x)$ is $x = \frac{n-\lfloor(k+\frac{1}{2})s\rfloor-3}{2} \leq n-k-2$. Then $f(x)$ is increasing on $x \geq n-k-2$. Since $s \geq 2$, $\lfloor(k+\frac{1}{2})s\rfloor \geq 2k+1$. Combining $n \geq s+\lfloor(k+\frac{1}{2})s\rfloor+2$ and $\lfloor(k+\frac{1}{2})s\rfloor \geq sk+\frac{s-1}{2}$, we have

$$\begin{aligned} f(n-k-2) &= \Big(\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor-k\Big)n-(k+s+1)\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor+k^2+k-s \\ &\geq \Big(\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor-k\Big)\Big(s+\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor+2\Big)-(k+s+1)\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor+k^2+k-s \\ &= \Big(\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor-2k+1\Big)\Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor+k^2-(s+1)k-s \\ &\geq 2\Big(sk+\frac{s-1}{2}\Big)+k^2-(s+1)k-s \\ &\geq k^2+k-1 \\ &> 0. \end{aligned}$$

Hence, $\rho(G') \leq \lambda_1(B_1) < n-k-2$.

Case 2. $n = s+\lfloor(k+\frac{1}{2})s\rfloor+1$.

In this case, $G' \cong K_s \vee (\lfloor(k+\frac{1}{2})s\rfloor+1)K_1$. Hence, the quotient matrix $R(A(G'))$ of $A(G')$ is

$$B_2 = \begin{bmatrix} s-1 & 1 \\ s & 0 \end{bmatrix}.$$

By Lemma 2.1, we have $\rho(G') = \lambda_1(B_2) = s$. Note that

$$n-k-2 \geq s+\lfloor 2(k+\frac{1}{2})\rfloor-k-2 \geq s+k-1 > s.$$

Then $\rho(G') \leq \lambda_1(B_2) = s < n-k-2$. □

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that G has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, where $k \geq 2$. By Theorem 1.1, there exists a vertex subset $S \subseteq V(G)$ such that

$$iso(G-S) \geq \Big\lfloor\Big(k+\frac{1}{2}\Big)s\Big\rfloor+1. \tag{3.2}$$

First, we claim that $S \neq \emptyset$. Suppose to the contrary that $S = \emptyset$. By (3.2), we have $iso(G) \geq 1$, which contradicts that G is connected. Hence, $|S| = s \geq 1$. Then G is a spanning subgraph of $G' = K_s \vee [K_{n-s-\lfloor(k+\frac{1}{2})s\rfloor-1} \cup (\lfloor(k+\frac{1}{2})s\rfloor+1)K_1]$. Combining Lemma 2.2, we have

$$\rho(G) \leq \rho(G'). \tag{3.3}$$

with equality if and only if $G \cong G'$.

Case 1. $s = 1$.

Then $G' \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. By (3.3), we have $\rho(G) \leq \rho(G') = \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))$. By the assumption, one can obtain that $\rho(G) \geq \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))$. Hence, $\rho(G) = \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))$. By Lemma 3.1, we have $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. Hence, $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.

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CASE 2. $s \geq 2$.

By (3.3) and Lemma 3.2, we have $\rho(G) \leq \rho(G') \leq n-k-2$. By the assumption and Lemma 3.1, we have $\rho(G) \geq \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)) > n-k-2$, a contradiction. □

4. Proof of Theorem 1.3. We first put forward an important lemma, which will be used to prove Theorem 1.3.

LEMMA 4.1. *Let $k \geq 2$ be an integer. If $s \geq 2$, then*

$$\lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))) < \lambda_1(D(K_s \vee (K_{n-s-\lfloor (k+\frac{1}{2})s \rfloor -1} \cup (\lfloor (k+\frac{1}{2})s \rfloor + 1)K_1))).$$

Proof. Let $t = \lfloor (k+\frac{1}{2})s \rfloor + 1$, $G^* = K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ and $G' = K_s \vee (K_{n-s-t} \cup tK_1)$, where $k \geq 2$ and $s \geq 2$ (see Fig. 2). We partition the vertex set of G' into $V(G') = V(K_s) \cup V(K_{n-s-t}) \cup V(tK_1)$. Then the distance matrix $D(G')$ of G' can be written as

$$\begin{array}{c} \\ s \\ n-s-t \\ t \end{array} \begin{array}{c} \begin{array}{ccc} s & n-s-t & t \end{array} \\ \begin{pmatrix} J-I & J & J \\ J & J-I & 2J \\ J & 2J & 2(J-I) \end{pmatrix}_{n\times n} \end{array}.$$

The quotient matrix of $D(G')$ with respect to the above equitable partition is

$$R(D(G')) = \begin{pmatrix} s-1 & n-s-t & t \\ s & n-s-t-1 & 2t \\ s & 2(n-s-t) & 2(t-1) \end{pmatrix}.$$

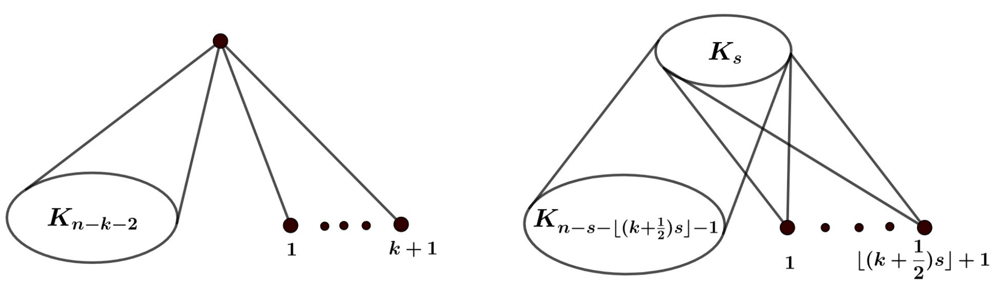

FIGURE 2. *Graphs $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ and $K_s \vee (K_{n-s-\lfloor (k+\frac{1}{2})s \rfloor -1} \cup (\lfloor (k+\frac{1}{2})s \rfloor + 1)K_1)$.*

Hence, the characteristic polynomial of $R(D(G'))$ is

$$\begin{aligned} f_1(\lambda) = {} & \lambda^3 - (n+t-4)\lambda^2 - (2nt+3n-2t^2-(3s-1)t-5)\lambda + st(n-s-t+3) \\ & -2n(t+1) + 2t^2 + 2. \end{aligned} \tag{4.4}$$

By Lemma 2.1, we have $\lambda_1(D(G')) = \lambda_1(R(D(G')))$.

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Moreover, the distance matrix $D(G^*)$ of G^* is

$$\begin{array}{c} \\ 1 \\ n-k-2 \\ k+1 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & n-k-2 & k+1 \end{array} \\ \begin{pmatrix} 0 & J & J \\ J & J-I & 2J \\ J & 2J & 2(J-I) \end{pmatrix}_{n\times n} \end{array}.$$

Then the quotient matrix $R(D(G^*))$ of $D(G^*)$ with respect to the partition $V(G^*) = V(K_1) \cup V(K_{n-k-2}) \cup V((k+1)K_1)$ is

$$R(D(G^*)) = \begin{pmatrix} 0 & n-k-2 & k+1 \\ 1 & n-k-3 & 2(k+1) \\ 1 & 2(n-k-2) & 2k \end{pmatrix}.$$

By Lemma 2.1, we obtain that $\lambda_1(D(G^*)) = \lambda_1(R(D(G^*)))$. The characteristic polynomial of $R(D(G^*))$ is

$$f_2(\lambda) = \lambda^3 - (n+k-3)\lambda^2 - (2kn + 5n - 2k^2 - 6k - 9)\lambda - n(k+3) + k^2 + 4k + 5. \tag{4.5}$$

Note that the Wiener index of G^* is

$$\begin{aligned} W(G^*) &= \sum_{i<j} d_{ij}(G^*) \\ &= n - 1 + \frac{(n-k-2)(n-k-3)}{2} + 2(n-k-2)(k+1) + k(k+1) \\ &= \frac{1}{2}(n^2 + (2k+1)n - k^2 - 5k - 4). \end{aligned}$$

Since $s \geq 2$, $n \geq s + \lfloor (k + \frac{1}{2})s \rfloor + 1 \geq 2k + 4$. By (2.1), we have

$$\begin{aligned} \lambda_1(D(G^*)) &\geq \frac{2W(G^*)}{n} \\ &= \frac{n^2 + (2k+1)n - k^2 - 5k - 4}{n} \\ &= n + 2k + 1 - \frac{k^2 + 5k + 4}{n} \\ &> n + k. \end{aligned}$$

Let $h(\lambda) = f_2(\lambda) - f_1(\lambda)$. By (4.4) and (4.5), we have

$$\begin{aligned} h(\lambda) &= (t-k-1)\lambda^2 + ((2t-2k-2)n - 3st - 2t^2 + t + 2k^2 + 6k + 4)\lambda \\ &\quad -(n-s-t+3)st - (k+1)n + 2nt - 2t^2 + k^2 + 4k + 3. \end{aligned} \tag{4.6}$$

Then we have

$$h'(\lambda) = 2(t-k-1)\lambda + (2t-2k-2)n - 3st - 2t^2 + t + 2k^2 + 6k + 4.$$

For $\lambda > n+k$, we analyze the monotonicity of $h(\lambda)$. Note that $n \geq s+t$ and $t = \lfloor (k+\frac{1}{2})s \rfloor + 1 \geq (k+\frac{1}{2})s + \frac{1}{2}$. Then

$$\begin{aligned} h'(\lambda) &> 2(t-k-1)(n+k) + (2t-2k-2)n - 3st - 2t^2 + t + 2k^2 + 6k + 4 \\ &= 4n(t-k-1) + 2kt - 3st - 2t^2 + t + 4k + 4 \\ &\geq 4(s+t)(t-k-1) + 2kt - 3st - 2t^2 + t + 4k + 4 \\ &= 2t^2 + (s-2k-3)t - 4ks + 4k - 4s + 4. \end{aligned} \tag{4.7}$$

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Let $g(t) = 2t^2 + (s - 2k - 3)t - 4ks + 4k - 4s + 4$. Then the axis of symmetry of $g(t)$ is $t = \frac{2k+3-s}{4}$. Note that $t \geq (k + \frac{1}{2})s + \frac{1}{2} > \frac{2k+3-s}{4}$. Then $g(t)$ is increasing on $t \geq (k + \frac{1}{2})s + \frac{1}{2}$. Combining (4.7), $s \geq 2$ and $k \geq 2$, we have

$$\begin{aligned} h'(\lambda) &> g(t) \\ &\geq 2((k + \frac{1}{2})s + \frac{1}{2})^2 + (s - 2k - 3)((k + \frac{1}{2})s + \frac{1}{2}) - 4ks + 4k - 4s + 4 \\ &= (2k^2 + 3k + 1)s^2 - (2k^2 + 6k + 4)s + 3k + 3 \\ &\geq 4k^2 + 3k - 1 \\ &> 0, \end{aligned}$$

which implies that $h(\lambda)$ is increasing with respect to $\lambda > n + k$. Combining (4.6) and $\lambda > n + k$, we have

$$\begin{aligned} h(\lambda) &> (t - k - 1)(n + k)^2 + ((2t - 2k - 2)n - 3st - 2t^2 + t + 2k^2 + 6k + 4)(n + k) \\ &\quad -(n - s - t + 3)st - (k + 1)n + 2nt - 2t^2 + k^2 + 4k + 3 \\ &= (3t - 3k - 3)n^2 - (2t^2 + (4s - 4k - 3)t + 2k^2 - k - 3)n + (s + t - 3k)st \\ &\quad -(2k + 2)t^2 + (k^2 + k)t + k^3 + 6k^2 + 8k + 3. \end{aligned}$$

Let $h_1(n) = (3t - 3k - 3)n^2 - (2t^2 + (4s - 4k - 3)t + 2k^2 - k - 3)n + (s + t - 3k)st - (2k + 2)t^2 + (k^2 + k)t + k^3 + 6k^2 + 8k + 3$. Then

$$h_1'(n) = 6n(t - k - 1) - 2t^2 - (4s - 4k - 3)t - 2k^2 + k + 3.$$

Combining $n \geq s + t$, $t \geq (k + \frac{1}{2})s + \frac{1}{2}$, $s \geq 2$ and $k \geq 2$, we have

$$\begin{aligned} h_1'(n) &> 6(s + t)(t - k - 1) - 2t^2 + (-4s + 4k + 3)t - 2k^2 + k + 3 \\ &= 4t^2 + (2s - 2k - 3)t - (6k + 6)s - 2k^2 + k + 3 \\ &\geq 4((k + \frac{1}{2})s + \frac{1}{2})^2 + (2s - 2k - 3)((k + \frac{1}{2})s + \frac{1}{2}) - (6k + 6)s - 2k^2 + k + 3 \\ &= (4k^2 + 6k + 2)s^2 - (2k^2 + 6k + \frac{9}{2})s - 2k^2 + \frac{5}{2} \\ &\geq 10k^2 + 12k + \frac{3}{2} \\ &> 0. \end{aligned}$$

Hence, $h_1(n)$ is increasing with respect to $n \geq s + t$. Then

$$\begin{aligned} h_1(n) &\geq (3t - 3k - 3)(s + t)^2 + (-2t^2 + (-4s + 4k + 3)t - 2k^2 + k + 3)(s + t) \\ &\quad +(s + t - 3k)st - (2k + 2)t^2 + (k^2 + k)t + k^3 + 6k^2 + 8k + 3 \\ &= t^3 + (s - k - 2)t^2 - (5ks + 6s + k^2 - 2k - 3)t - (3k + 3)s^2 - (2k^2 - k - 3)s \\ &\quad +k^3 + 6k^2 + 8k + 3. \end{aligned}$$

Let $h_2(t) = t^3 + (s - k - 2)t^2 - (5ks + 6s + k^2 - 2k - 3)t - (3k + 3)s^2 - (2k^2 - k - 3)s + k^3 + 6k^2 + 8k + 3$. Then

$$h_2'(t) = 3t^2 + (2s - 2k - 4)t - (5k + 6)s - k^2 + 2k + 3.$$

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Note that the axis of symmetry of $h_2'(t)$ is $t = \frac{k+2-s}{3}$. By $t \geq (k+\frac{1}{2})s + \frac{1}{2}$, $s \geq 2$ and $k \geq 2$, we have

$$\begin{aligned} h_2'(t) &\geq 3((k+\frac{1}{2})s+\frac{1}{2})^2 + (2s-2k-4)((k+\frac{1}{2})s+\frac{1}{2}) - (5k+6)s - k^2 + 2k + 3 \\ &= (3k^2+5k+\frac{7}{4})s^2 - (2k^2+7k+\frac{11}{2})s - k^2 + k + \frac{7}{4} \\ &\geq 7k^2 + 7k - \frac{9}{4} \\ &> 0, \end{aligned}$$

which implies that $h_2(t)$ is increasing on $t \geq (k+\frac{1}{2})s+\frac{1}{2}$. Hence, we have $h_2(t) \geq h_2(\lfloor (k+\frac{1}{2})s \rfloor + 1)$ $\geq h_2((k+\frac{1}{2})s+\frac{1}{2})$.

Claim 1. $h_2(\lfloor (k+\frac{1}{2})s \rfloor + 1) > 0$.

Proof. Let $h_3(s) = h_2(\lfloor (k+\frac{1}{2})s \rfloor + 1)$. Suppose that s is odd. Then we have $s \geq 3$ and

$$\begin{aligned} h_3(s) &= h_2((k+\frac{1}{2})s+\frac{1}{2}) \\ &= (\frac{3}{8}+k^3+\frac{5}{2}k^2+\frac{7}{4}k)s^3 - (k^3+\frac{45}{8}+\frac{13}{2}k^2+\frac{45}{4}k)s^2 + (\frac{9}{8}-k^3+\frac{3}{4}k-\frac{3}{2}k^2)s \\ &\quad + \frac{35k}{4} + k^3 + \frac{33}{8} + \frac{11k^2}{2}. \end{aligned}$$

Note that

$$\begin{aligned} h_3'(s) &= 3(\frac{3}{8}+k^3+\frac{5}{2}k^2+\frac{7}{4}k)s^2 - 2(k^3+\frac{45}{8}-\frac{13}{2}k^2-\frac{45}{4}k)s + \frac{9}{8} - k^3 + \frac{3}{4}k - \frac{3}{2}k^2 \\ &\geq -\frac{45}{2} + 20k^3 + 27k^2 - \frac{39}{2}k \\ &> 0. \end{aligned}$$

Hence, $h_3(s)$ is increasing on $s \geq 3$. Then

$$\begin{aligned} h_3(s) &\geq h_3(3) \\ &= 16k^3 + 10k^2 - 43k - 33 \\ &> 0. \end{aligned}$$

Hence, we obtain that $h_2((k+\frac{1}{2})s+\frac{1}{2}) \geq h_3(3) > 0$.

Suppose that s is even. Then we have $s \geq 2$ and $t \geq (k+\frac{1}{2})s + 1$. Hence,

$$\begin{aligned} h_3(s) &= h_2((k+\frac{1}{2})s+1) \\ &= (\frac{3}{8}+k^3+\frac{5}{2}k^2+\frac{7}{4}k)s^3 - (k^3+\frac{19}{4}-5k^2-\frac{35}{4}k)s^2 - (1+k^3+2k+\frac{5}{2}k^2)s + k^3 \\ &\quad + 5k^2 + 9k + 5. \end{aligned}$$

Note that

$$h_3'(s) = 3(\frac{3}{8}+k^3+\frac{5}{2}k^2+\frac{7}{4}k)s^2 - 2(k^3-5k^2-\frac{35}{4}k+\frac{19}{4})s - (k^3+\frac{5}{2}k^2+2k+1).$$

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Combining $s \geq 2$ and $k \geq 2$, one can obtain that $h_3'(s) > 0$. Hence, $h_3(s)$ is increasing on $s \geq 2$. Then we have

$$\begin{aligned} h_3(s) &\geq h_3(2) \\ &= 12(\frac{3}{8} + k^3 + \frac{5}{2}k^2 + \frac{7}{4}k) - 4(k^3 - 5k^2 - \frac{35}{4}k + \frac{19}{4}) - (k^3 + \frac{5}{2}k^2 + 2k + 1) \\ &> 0. \end{aligned}$$

Hence, $h_2(\lfloor (k + \frac{1}{2})s \rfloor + 1) = h_3(s) > h_3(2) > 0$. ∎

Note that $h(\lambda) > h_1(n) \geq h_2(t) \geq h_2(\lfloor (k + \frac{1}{2})s \rfloor + 1)$. Combining Claim 1, we have $h(\lambda) > h_2(\lfloor (k + \frac{1}{2})s \rfloor + 1) > 0$. This implies that $f_2(\lambda) > f_1(\lambda)$ with $\lambda > n + k$. Recall that $\lambda_1(D(G^*)) > n + k$. Then we have $\lambda_1(D(G^*)) < \lambda_1(D(G'))$. ∎

Proof of Theorem 1.3. Let G be a connected graph of order n. Suppose to the contrary that G contains no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k + 1)\}$-factor, where $k \geq 2$. By Theorem 1.1, there exists a vertex subset $S \subseteq V(G)$ such that

$$iso(G - S) \geq \lfloor (k + \frac{1}{2})s \rfloor + 1. \tag{4.8}$$

First, we claim that $S \neq \emptyset$. Suppose to the contrary that $S = \emptyset$. By (4.8), we have $iso(G) \geq 1$, which contradicts that G is connected. Hence, $s \geq 1$. Obviously, G is a spanning subgraph of $G' = K_s \vee ((\lfloor (k + \frac{1}{2})s \rfloor + 1)K_1 \cup K_{n-s-\lfloor (k+\frac{1}{2})s \rfloor - 1})$. Combining Lemma 2.4, we obtain that

$$\lambda_1(D(G)) \geq \lambda_1(D(G')), \tag{4.9}$$

with equality if and only if $G \cong G'$.

Next we distinguish our proof into two cases.

Case 1. $s = 1$.

At this moment, $G' = K_1 \vee (K_{n-k-2} \cup (k + 1)K_1)$. By (4.9), we have

$$\lambda_1(D(G)) \geq \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k + 1)K_1))).$$

By the assumption, we have $\lambda_1(D(G)) \leq \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)))$. Hence, $\lambda_1(D(G)) = \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)))$. Note also that $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. Hence, $G \cong K_1 \vee (K_{n-k-2} \cup (k + 1)K_1)$.

Case 2. $s \geq 2$.

By (4.9) and Lemma 4.1, we have

$$\lambda_1(D(G)) \geq \lambda_1(D(G')) > \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k + 1)K_1))).$$

By the assumption, we have $\lambda_1(D(G)) \leq \lambda_1(D(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)))$, a contradiction. This complete the proof of Theorem 1.3. □

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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