



## TOUGHNESS AND SIGNLESS LAPLACIAN EIGENVALUES IN GRAPHS\*

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**Abstract.** For a non-complete connected graph  $G$ , the *toughness* of  $G$  is defined as  $t(G) = \min \left\{ \frac{|S|}{c(G-S)} \right\}$  by Chvátal, in which the minimum is taken over all proper sets  $S \subset V(G)$ , where  $c(G-S)$  denotes the number of components of  $G-S$  and  $c(G-S) \geq 2$ . A graph  $G$  is called  $t$ -*tough* if  $t(G) \geq t$ . Incorporating the toughness and signless Laplacian eigenvalues of a regular graph, we provide a sufficient spectral radius condition for a connected regular graph to be  $\frac{1}{b}$ -tough, where  $b \geq 1$  is an integer. Particularly, we obtain the second largest eigenvalue condition for regular graphs to be  $\frac{1}{b}$ -tough. Moreover, we give a sufficient condition for a connected graph to be  $\frac{1}{b}$ -tough with fixed minimum degree  $\delta$  in terms of the signless Laplacian spectral radius and spectral radius, which extends a significant result by Fan, Lin and Lu [*Eur. J. Comb.* 110:Paper No. 103701, 2023].

**Key words.** Toughness; Eigenvalues; Regular graph.

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**1. Introduction.** All graphs considered in this paper are finite, undirected, and simple connected graphs. Let  $G$  be a graph with vertex set  $V(G) = \{v_1, \dots, v_n\}$  and edge set  $E(G) = \{e_1, \dots, e_m\}$ , and let  $m(G) = |E(G)|$  be the number of edges in  $G$ . For any  $v \in V(G)$ , let  $N_G(v)$  ( $N(v)$ ) and  $d_G(v)$  ( $d(v)$ ) denote the neighborhood and degree of  $v$ , respectively. The *adjacency matrix* of  $G$  is defined as the matrix  $A(G) = (a_{u,v})_{u,v \in V(G)}$  with  $a_{u,v} = 1$  if  $u$  and  $v$  are adjacent, and  $a_{u,v} = 0$  otherwise. The *signless Laplacian matrix* of  $G$  is defined as  $Q(G) = D(G) + A(G)$ , where  $D(G)$  is the diagonal matrix of vertex degrees of  $G$ . The largest eigenvalues of  $A(G)$  and  $Q(G)$ , denoted by  $\rho(G)$  and  $q(G)$ , are called the *spectral radius* and *signless Laplacian spectral radius* of  $G$ , respectively. We denote by  $\rho(G) = \lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$  and  $q(G) = q_1(G) \geq q_2(G) \geq \dots \geq q_n(G)$  the eigenvalues of its adjacency matrix and signless Laplacian matrix, respectively.

Toughness is an important invariant in graphs, and there has been extensive work on toughness since Chvátal [9] introduced the concept in 1973. For a non-complete graph  $G$ , the *toughness* of  $G$  is defined as  $t(G) = \min \left\{ \frac{|S|}{c(G-S)} \right\}$ , in which the minimum is taken over all proper sets  $S \subset G$ , where  $c(G-S)$  denotes the number of components of  $G-S$  and  $c(G-S) \geq 2$ . The toughness of a graph is related to many other important properties such as hamiltonicity [23, 29], pancyclicity [3, 16], and the existence of various factors [5, 8, 13, 15, 17, 18, 27, 30], cycles [1, 21], and spanning trees [2, 25]. The relationship between Laplacian eigenvalues and toughness has also been widely examined [10, 11, 25, 28]. For a comprehensive survey on toughness, we refer the reader to [4].

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There are many studies on the relationship between toughness and eigenvalues. It was initiated by Alon [1] who showed that for any connected  $d$ -regular graph  $G$ ,  $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+1} - 1)$ , where  $\lambda = \max\{\lambda_2(G), |\lambda_n(G)|\}$ . Moreover, Brouwer [6] independently proved  $t(G) > \frac{d}{\lambda} - 2$ , and he further conjectured in [26] that  $t(G) > \frac{d}{\lambda} - 1$ . Then, it was completely proved by Gu [24] recently. From Gu's result, to guarantee 1-toughness of a  $d$ -regular graph  $G$ , it is necessary that  $\lambda \leq \frac{d}{2}$ . This indicates  $\lambda_2(G) \leq \frac{d}{2}$  (it is trivial for complete graphs whose toughness is  $\infty$ ). In [28], Liu and Chen improved the eigenvalue condition for 1-tough  $d$ -regular graphs. More exciting results can be found in Cioabă and Wong [11], Cioabă and Gu [10].

In this paper, we first focus on the relationship between toughness of a regular graph and its signless Laplacian eigenvalues. We determine a signless Laplacian eigenvalue condition for a  $d$ -regular graph to be  $\frac{1}{b}$ -tough where  $b$  is a positive integer and  $d \geq 2(b+1)$ .

**THEOREM 1.1.** *Let  $b \geq 1$  be an integers and  $d \geq 2(b+1)$ . If  $G$  is a connected  $d$ -regular graph and*

$$q_{b+1}(G) < \begin{cases} \frac{3(d-1)+\sqrt{d^2-2d+8b+9}}{2} & \text{when } d, b \text{ have the same parity,} \\ \frac{3(d-1)+\sqrt{d^2-2d+8b+17}}{2} & \text{when } d, b \text{ have different parity,} \end{cases}$$

then  $t(G) \geq \frac{1}{b}$ .

Moreover, we provide a sufficient condition in terms of  $q_2(G)$  for a  $d$ -regular graph  $G$  to be  $\frac{1}{b}$ -tough ( $b \geq 2$  and  $d \leq 2(b-1)$ ).

**THEOREM 1.2.** *Let  $G$  be a connected  $d$ -regular graph with integer  $d \leq 2(b-1)$  and  $b \geq 2$ , if*

$$q_2(G) < \begin{cases} \frac{3(d-1)+\sqrt{d^2-8b+6d+17}}{2} & \text{when } d \text{ is odd,} \\ \frac{3(d-1)+\sqrt{d^2-8b+6d+9}}{2} & \text{when } d \text{ is even,} \end{cases}$$

then  $t(G) \geq \frac{1}{b}$ .

Fan, Lin, and Lu [19] provided spectral conditions for a graph with minimum degree to be 1-tough and to be  $t$ -tough, respectively. Very recently, Chen, Fan, and Lin [14] provided spectral conditions for a graph to be  $t$ -tough. In this paper, we improve some toughness results of [19] in terms of spectral radius. Moreover, a signless Laplacian spectral radius condition for a graph with minimum degree  $\delta$  to be  $\frac{1}{b}$ -tough is also considered.

If  $G$  and  $H$  are two vertex disjoint graphs, the *join*  $G \nabla H$  is the graph obtained by taking the union of  $G$  and  $H$  and adding all the edges between the vertex set of  $G$  and the vertex set of  $H$ .

The *union*  $G_1 \cup G_2$  is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ . The *complement* of a graph  $G$  is denoted by  $\overline{G}$ . Let  $\delta(G)$  be the minimum degree of  $G$ .

**THEOREM 1.3.** *Let  $b \geq 1$  be a positive integer. Suppose that  $G$  is a connected graph of order  $n$  with minimum degree  $\delta(G) = \delta$ . Then, we have the following statements.*

- (i) *If  $n \geq \max\{5\delta, b\delta^3 + \delta\}$  and  $\rho(G) \geq \rho(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1))$ , then  $G$  is a  $\frac{1}{b}$ -tough graph, unless  $G \cong K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)$ .*
- (ii) *If  $n \geq \max\{\lceil \frac{22\delta}{3} \rceil, 2(b\delta + \delta + 1), b\delta^3 + \delta\}$  and  $q(G) \geq q(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1))$ , then  $G$  is a  $\frac{1}{b}$ -tough graph, unless  $G \cong K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)$ .*

In order to prove our results, we need some tools. Let  $A$  be a real symmetric matrix of order  $n$  and let  $X = \{1, 2, \dots, n\}$ . For any partition  $\Pi$  of  $X : X_1 \cup X_2 \cup \dots \cup X_m$ , the matrix  $A$  can be partitioned correspondingly.

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,k} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k,1} & A_{k,2} & \cdots & A_{k,k} \end{pmatrix}.$$

The *characteristic matrix* of  $\Pi$  is the  $n \times m$  matrix  $\chi_\Pi$  whose columns are the characteristic vectors of  $X_1, \dots, X_m$ , and *quotient matrix* of  $A$  with respect to  $\Pi$  is the matrix  $B_\Pi = (b_{i,j})_{m \times m}$  with  $b_{i,j} = \frac{1}{|X_i|} e_{|X_i|}^T A_{i,j} e_{|X_j|}$ , where  $e_{|X_i|}$  and  $e_{|X_j|}$  are the all ones  $|X_i|$ - and  $|X_j|$ -vectors, respectively. In particular, the partition  $\Pi$  is called *equitable* if each block  $A_{i,j}$  has constant row sum.

LEMMA 1.4. (Brouwer and Haemers [7, p. 30]; Godsil and Royle [22, pp. 196–198]) *Let  $M$  be a real symmetric matrix and let  $\lambda_1(M)$  be the largest eigenvalue of  $M$ . If  $B_\Pi$  is a quotient matrix of  $M$ , then the eigenvalues of  $B_\Pi$  interlace those of  $M$ . Furthermore, if the partition  $\Pi$  is equitable, then all eigenvalues of  $B_\Pi$  are also eigenvalues of  $M$ ; and for any eigenvalue of  $B_\Pi$ , the corresponding eigenvector has constant element on each vertex in a common part.*

**2. Proof of Theorems 1.1 and 1.2.** We first give some lemmas that will be used in our proof.

LEMMA 2.1. (see [12]) *Let  $G$  be a graph of order  $n$  and  $v \in V(G)$ . Then, for  $i = 1, \dots, n - 1$ ,*

$$q_{i+1}(G) - 1 \leq q_i(G - v) \leq q_i(G).$$

where the right inequality holds if and only if  $v$  is an isolated vertex.

LEMMA 2.2. *Let  $G$  be a connected graph with  $n$  vertices and  $m$  edges. Then,*

$$q(G) \geq \frac{4m}{n},$$

and the equality holds if and only if  $G$  is a regular graph.

If  $t$  is a positive even integer, denote by  $M_t$  the union of  $\frac{t}{2}$  disjoint edges. Let  $d, b$  be two integers and  $d \geq 2(b + 1), b \geq 1$ . Denote by  $\mathbb{G}_{d,b}^1$  the family of all connected irregular graphs with maximum degree  $d$ , size  $m$  with  $2m \geq dn - d + b$  that have at least  $b + 1$  vertices of degree  $d$  when  $d, b$  have the same parity and at least  $b + 2$  vertices of degree  $d$  when  $d, b$  have different parity. Let  $G_{d,b} = \overline{M_{d-b}} \nabla K_{b+1}$ , when  $d, b$  have the same parity, and  $G_{d,b} = \overline{M_{d-b-1}} \nabla K_{b+2}$ , when  $d, b$  have different parity. Obviously,  $G_{d,b} \in \mathbb{G}_{d,b}^1$ .

LEMMA 2.3. *Let  $d \geq 2(b + 1)$  be an integer. If  $G \in \mathbb{G}_{d,b}^1$ , then we have*

$$q(G) \geq q(G_{d,b}) = \begin{cases} \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2} & \text{if } d, b \text{ have the same parity,} \\ \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 17}}{2} & \text{if } d, b \text{ have different parity,} \end{cases}$$

with equality if and only if  $G \cong G_{d,b}$ .

*Proof.* Let  $d, b$  have the same parity. Then, the partition of the vertex set of  $\overline{M_{d-b}} \nabla K_{b+1}$  into the  $b + 1$  vertices of degree  $d$  and the  $d - b$  vertices of degree  $d - 1$  is an equitable partition whose quotient matrix is  $\begin{bmatrix} d + b & d - b \\ b + 1 & 2d - b - 3 \end{bmatrix}$ . By a simple calculation, its largest eigenvalue is  $\frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2}$ . Then, we have  $q(\overline{M_{d-b}} \nabla K_{b+1}) = \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2}$  by Lemma 1.4.

Let  $G \in \mathbb{G}_{d,b}^1$ . If  $n \geq d + 2$ , then by Lemma 2.2, after some straightforward calculations, we obtain that

$$\begin{aligned} q(G) &\geq \frac{4m}{n} \geq 2 \left( d - \frac{d-b}{n} \right) \geq 2 \left( d - \frac{d-b}{d+2} \right) \\ &\geq \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2} \quad (\text{since } d \geq 2(b+1)) \\ &= q(G_{d,b}). \end{aligned}$$

If  $n = d + 1$ , then  $2m \geq dn - d + b = d^2 + b$ . When  $2m \geq d^2 + b + 2$ , we can obtain that

$$q(G) \geq \frac{4m}{n} \geq 2 \left( \frac{d^2 + b + 2}{d + 1} \right) > \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2} = q(G_{d,b}).$$

The only remaining case is  $n = d + 1$  and  $2m = d^2 + b$ . Let  $V_1$  be a subset of  $b + 1$  vertices of degree  $d$  and let  $V_2$  be the complement of  $V_1$ . The quotient matrix of the partition of the vertex set into  $V_1$  and  $V_2$  is

$$B = \begin{bmatrix} d+b & d-b \\ b+1 & 2d-b-3 \end{bmatrix}.$$

By Lemma 2.1, we have  $q(G) \geq \lambda_1(B) = \frac{3(d-1) + \sqrt{d^2 - 2d + 8b + 9}}{2} = q(G_{d,b})$ , with equality if and only if the partition  $V_1 \cup V_2$  is equitable. This happens if and only if  $G \cong G_{d,b}$ . If  $d, b$  have different parity, the proof is similar. This completes the proof.  $\square$

Let  $\mathbb{G}_{d,b}^2$  with  $d \geq b + 1 \geq 3$  be the family of graphs with maximum degree  $d$ , size  $m$  with  $2m \geq dn - b$  that have at least  $d + 1 - b$  vertices of degree  $d$  when  $b$  is even, and at least  $d + 2 - b$  vertices of degree  $d$  when  $b$  is odd. Let  $G'_{d,b} = \overline{M}_b \nabla K_{d+1-b}$ , when  $b$  is even, and  $G'_{d,b} = \overline{M}_{b-1} \nabla K_{d+2-b}$ , when  $b$  is odd. Obviously,  $G'_{d,b} \in \mathbb{G}_{d,b}^2$ . Similar to the proof of Lemma 2.3, we have following result.

LEMMA 2.4. *Let  $d \leq 2(b-1)$  and  $G \in \mathbb{G}_{d,b}^2$ , then*

$$q(G) \geq q(G'_{d,b}) \geq \begin{cases} \frac{3(d-1) + \sqrt{d^2 - 8b + 6d + 9}}{2} & \text{if } b \text{ is even,} \\ \frac{3(d-1) + \sqrt{d^2 - 8b + 6d + 17}}{2} & \text{if } b \text{ is odd,} \end{cases}$$

with equality if and only if  $G \cong G'_{d,b}$ .

Now, we present the proof of Theorem 1.1.

*Proof.* We suppose to the contrary that  $G$  is not a  $\frac{1}{b}$ -tough graph. Then, there exists some nonempty subset  $S$  of  $V(G)$  such that  $|S| < \frac{1}{b}c(G - S)$ . Choose  $|S|$  to be as small as possible. Let  $|S| = s$  and  $c(G - S) = c$ , then  $c \geq bs + 1$ . We denote by  $G_1, G_2, \dots, G_c$  the  $c$  components of  $G - S$ . For  $1 \leq i \leq c$ , let  $e_i$  be the number of edges in  $G$  between  $S$  and  $G_i$ . Since  $G$  is connected, it is clear that  $e_i \geq 1$  and  $\sum_{i=1}^c e_i \leq ds$ . We have the following claims.

CLAIM 1.  $d \geq b + 1$ .

*Proof.* Each vertex in  $S$  is connected to at least one component in  $\{G_1, G_2, \dots, G_c\}$ . Otherwise, there exists a vertex  $v$  of  $S$  whose neighbors lie in  $S$ . Let  $S' = S - \{v\}$ , then we have  $c(G - S') = c + 1$ , which contradicts the minimality of  $S$ . Since  $G$  is  $d$ -regular,  $c \geq bs + 1$  and  $s \geq 1$ . There exists a vertex in  $S$ , say  $u$ , satisfies

$$d \geq d_{G-S}(u) \geq \frac{c}{s} \geq \frac{bs + 1}{s} = b + \frac{1}{s}.$$

Thus, the claim holds.  $\square$

Without loss of generality, the number of edges  $e_i$  of  $S$  and  $G_i$  can be sorted in ascending order.

CLAIM 2.  $e_i \leq d - b$ , for  $i = 1, \dots, b + 1$ .

*Proof.* Otherwise, suppose that  $e_i \geq d - b + 1$  for  $i = b + 1, \dots, c$ . Thus, we have

$$\begin{aligned} ds &\geq \sum_{i=1}^c e_i \geq b + (d - b + 1)(c - b) \\ &\geq b + (d - b + 1)(bs + 1) - (d - b + 1)b \quad (\text{since } c \geq bs + 1) \\ &= b + ds + (b - 1)((s - 1)(d - b) - 1) \\ &> ds \quad (\text{since } b \geq 1, s \geq 1, d \geq b + 1), \end{aligned}$$

a contradiction. Thus, the claim holds. □

Suppose that  $q(G_1) \geq q(G_2) \geq \dots \geq q(G_{b+1}) \geq \dots \geq q(G_c)$ . By Lemma 2.1, we have

$$(2.1) \quad q_{b+1}(G) \geq q_{b+1}(G - v) \geq q_{b+1}(G - S) = q(G_{b+1}).$$

Let  $|V(G_{b+1})| = n_{b+1}$  and  $|E(G_{b+1})| = m_{b+1}$ . We have  $n_{b+1} \geq d + 1$ . Otherwise, if  $n_{b+1} \leq d$ , then

$$(2.2) \quad \begin{aligned} dn_{b+1} &= e_{b+1} + 2m_{b+1} \\ &\leq e_{b+1} + n_{b+1}(n_{b+1} - 1) \\ &\leq e_{b+1} + d(n_{b+1} - 1) \quad (\text{since } n_{b+1} \leq d). \end{aligned}$$

This implies  $e_{b+1} \geq d$ , contrary to  $e_{b+1} \leq d - b$ . Thus,  $G_{b+1}$  contains at least  $b + 1$  vertices of degree  $d$ .

If  $d$  and  $b$  have the same parity, then  $d \geq b + 2$ , implying  $G_{b+1} \in \mathbb{G}_{d,b}^1$ . By (2.1) and Lemma 2.3, we have

$$q_{b+1}(G) \geq q(G_{b+1}) \geq \frac{3(d - 1) + \sqrt{d^2 - 2d + 8b + 9}}{2}.$$

If  $d$  is even and  $b$  is odd, then  $e_{b+1}$  must be even by (2.2) and so  $2 \leq e_{b+1} \leq d - b - 1$  by Claim 2. If  $d$  is odd,  $b$  is even, and  $dn_{b+1}$  is even, we know  $e_{b+1}$  must be even by (2.2) and so  $2 \leq e_{b+1} \leq d - b - 1$ . Each of these cases implies  $G_{b+1}$  contains at least  $n_{b+1} - e_{b+1} \geq b + 2$  vertices of degree  $d$ . So  $G_{b+1} \in \mathbb{G}_{d,b}^1$ . By (2.1) and Lemma 2.3, we have

$$q_{b+1}(G) \geq q(G_{b+1}) \geq \frac{3(d - 1) + \sqrt{d^2 - 2d + 8b + 17}}{2}.$$

If  $d$  is odd,  $b$  is even, and  $dn_{b+1}$  is odd. We know  $n_{b+1}$  is odd. So  $n_{b+1} \geq d + 2$  and  $G_{b+1}$  contains at least  $n_{b+1} - e_{b+1} \geq b + 2$  vertices of degree  $d$ . Then,  $G_{b+1} \in \mathbb{G}_{d,b}^1$ . By (2.1) and Lemma 2.3, we obtain

$$q_{b+1}(G) \geq q(G_{b+1}) \geq \frac{3(d - 1) + \sqrt{d^2 - 2d + 8b + 17}}{2}.$$

This completes the proof. □

Next, we present the proof of Theorem 1.2.

*Proof.* Suppose that  $G$  is not a  $\frac{1}{b}$ -tough graph. There exists some nonempty subset  $S$  of  $V(G)$  such that  $|S| < \frac{1}{b}c(G - S)$ . We choose  $|S|$  to be as small as possible. Let  $|S| = s$  and  $c(G - S) = c$ , then  $c \geq bs + 1$ . Write the  $c$  components of  $G - S$  as  $G_1, G_2, \dots, G_c$ . Let  $e_i$  be the number of edges in  $G$  between  $S$  and  $G_i$ . Since  $G$  is connected, it is clear that  $e_i \geq 1$  and  $\sum_{i=1}^c e_i \leq ds$ . Similar to the proof of Claim 1 in Theorem 1.1 above, we have  $d \geq b + 1$ . Now, we claim that  $e_i \leq b$  for at least 2 candidates of  $i$ . Otherwise, let  $e_i \geq b + 1$  for  $i = 2, 3, \dots, c$ . Then, we have

$$\begin{aligned} ds &\geq \sum_{i=1}^c e_i \geq 1 + (b + 1)(c - 1) \\ &\geq 1 + bs(b + 1) \quad (\text{since } c \geq bs + 1) \\ &= 1 + s(b^2 + b) \\ &> s(b^2 + b) \\ &\geq ds \quad (\text{since } d \leq 2(b - 1) \leq b^2 + b), \end{aligned}$$

a contradiction. Without loss of generality, we assume that  $e_i \leq b, i = 1, 2$  and  $q(G_1) \geq q(G_2)$ . By Lemma 2.1, we obtain

$$(2.3) \quad q_2(G) \geq q_2(G - v) \geq q_2(G - S) = q(G_2).$$

Let  $|V(G_2)| = n_2$  and  $|E(G_2)| = m_2$ . We have  $n_2 \geq d + 1$ . Otherwise,  $n_2 \leq d$ , then

$$(2.4) \quad \begin{aligned} dn_2 &= e_2 + 2m_2 \\ &\leq e_2 + n_2(n_2 - 1) \\ &\leq e_2 + d(n_2 - 1) \quad (\text{since } n_2 \leq d). \end{aligned}$$

This implies  $e_2 \geq d \geq b + 1 > b$ , contrary to  $e_2 \leq b$ . Thus,  $G_2$  contains at least  $d + 1 - b$  vertices of degree  $d$ . If  $b$  is even, then by (2.3) and Lemma 2.4, we know

$$q_2(G) \geq q(G_2) \geq \frac{3d - 3 + \sqrt{d^2 - 8b + 6d + 9}}{2}.$$

Now let  $b$  be odd. If  $d$  is even, then  $e_2$  must be even by (2.4) and so  $e_2 \leq b - 1$ . Thus,  $G_2$  contains at least  $n_2 - e_2 \geq d + 2 - b$  vertices of degree  $d$ . This implies  $G_2 \in \mathbb{G}_{d,b}^2$ . If  $d$  is odd and  $dn_2$  is even, we know  $e_2$  is even by (2.4) and so  $e_2 \leq b - 1$ . Similarly, this gives  $G_2 \in \mathbb{G}_{d,b}^2$ . If  $d$  is odd and  $dn_2$  is odd, we know  $n_2$  is odd and so  $n_2 \geq d + 2$ . This indicates  $G_2$  contains at least  $n_2 - e_2 \geq d + 2 - b$  vertices of degree  $d$ . Note that  $2m_2 \geq dn_2 - b$ . So  $G_2 \in \mathbb{G}_{d,b}^2$ . By (2.3) and Lemma 2.4, we know each of these cases has

$$q_2(G) \geq q(G_2) \geq \frac{3d - 3 + \sqrt{d^2 - 8b + 6d + 17}}{2}.$$

This completes the proof. □

**3. Proof of Theorem 1.3.** In order to prove the Theorem 1.3 for  $\rho(G)$  and  $q(G)$  simultaneously, we introduce a new matrix  $A_a(G) = aD(G) + A(G)$  and denote by  $\rho_a(G)$  the largest eigenvalue of  $A_a(G)$ , where  $a \geq 0$ . Clearly,  $A_0(G) = A(G)$  (resp.  $A_1(G) = Q(G)$ ) and  $\rho_0(G) = \rho(G)$  (resp.  $\rho_1(G) = q(G)$ ).

By the well-known Perron-Frobenius theorem ([22, Section 8.8]), we can easily deduce the following result.

LEMMA 3.1. (see [20]) Let  $a \geq 0$ . If  $H$  is a spanning subgraph of a connected graph  $G$ , then  $\rho_a(H) \leq \rho_a(G)$ , with equality if and only if  $H \cong G$ .

LEMMA 3.2. (see [20]) Let  $a \geq 0$  and  $n = \sum_{i=1}^t n_i + s$ . If  $n_1 \geq n_2 \geq \dots \geq n_t \geq p$  and  $n_1 < n - s - p(t-1)$ , then  $\rho_a(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_t})) < \rho_a(K_s \nabla (K_{n-s-p(t-1)} \cup (t-1)K_p))$ .

Now, we present the proof of Theorem 1.3.

*Proof.* Suppose that  $G$  is not a  $\frac{1}{b}$ -tough graph, we shall prove that  $\rho_a(G) \leq \rho_a(K_s \nabla (K_{n-(b+1)\delta} \cup b\delta K_1))$ . Note that  $t(G) < \frac{1}{b}$ , there exists some nonempty subset  $S$  of  $V(G)$  such that  $|S| < \frac{1}{b}c(G-S)$ . Take  $|S| = s$  and  $c(G-S) = c$ . Notice that  $b \geq 1$  is an integer. Hence,  $c \geq bs + 1$ . One can verify that  $G$  is a spanning subgraph of  $G_1 = K_s \nabla (K_{c_1} \cup K_{c_2} \cup \dots \cup K_{c_l})$  for some integers  $c_1 \geq c_2 \geq \dots \geq c_l$  with  $l = bs + 1$  and  $\sum_{i=1}^l c_i = n - s$ . Let  $a \in \{0, 1\}$ . By Lemma 3.1, we have

$$(3.5) \quad \rho_a(G) \leq \rho_a(G_1),$$

with equality if and only if  $G \cong G_1$ . Then, we shall divide the proof into the following two cases.

CASE 1.  $s \geq \delta + 1$ .

Let  $G_2 = K_s \nabla (K_{n-(b+1)s} \cup bsK_1)$ . Note that  $s \geq \delta + 1$ . Then, by Lemma 3.2, we have

$$(3.6) \quad \rho_a(G_1) \leq \rho_a(G_2),$$

with equality if and only if  $(c_1, \dots, c_l) = (n - (b + 1)s, 1, \dots, 1)$ . Notice that  $A_a(G_2)$  has the equitable quotient matrix

$$A_{\Pi}^a(s) = \begin{bmatrix} a(n-1) + s - 1 & bs & n - (b+1)s \\ s & as & 0 \\ s & 0 & a(n-bs-1) + n - (b+1)s - 1 \end{bmatrix}.$$

(i) Let  $a = 0$ . Then,  $A_{\Pi}^0(s)$  is the equitable quotient matrix of  $A(G_2)$ . By a simple calculation, the characteristic polynomial of  $A_{\Pi}^0(s)$  is

$$\phi(A_{\Pi}^0(s), x) = x^3 - (n - bs - 2)x^2 - (n + bs^2 - bs - 1)x - bs^2(bs - n + s + 1).$$

Note that  $A_0(K_s \nabla (K_{n-(b+1)\delta} \cup b\delta K_1))$  has the equitable quotient matrix  $A_{\Pi}^0(\delta)$ , which is obtained by replacing  $s$  with  $\delta$  in  $A_{\Pi}^0(s)$ . Then,

$$(3.7) \quad \phi(A_{\Pi}^0(s), x) - \phi(A_{\Pi}^0(\delta), x) = (s - \delta)\varphi(x),$$

where

$$\varphi(x) = bx^2 - (\delta + s - 1)bx - (\delta^2 + \delta s + s^2)b^2 - (\delta^2 - \delta n + \delta s - sn + s^2 + \delta + s)b.$$

In what follows, we shall prove that  $\varphi(x) > 0$ . Since  $n \geq (b + 1)s + 1$ ,  $s \geq \delta + 1$  and  $b \geq 1$ , the symmetry axis of parabola  $\varphi(x)$  is

$$x = \frac{\delta + s - 1}{2} < n - b\delta - 1.$$

This implies that  $\varphi(x)$  is increasing with respect to  $x \geq n - b\delta - 1$ . Since  $n \geq (b + 1)s + 1$ , we have  $s \leq \frac{n-1}{b+1}$ . Hence,

$$\begin{aligned}
 \varphi(x) &\geq \varphi(n - b\delta - 1) \\
 (3.8) \quad &= -(b^2 + b)s^2 - \delta bs + b((b^2 - 1)\delta^2 - (2n - 1)b\delta + n^2 - n) \\
 &\geq \frac{1}{b + 1}g(n) \quad (\text{since } s \leq \frac{n-1}{b+1}),
 \end{aligned}$$

where

$$g(n) = b^2n^2 - (2b^2\delta + 2b\delta + b + \delta - 1)bn + b(b^3\delta^2 + b^2\delta^2 + b^2\delta - b\delta^2 + \delta b - \delta^2 + \delta - 1).$$

If  $b = 1$ , since  $n \geq 5\delta$  and  $\delta \geq 1$ , we have  $g(n) = n(n - 5\delta) + 3\delta - 1 > 0$ . Combining this with (3.8), we have  $\varphi(x) > 0$ . If  $b \geq 2$  and  $\delta = 1$ , then  $g(n) = b^2n^2 - (2b^3 + 3b^2)n + b(b^3 + 2b^2 - 1)$ . Observe that  $s \geq \delta + 1 = 2$  and  $n \geq (b + 1)s + 1 \geq 2b + 3$ . The symmetry axis of parabola  $g(n)$  is

$$\frac{2b^3 + 3b^2}{2b^2} = b + \frac{3}{2} < 2b + 3.$$

Then,  $g(n)$  is increasing with respect to  $n \geq 2b + 3$ , and hence  $g(n) \geq g(2b + 3) = b(b + 1)(b^2 + b - 1) > 0$ . Thus, we have  $\varphi(x) > 0$  by (3.8). If  $b \geq 2$  and  $\delta \geq 2$ , since the symmetry axis of parabola  $g(n)$  is

$$b\delta + \delta + \frac{1}{2} + \frac{\delta}{2b} - \frac{1}{2b} < b\delta^3 + \delta,$$

we conclude that  $g(n)$  is increasing with respect to  $n \geq b\delta^3 + \delta$ . For  $n \geq b\delta^3 + \delta$ , we get

$$\begin{aligned}
 g(n) &\geq g(b\delta^3 + \delta) \\
 &= \delta^3b^2(\delta^3b^2 - 2\delta b^2 - \delta - b) + ((b^2 - b + \delta - 2)\delta^2 + \delta b)b^2 - (2\delta^2 - 2\delta + 1)b \\
 &\geq \delta^3b^2(2\delta b^2 - \delta - b) - (2\delta^2 - 2\delta + 1)b \\
 &= \delta^3b^2(2(\delta - 2)(b - 2) + 2\delta b(b - 1) + 3\delta + 3b - 8) - (2\delta^2 - 2\delta + 1)b \\
 &\geq b(2\delta^4b^2 - 2\delta^2 + 2\delta - 1) \\
 &> 0,
 \end{aligned}$$

where all the inequalities follows from the fact that  $\delta \geq 2$  and  $b \geq 2$ . Thus, we also have  $\varphi(x) > 0$  by (3.8). Combining this with (3.7), we have  $\phi(A_{\Pi}^0(s), x) > \phi(A_{\Pi}^0(\delta), x)$  for  $x > n - b\delta - 1$ . Since  $K_{n-b\delta}$  is a proper subgraph of  $K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)$ , we have

$$\rho(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)) > \rho(K_{n-b\delta}) = n - b\delta - 1,$$

and hence  $\lambda_1(A_{\Pi}^0(\delta)) > \lambda_1(A_{\Pi}^0(s))$ . Combining this with Lemma 1.4, (3.5) and (3.6), we have

$$\rho(G) \leq \rho(G_1) \leq \rho(G_2) < \rho(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)).$$

(ii) Let  $a = 1$ . Then,  $A_{\Pi}^1(s)$  is the equitable quotient matrix of  $Q(G_2)$ . And the characteristic polynomial of  $A_{\Pi}^1(s)$  is

$$\begin{aligned}
 \psi(A_{\Pi}^1(s), x) &= x^3 + (2bs - 3n - s + 4)x^2 - (2bns + 4bs^2 - 4bs - 2n^2 - 3ns + 6n + 4s - 4)x \\
 &\quad - 2s(bs - n + 1)(bs - n + 2).
 \end{aligned}$$

Also, since  $A_1(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1))$  has the equitable quotient matrix  $A_{\Pi}^1(\delta)$ , which is obtained by replacing  $s$  with  $\delta$  in  $A_{\Pi}^1(s)$ . Then,

$$(3.9) \quad \psi(A_{\Pi}^1(s), x) - \psi(A_{\Pi}^1(\delta), x) = (s - \delta)\gamma(x),$$

where

$$\begin{aligned} \gamma(x) &= (2b - 1)x^2 - ((4\delta + 2n + 4s - 4)b - 3n + 4)x \\ &\quad - ((2\delta^2 + 2\delta s + 2s^2)b^2 - (4\delta n + 4sn - 6\delta - 6s)b + 2n^2 - 6n + 4). \end{aligned}$$

In what follows, we shall prove that  $\gamma(x) > 0$ . The symmetry axis of parabola  $\gamma(x)$  is

$$x = \frac{(4\delta + 2n + 4s - 4)b - 3n + 4}{2(2b - 1)} < 2(n - b\delta - 1),$$

due to  $n \geq 2(b\delta + \delta + 1)$ ,  $\delta + 1 \leq s \leq \frac{n-1}{b+1}$  and  $b \geq 1$ . This implies that  $\gamma(x)$  is increasing with respect to  $x \geq 2(n - b\delta - 1)$ . Since  $s \leq \frac{n-1}{b+1}$ . Hence,

$$(3.10) \quad \begin{aligned} \gamma(x) &\geq \gamma(2(n - b\delta - 1)) \\ &= -2b^2s^2 - 2b(2n - 3b\delta - 1)s + 2b(b\delta - n + 1)(4b\delta + \delta - 2n) \\ &\geq \frac{2b}{(b + 1)^2}h(n) \quad (\text{since } s \leq \frac{n-1}{b+1}), \end{aligned}$$

where

$$\begin{aligned} h(n) &= (2b^2 + b)n^2 - (6\delta b^3 + (10\delta + 2)b^2 + (5\delta - 1)b + \delta - 1)n + 4\delta^2b^4 \\ &\quad + (9\delta^2 + 4\delta)b^3 + (6\delta^2 + 6\delta)b^2 + (\delta^2 + 3\delta - 2)b + \delta - 1. \end{aligned}$$

If  $b = 1$ , since  $n \geq \lceil \frac{22\delta}{3} \rceil$  and  $\delta \geq 1$ , we have  $h(n) = 3n^2 - 22\delta n + 20\delta^2 + 14\delta - 3 > 0$ . Combining this with (3.10), we have  $\gamma(x) > 0$ . If  $b \geq 2$  and  $\delta = 1$ , then  $h(n) = (2b^2 + b)n^2 - (6b^3 + 12b^2 + 4b)n + 4b^4 + 13b^3 + 12b^2 + 2b$ . Observe that  $n \geq 2(b\delta + \delta + 1) \geq 2b + 4$ . The symmetry axis of parabola  $h(n)$  is

$$\frac{6b^3 + 12b^2 + 4b}{2(2b^2 + b)} < 2b + 4.$$

Then,  $h(n)$  is increasing with respect to  $n \geq 2b + 4$ , and hence  $h(n) \geq h(2b + 4) = b(b^2 + 4b + 2) > 0$ . Thus, we have  $\gamma(x) > 0$  by (3.10). If  $b \geq 2$  and  $\delta \geq 2$ , since the symmetry axis of parabola  $h(n)$  is

$$\frac{6\delta b^3 + 10\delta b^2 + 2b^2 + 5\delta b - b + \delta - 1}{2(2b^2 + b)} < b\delta^3 + \delta,$$

we conclude that  $h(n)$  is increasing with respect to  $n \geq b\delta^3 + \delta$ . For  $n \geq b\delta^3 + \delta$ , if  $\delta = 2$ , we obtain  $h(n) \geq h(b\delta^3 + \delta) = h(8b + 2) = 48b^4 - 28b^3 - 40b^2 - 14b - 1 > 0$ , and so the result holds. If  $\delta \geq 3$ , we get

$$\begin{aligned} h(n) &\geq h(b\delta^3 + \delta) \\ &= \delta^3b^2(2\delta^3b^2 - 6\delta b^2 + \delta^3b - 6\delta b - 2b - 3\delta) + (4\delta^2b^2 + 3\delta^2b + 4\delta b + \delta^3 - 2\delta^2 + 4\delta)b^2 \\ &\quad - (\delta^4 - \delta^3 + 3\delta^2 - 4\delta + 2)b - \delta^2 + 2\delta - 1 \\ &\geq \delta^3b^2(12\delta b^2 - 3\delta - 2b) - (\delta^4 - \delta^3 + 3\delta^2 - 4\delta + 2)b \\ &= \delta^3b^2(12(\delta - 2)(b - 2) + 12\delta b(b - 1) + 21\delta + 22b - 48) - (\delta^4 - \delta^3 + 3\delta^2 - 4\delta + 2)b \\ &\geq 12\delta^4b^3 - (\delta^4 - \delta^3 + 3\delta^2 - 4\delta + 2)b \\ &= b(12\delta^4b^2 - \delta^4 + \delta^3 - 3\delta^2 + 4\delta - 2) \\ &> 0, \end{aligned}$$

where all the inequalities follows from the fact that  $\delta \geq 3$  and  $b \geq 2$ . Thus, we also have  $\gamma(x) > 0$  by (3.10). Combining this with (3.9), we have  $\psi(A_{\Pi}^1(s), x) > \psi(A_{\Pi}^1(\delta), x)$  for  $x > 2(n - b\delta - 1)$ . Since  $K_{n-b\delta}$  is a proper subgraph of  $K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)$ , we have

$$q(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)) > q(K_{n-b\delta}) = 2(n - b\delta - 1),$$

and hence  $\lambda_1(A_{\Pi}^1(\delta)) > \lambda_1(A_{\Pi}^1(s))$ . Combining this with Lemma 1.4, (3.5) and (3.6), we have

$$q(G) \leq q(G_1) \leq q(G_2) < q(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)).$$

CASE 2.  $s \leq \delta$ .

If  $s = \delta$ , then by Lemma 3.2, we have

$$\rho_a(G_1) \leq \rho_a(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)),$$

with equality if and only if  $G_1 \cong K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)$ . Combining this with (3.5), we deduce that

$$\rho_a(G) \leq \rho_a(K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)),$$

with equality if and only if  $G \cong K_{\delta}\nabla(K_{n-(b+1)\delta} \cup b\delta K_1)$ , and so the result holds.

If  $s \leq \delta - 1$ , let  $G_3 = K_s\nabla(K_{n-s-(\delta-s+1)bs} \cup bsK_{\delta-s+1})$ . Recall that  $c_i \geq c_{i+1}$  for  $1 \leq i \leq l - 1$ . Note that  $d_G(u) \geq \delta$  for any  $u \in V(K_{c_l})$ . So  $\delta \leq s + c_l - 1$ . That is,  $c_l \geq \delta - s + 1$ . By Lemma 3.2, we have

$$(3.11) \quad \rho_a(G_1) \leq \rho_a(G_3),$$

with equality if and only if  $G_1 \cong G_3$ . In the following, we divide the proof for  $a = 0$  and  $a = 1$ .

(i) Let  $a = 0$ . We first prove that  $\rho(G_3) \leq n - b\delta - 1$ . Assume that  $\rho(G_3) = \rho^* \geq n - (\delta + 1 - s)bs$ . Let  $x$  be the Perron vector of  $A(G_3)$ . By symmetry,  $x$  takes the same values  $x_1, x_2$ , and  $x_3$  on the vertices of  $V(K_{n-s-(\delta+1-s)bs})$ ,  $V(K_{\delta+1-s})$  and  $V(K_s)$ , respectively. Then, by  $A(G_3)x = \rho^*x$ , we have

$$(3.12) \quad \rho^*x_1 = (n - s - (\delta + 1 - s)bs - 1)x_1 + sx_3,$$

$$(3.13) \quad \rho^*x_2 = (\delta - s)x_2 + sx_3,$$

$$(3.14) \quad \rho^*x_3 = (n - s - (\delta + 1 - s)bs)x_1 + (\delta + 1 - s)bsx_2 + (s - 1)x_3.$$

From (3.12) and (3.13), we obtain

$$(3.15) \quad \begin{cases} x_1 = \frac{sx_3}{\rho^* - (n - (\delta + 1 - s)bs - 1) + s}, \\ x_2 = \frac{sx_3}{\rho^* - \delta + s}. \end{cases}$$

Note that  $n \geq b\delta^3 + \delta$ ,  $b \geq 1$  and  $\delta \geq s + 1 \geq 2$ . Then,  $\rho^* \geq n - (\delta + 1 - s)bs > \delta + 1$ . Putting (3.15) into (3.14), and considering that  $\rho^* \geq n - (\delta + 1 - s)bs > \delta + 1$ , we get

$$\begin{aligned}
 & \rho^* + 1 \\
 &= s + \frac{s(n-s-(\delta+1-s)bs)}{\rho^* - (n-(\delta+1-s)bs-1) + s} + \frac{s(\delta+1-s)bs}{\rho^* - \delta + s} \\
 &< s + \frac{s(n-s)}{s+1} \quad (\text{since } \rho^* \geq n - (\delta+1-s)bs > \delta+1) \\
 &= n - (\delta+1-s)bs - \frac{n-s-(\delta+1-s)bs(s+1)}{s+1} \\
 &\leq n - (\delta+1-s)bs - \frac{b\delta^3 + \delta - s - (\delta+1-s)bs(s+1)}{s+1} \quad (\text{since } n \geq b\delta^3 + \delta) \\
 &< n - (\delta+1-s)bs \quad (\text{since } \delta \geq s+1, b \geq 1 \text{ and } s \geq 1) \\
 &\leq \rho^*,
 \end{aligned}$$

which is impossible. Thus, we have

$$\begin{aligned}
 (3.16) \quad & \rho^* < n - (\delta+1-s)bs \\
 &= n - b\delta - 1 - ((s-1)(\delta-s)b-1) \\
 &\leq n - b\delta \quad (\text{since } \delta \geq s+1, b \geq 1 \text{ and } s \geq 1).
 \end{aligned}$$

Since  $K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)$  contains  $K_{n-b\delta}$  as a proper subgraph, we have

$$\rho(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)) > \rho(K_{n-b\delta}) = n - b\delta - 1.$$

Combining this with (3.5), (3.11) and (3.16), we may conclude that

$$\rho(G) \leq \rho(G_1) \leq \rho(G_3) \leq n - b\delta - 1 < \rho(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)).$$

(ii) Let  $a = 1$ . We first prove  $q(G_3) < 2(n - b\delta)$ . Similarly, assume that  $q(G_3) = q^* \geq 2(n - (\delta+1-s)bs)$ . Let  $y$  be the Perron vector of  $Q(G_3)$ . By symmetry,  $y$  takes the same values  $y_1, y_2$ , and  $y_3$  on the vertices of  $V(K_{n-s-(\delta+1-s)bs}), V(K_{\delta+1-s})$  and  $V(K_s)$ , respectively. Then, by  $Q(G_3)y = q^*y$ , we have

$$(3.17) \quad q^*y_1 = (n-s-(\delta+1-s)bs-1)y_1 + sy_3 + (n-(\delta+1-s)bs-1)y_1,$$

$$(3.18) \quad q^*y_2 = (\delta-s)y_2 + sy_3 + \delta y_2,$$

$$(3.19) \quad q^*y_3 = (n-s-(\delta+1-s)bs)y_1 + (\delta+1-s)bsy_2 + (s-1)y_3 + (n-1)y_3.$$

From (3.17) and (3.18), we obtain

$$(3.20) \quad \begin{cases} y_1 = \frac{sy_3}{q^* - (n-(\delta+1-s)bs-1) + s - (n-(\delta+1-s)bs-1)}, \\ y_2 = \frac{sy_3}{q^* - 2\delta + s}. \end{cases}$$

Note that  $n \geq b\delta^3 + \delta, b \geq 1$  and  $\delta \geq s+1 \geq 2$ . Then,  $q^* \geq 2(n - (\delta+1-s)bs) > 2\delta + 2$ . Putting (3.20) into (3.19), and considering that  $q^* \geq 2(n - (\delta+1-s)bs) > 2\delta + 2$ , we get

$$\begin{aligned}
 & q^* + 1 \\
 &= s + \frac{s(n-s-(\delta+1-s)bs)}{q^* - (n-(\delta+1-s)bs-1) + s - (n-(\delta+1-s)bs-1)} + \frac{(\delta+1-s)bs^2}{q^* - 2\delta + s} + n - 1 \\
 &< s + \frac{s(n-s)}{s+2} + n - 1 \quad (\text{since } q^* \geq 2(n - (\delta + 1 - s)bs) > 2\delta + 2) \\
 &= 2(n - (\delta + 1 - s)bs) - \frac{2n - s - 2bs(\delta + 1 - s)(s + 2) + 2}{s + 2} \\
 &\leq 2(n - (\delta + 1 - s)bs) - \frac{2(b\delta^3 + \delta) - s - 2bs(\delta + 1 - s)(s + 2) + 2}{s + 2} \quad (\text{since } n \geq b\delta^3 + \delta) \\
 &< 2(n - (\delta + 1 - s)bs) \quad (\text{since } \delta \geq s + 1, b \geq 1 \text{ and } s \geq 1) \\
 &\leq q^*,
 \end{aligned}$$

which is impossible. Thus, we have

$$\begin{aligned}
 (3.21) \quad & q^* < 2(n - (\delta + 1 - s)bs) \\
 &= 2(n - b\delta - 1 - ((s - 1)(\delta - s)b - 1)) \\
 &\leq 2(n - b\delta) \quad (\text{since } \delta \geq s + 1, b \geq 1 \text{ and } s \geq 1).
 \end{aligned}$$

Since  $K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)$  contains  $K_{n-b\delta}$  as a proper subgraph, we have

$$q(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)) > q(K_{n-b\delta}) = 2(n - b\delta - 1).$$

Combining this with (3.5), (3.11), and (3.21), we may conclude that

$$q(G) \leq q(G_1) \leq q(G_3) \leq 2(n - b\delta) - 1 < q(K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)).$$

It is easy to verify that  $K_\delta \nabla (K_{n-(b+1)\delta} \cup b\delta K_1)$  is not  $\frac{1}{b}$ -tough. This completes the proof of Theorem 1.3.  $\square$

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