



SYMMETRIC INVERSE GENERALIZED EIGENVALUE PROBLEM WITH SUBMATRIX CONSTRAINTS IN STRUCTURAL DYNAMIC MODEL UPDATING*

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Abstract. In this literature, the symmetric inverse generalized eigenvalue problem with submatrix constraints and its corresponding optimal approximation problem are studied. A necessary and sufficient condition for solvability is derived, and when solvable, the general solutions are presented.

Key words. Symmetric inverse generalized eigenvalue problem, Submatrix constraint, Optimal approximation.

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1. Introduction. Let $\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ real matrices, $\mathbb{S}\mathbb{R}^{n \times n}$ the set of all $n \times n$ real symmetric matrices, and \mathbb{C}^n the n -dimensional complex vector space. The dynamic analysis of a mechanical or civil structure by the finite element technique can be modelled by a generalized eigenvalue problem

$$(1.1) \quad K_a x = \lambda M_a x,$$

where $K_a, M_a \in \mathbb{S}\mathbb{R}^{n \times n}$ are the analytical stiffness matrix and mass matrix, respectively, and $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$ are the generalized eigenvalue and corresponding generalized eigenvector. It's well known that the natural frequencies and mode shapes of such a finite element model and ones experimentally measured by a vibration test do not match very well ([4], [20]). The finite element model updating problem is to determine how to update the model to closely match the experimental model data that gives an incomplete set of eigenpairs.

Let $x_1, \dots, x_p \in \mathbb{C}^n$ be the measured modal vectors and $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ be the

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measured natural frequencies, where $p \leq n$. The measured mode shapes and frequencies are assumed to be correct, and the mass matrix $M \in \mathbb{S}\mathbb{R}^{n \times n}$ and stiffness matrix $K \in \mathbb{S}\mathbb{R}^{n \times n}$ to be constructed have to satisfy the dynamic equation

$$(1.2) \quad Kx_i = Mx_i\lambda_i, \quad i = 1, \dots, p$$

(see e.g. [1], [2], [6]–[8], [11], [12], [16], [17], [22]). Since we are only interested in real matrices, the prescribed eigenpairs must be closed under complex conjugation. Without causing ambiguity, the p prescribed eigenpairs are represented by the matrix form (X, Λ) , where each block of $\Lambda \in \mathbb{R}^{p \times p}$ is either a 1-by-1 matrix or a 2-by-2 matrix whose eigenvalues are a complex conjugate pair, and $X \in \mathbb{R}^{n \times p}$ represents the “eigenvector matrix” in the sense that each pair of column vectors associated with a 2-by-2 block in Λ retains the real and the imaginary part, respectively, of the original complex eigenvector. Then (1.2) is equivalent to $KX = MX\Lambda$. As model errors can be localized by sensitivity analysis ([10], [19]), residual force approach ([9]), least-squares approach ([13]) and assigned eigenstructure ([5]), it is a usual practice to adjust partial elements of the analytical mass and stiffness matrices. Mathematically, such a partially updating problem can be described as the following two problems:

PROBLEM 1.1. Given p eigenpairs (X, Λ) , where $X \in \mathbb{R}^{n \times p}$ and $\Lambda \in \mathbb{R}^{p \times p}$ are described as the above and $K_0, M_0 \in \mathbb{S}\mathbb{R}^{r \times r}$, $1 \leq p, r \leq n, p + r \leq n$, find real matrices $K, M \in \mathbb{S}\mathbb{R}^{n \times n}$ such that

$$(1.3) \quad KX = MX\Lambda, \quad K([1, r]) = K_0, \quad M([1, r]) = M_0,$$

where $K([1, r])$ and $M([1, r])$ are the $r \times r$ leading principal submatrices of K and M , respectively.

PROBLEM 1.2. Given $K_a, M_a \in \mathbb{S}\mathbb{R}^{n \times n}$ with $K_a([1, r]) = K_0, M_a([1, r]) = M_0$, find $\hat{K}, \hat{M} \in S_E$ such that

$$(1.4) \quad \|(K_a, M_a) - (K, M)\| = \inf_{(\hat{K}, \hat{M}) \in S_E} \|(K_a, M_a) - (\hat{K}, \hat{M})\|,$$

where S_E is the solution set of Problem 1.1. Here $\|\cdot\|$ denotes the Frobenius norm.

The second problem is to find the best approximation for a given symmetric matrix pencil under a given spectral constraint and a symmetric submatrix pencil constraint. Such a problem always arises in structural dynamic model updating.

Y. Yuan and H. Dai [21] solved the above two problems, where K_a, K_0, K, M_a, M_0 , and M are free from the symmetry constraint conditions. In this paper, we consider the symmetric cases in the dynamic analysis of a mechanical or civil structure by the finite element technique. Only by using the Moore-Penrose generalized inverse and the singular value decomposition, we present the solvability condition and the

expression for the solutions of Problems 1.1 and 1.2. For a special case, we derive a formula of the general solution which is inexpensive to compute and can be used routinely in practice.

Throughout this paper $O(n)$ stands for the set of all $n \times n$ orthogonal matrices; A^T the transpose of a real matrix A ; A^\dagger the Moore-Penrose generalized inverse of A ; $P_A = AA^\dagger$, $P_A^\perp = I - P_A$; $\text{tr}(A)$ the trace of the matrix $A \in \mathbb{R}^{n \times n}$; and $\text{rank}(A)$ the rank of A . For $A, B \in \mathbb{R}^{m \times n}$, $(A, B) = \text{tr}(B^T A)$ denotes an inner product in $\mathbb{R}^{m \times n}$.

2. General solutions of Problems 1.1 and 1.2. To facilitate the following discussion, we first make some notations. Let $X \in \mathbb{R}^{n \times p}$, $K, M \in \mathbb{S}\mathbb{R}^{n \times n}$ have the following partitions:

$$(2.1) \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_0 & K_1 \\ K_1^T & K_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_0 & M_1 \\ M_1^T & M_2 \end{pmatrix},$$

where X_1 and X_2 have r and $n - r$ rows respectively, $K_0, M_0 \in \mathbb{R}^{r \times r}$, $K_1, M_1 \in \mathbb{R}^{r \times (n-r)}$, and $K_2, M_2 \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}$. Then $KX = MX\Lambda$ can be rewritten as

$$(2.2) \quad \begin{pmatrix} K_0 & K_1 \\ K_1^T & K_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} M_0 & M_1 \\ M_1^T & M_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Lambda.$$

That implies that (1.3) is equivalent to

$$(2.3) \quad K_1 X_2 = M_0 X_1 \Lambda - K_0 X_1 + M_1 X_2 \Lambda,$$

$$(2.4) \quad K_1^T X_1 = M_1^T X_1 \Lambda - K_2 X_2 + M_2 X_2 \Lambda.$$

Problem 1.1 can be solved by computing the solutions K_1, M_1, K_2 and M_2 of (2.3) and (2.4).

Suppose that the singular value decomposition of X_2 has been computed

$$(2.5) \quad X_2 = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T,$$

where $U = [U_1, U_2] \in O(n - r)$, $V = [V_1, V_2] \in O(p)$, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_s)$, $\sigma_i > 0$, $i = 1, \dots, s$, $s = \text{rank}(X_2)$, $U_1 \in \mathbb{R}^{(n-r) \times s}$, $V_1 \in \mathbb{R}^{p \times s}$. Similarly, suppose that the following singular value decompositions have been computed

$$(2.6) \quad X_2 \Lambda V_2 = [P_1, P_2] \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} [Q_1, Q_2]^T,$$

$$(2.7) \quad X_1 V_2 Q_2 = [\tilde{P}_1, \tilde{P}_2] \begin{pmatrix} \tilde{\Omega} & 0 \\ 0 & 0 \end{pmatrix} [\tilde{Q}_1, \tilde{Q}_2]^T,$$

$$(2.8) \quad (X_2 \Lambda X_2^\dagger)^T = [\hat{U}_1, \hat{U}_2] \begin{pmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} [\hat{V}_1, \hat{V}_2]^T.$$

Now we recall some known results, which will be used repeatedly in this paper.

LEMMA 2.1. [14] *Let $Y, B \in \mathbb{R}^{m \times k}$, $m \geq k$, be given. Then $AY = B$ has a symmetric solution $A \in \mathbb{S}\mathbb{R}^{m \times m}$ if and only if $BP_{Y^T} = B$ and $(P_Y BY^\dagger)^T = P_Y BY^\dagger$. Moreover, the solution set is*

$$S = \{BY^\dagger + (BY^\dagger)^T P_Y^\perp + P_Y^\perp H P_Y^\perp : H \in \mathbb{S}\mathbb{R}^{m \times m}\},$$

and the minimal solution under the Frobenius norm is

$$A_{opt} = BY^\dagger + (BY^\dagger)^T P_Y^\perp.$$

LEMMA 2.2. [3] *If $E \in \mathbb{R}^{m \times n}$, $F \in \mathbb{R}^{p \times q}$ and $G \in \mathbb{R}^{m \times q}$,*

$$(2.9) \quad EXF = G$$

has a solution $X \in \mathbb{R}^{p \times q}$ if and only if

$$EE^\dagger GF^\dagger F = G,$$

in which case, the general solution of (2.9) can be expressed as

$$X = E^\dagger GF^\dagger + Y - E^\dagger EYFF^\dagger,$$

where $Y \in \mathbb{R}^{n \times p}$ is an arbitrary matrix.

The next lemma is extracted from the proof of Theorem 1 in [21].

LEMMA 2.3. [21] *Equation $K_0 X_1 + K_1 X_2 = M_0 X_1 \Lambda + M_1 X_2 \Lambda$ with respect to matrices K_1 and M_1 is solvable if and only if*

$$(2.10) \quad K_0 X_1 V_2 Q_2 = M_0 X_1 \Lambda V_2 Q_2,$$

in which case, we have

$$(2.11) \quad K_1 = K_{10} + LP_2^T X_2 \Lambda X_2^\dagger + WU_2^T,$$

$$(2.12) \quad M_1 = M_{10} + LP_2^T,$$

where

$$M_{10} = (K_0 X_1 V_2 - M_0 X_1 \Lambda V_2)(X_2 \Lambda V_2)^\dagger,$$

$$K_{10} = [M_0 X_1 \Lambda - K_0 X_1 + M_{10} X_2 \Lambda] X_2^\dagger,$$

and $L \in \mathbb{R}^{r \times (n-r-t)}$, $W \in \mathbb{R}^{r \times (n-r-s)}$ are arbitrary matrices.

As we discussed above, the general solution of Problem 1.1 can be derived by solving (2.3) and (2.4). Lemma 2.3 has presented the solvability condition and the general solution (if exists) of (2.3). Now we study (2.4). Define $\tilde{L} = P_2 L^T$, $\tilde{W} = U_2 W^T$, and

$$\Psi = M_{10}^T X_1 \Lambda - K_{10}^T X_1 + \tilde{L} X_1 \Lambda - (X_2 \Lambda X_2^\dagger)^T \tilde{L} X_1.$$

Substitute (2.11) and (2.12) into (2.4), to get

$$(2.13) \quad K_2 X_2 = \Psi - \tilde{W} X_1 + M_2 X_2 \Lambda$$

with respect to unknown real symmetric matrices K_2 and M_2 . Repeatedly applying Lemma 2.1 and equations (2.5)–(2.8), and (2.13) has a symmetric solution K_2 if and only if

$$(2.14) \quad M_2 X_2 \Lambda V_2 = [-\Psi + \tilde{W} X_1] V_2,$$

$$(2.15) \quad U_1 U_1^T [\Psi + M_2 X_2 \Lambda] X_2^\dagger \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}.$$

Equation (2.14) has a symmetric solution M_2 if and only if

$$(2.16) \quad (X_2 \Lambda X_2^\dagger)^T \tilde{L} X_1 V_2 Q_2 = -(K_{10}^T + \tilde{W}) X_1 V_2 Q_2,$$

$$(2.17) \quad P_1 P_1^T [-\Psi + \tilde{W} X_1] V_2 (X_1 \Lambda V_2)^\dagger \in \mathbb{S}\mathbb{R}^{r \times r}.$$

Now we prove that (2.16) always holds. Lemma 2.2 will be repeatedly used without being mentioned in the following analysis. Equation (2.16) has a solution \tilde{L} if and only if

$$(2.18) \quad [I - (X_2 \Lambda X_2^\dagger)^T ((X_2 \Lambda X_2^\dagger)^T)^\dagger] (K_{10}^T + \tilde{W}) X_1 V_2 Q_2 = 0.$$

Recalling the SVD of $(X_2 \Lambda X_2^\dagger)^T$ as in (2.8), (2.18) can be rewritten as

$$(2.19) \quad \hat{U}_2^T \tilde{W} X_1 V_2 Q_2 = -\hat{U}_2^T K_{10}^T X_1 V_2 Q_2.$$

We can see that (2.19) is always solvable and the expression of its general solution is

$$(2.20) \quad \tilde{W} = -\hat{U}_2 \hat{U}_2^T K_{10}^T (X_1 V_2 Q_2) (X_1 V_2 Q_2)^\dagger + \hat{Y} - \hat{U}_2 \hat{U}_2^T \hat{Y} (X_1 V_2 Q_2) (X_1 V_2 Q_2)^\dagger,$$

where $\hat{Y} \in \mathbb{R}^{(n-r) \times r}$ is arbitrary. That means (2.18) always holds and consequently there must exist a matrix \tilde{L} satisfying (2.16). Now substitute (2.20) into (2.16), to get

$$(2.21) \quad (X_2 \Lambda X_2^\dagger)^T \tilde{L} X_1 V_2 Q_2 = -\hat{U}_1 \hat{U}_1^T (K_{10}^T - \hat{Y}) X_1 V_2 Q_2.$$

The general solution of (2.21) has the form

$$(2.22) \quad \tilde{L} = -((X_2 \Lambda X_2^\dagger)^T)^\dagger (K_{10}^T - \hat{Y})(\tilde{P}_1 \tilde{P}_1^T) + \tilde{Y} - \hat{V}_1 \hat{V}_1^T \tilde{Y} (\tilde{P}_1 \tilde{P}_1^T),$$

where \tilde{Y} is arbitrary. So that (2.16) is always solvable and its general solution can be expressed as in (2.22).

By Lemma 2.1, we can conclude that (2.14) is solvable when (2.16) and (2.17) hold, and its general solution is

$$(2.23) \quad M_2 = B_{M_2} (X_2 \Lambda V_2)^\dagger + (B_{M_2} (X_2 \Lambda V_2)^\dagger)^T P_{(X_2 \Lambda V_2)}^\perp + P_{(X_2 \Lambda V_2)}^\perp H_{M_2} P_{(X_2 \Lambda V_2)}^\perp$$

in which $B_{M_2} = [-\Psi + (X_2 \Lambda X_2^\dagger)^T \tilde{L} X_1 + \tilde{W} X_1] V_2$ and $H_{M_2} \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}$ is arbitrary. Similarly, if (2.14) and (2.15) hold, then (2.13) has a symmetric solution

$$(2.24) \quad K_2 = B_{K_2} X_2^\dagger + (B_{K_2} X_2^\dagger)^T P_{X_2}^\perp + P_{X_2}^\perp H_{K_2} P_{X_2}^\perp,$$

where $B_{K_2} = \Psi - (X_2 \Lambda X_2^\dagger)^T \tilde{L} X_1 - \tilde{W} X_1 + M_2 X_2 \Lambda$ and $H_{K_2} \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}$ is arbitrary.

Now we suppose that there exist \hat{Y}, \tilde{Y} and H_{M_2} such that the symmetric conditions (2.15) and (2.17) hold after substituting (2.20), (2.22), and (2.23) into them. Then we can conclude that Problem 1.1 is solvable from the above analysis.

THEOREM 2.4. *Problem 1.1 is solvable if and only if (2.10), (2.15) and (2.17) hold. In this case, the general solution is*

$$(K, M) = \begin{pmatrix} K_0 & K_1 & M_0 & M_1 \\ K_1^T & K_2 & M_1^T & M_2 \end{pmatrix},$$

where K_1, M_1, M_2, K_2 are given by (2.11), (2.12), (2.23), and (2.24) in which \hat{Y}, \tilde{Y} , and H_{M_2} satisfy (2.15) and (2.17).

Now we study Problem 1.2 under the condition that the solution set S_E of Problem 1.1 is not empty. Firstly, recall an useful result in [15].

LEMMA 2.5. [15] *Suppose that $A \in \mathbb{R}^{q \times m}, \Delta \in \mathbb{R}^{q \times q}$, and $\Gamma \in \mathbb{R}^{m \times m}$, where $\Delta^2 = \Delta = \Delta^T$ and $\Gamma^2 = \Gamma = \Gamma^T$. Then*

$$\|A - \Delta D \Gamma\| = \min_{E \in \mathbb{R}^{q \times m}} \|A - \Delta E \Gamma\|$$

if and only if $\Delta(A - D)\Gamma = 0$, in which case,

$$\|A - \Delta D \Gamma\| = \|A - \Delta A \Gamma\|.$$

For given matrices $K_a \in \mathbb{R}^{n \times n}$ and $M_a \in \mathbb{R}^{n \times n}$ in Problem 1.2, define

$$(2.25) \quad K_a = \begin{pmatrix} K_0 & K_1^{(a)} \\ (K_1^{(a)})^T & K_2^{(a)} \end{pmatrix}, \quad M_a = \begin{pmatrix} M_0 & M_1^{(a)} \\ (M_1^{(a)})^T & M_2^{(a)} \end{pmatrix},$$

where $K_1^{(a)}, M_1^{(a)} \in \mathbb{R}^{r \times (n-r)}$, and $K_2^{(a)}, M_2^{(a)} \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}$. Given two matrices C, D and a matrix set S , the notation $\|C - A\|^2 + \|D - B\|^2 = \min$ means that $\|C - A\|^2 + \|D - B\|^2$ is minimized by $(A, B) \in S$; i.e.,

$$\|C - A\|^2 + \|D - B\|^2 = \min_{(\hat{A}, \hat{B}) \in S} (\|C - \hat{A}\|^2 + \|D - \hat{B}\|^2).$$

THEOREM 2.6. *Under the conditions of Theorem 2.4 and assuming that \hat{Y}, \tilde{Y} and H_{M_2} are unique solutions of (2.15) and (2.17), Problem 1.2 has a unique solution*

$$(K, M) = \begin{pmatrix} K_0 & K_1 & M_0 & M_1 \\ K_1^T & K_2 & M_1^T & M_2 \end{pmatrix},$$

where K_1, M_1, M_2, K_2 are given by (2.11), (2.12), (2.23), and (2.24), and H_{K_2} and H_{M_2} satisfy $P_{X_2}^\perp (K_2^{(a)} - H_{K_2}) P_{X_2}^\perp = 0$ and $P_{(X_2 \wedge V_2)}^\perp (M_2^{(a)} - H_{M_2}) P_{(X_2 \wedge V_2)}^\perp = 0$, respectively.

Proof. Equation (1.4) is equivalent to

$$(2.26) \quad \|K_a - K\|^2 + \|M_a - M\|^2 = \min.$$

With the partitions (2.1) and (2.25) in mind, we have

$$\|K_a - K\|^2 + \|M_a - M\|^2 = 2\|K_1^{(a)} - K_1\|^2 + \|K_2^{(a)} - K_2\|^2 + 2\|M_1^{(a)} - M_1\|^2 + \|M_2^{(a)} - M_2\|^2.$$

Then (2.26) holds if and only if

$$(2.27) \quad \|K_2^{(a)} - K_2\|^2 + \|M_2^{(a)} - M_2\|^2 = \min.$$

By Lemma 2.5, (2.27) holds if and only if

$$U_2 U_2^T (K_2^{(a)} - H_{K_2}) U_2 U_2^T = 0, \quad P_2 P_2^T (M_2^{(a)} - H_{M_2}) P_2 P_2^T = 0. \quad \square$$

REMARK 2.7. We have solved Problem 1.1 and Problem 1.2 under the assumption that (2.15) and (2.17) hold for some \hat{Y}, \tilde{Y} , and H_{M_2} . In general to find such \hat{Y}, \tilde{Y} , and H_{M_2} , we have to solve two matrix equations with three unknowns after substituting (2.20), (2.22), and (2.23) into (2.15) and (2.17).

To end this section, we consider a sufficient condition that

$$(2.28) \quad K_0 X_1 = M_0 X_1 \Lambda, \quad K_1 = 0, \quad M_1 = 0,$$

under which (2.2) is reduced to

$$\begin{pmatrix} K_0 & 0 \\ 0 & K_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} M_0 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Lambda;$$

i.e.,

$$(2.29) \quad K_2 X_2 = M_2 X_2 \Lambda.$$

To solve (2.29) for symmetric matrices K_2 and M_2 is the symmetric inverse eigenvalue problem without constraints. Dai [7] studied this problem and gave an efficient algorithm for it by using QR-like decomposition with column pivoting and least squares techniques. Here we present the expression of the general symmetric solution only by generalized inverse and SVD.

THEOREM 2.8. *Suppose X_2 and Λ are known and the singular value decomposition of X_2 is (2.5). Then (2.29) has a symmetric solution and*

$$(K_2, M_2) = U \begin{pmatrix} M_{11}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} & (M_{21}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1})^T & M_{11}^{(2)} & (M_{21}^{(2)})^T \\ M_{21}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} & K_{22}^{(2)} & M_{21}^{(2)} & M_{22}^{(2)} \end{pmatrix} \text{diag}(U, U)^T,$$

where $M_{11}^{(2)} = P_Y^\perp H P_Y^\perp$ with H being a symmetric matrix such that $P_Y^\perp H P_Y^\perp \Sigma \Lambda_{11}^{(2)} \Sigma^{-1}$ is symmetric, $Y = \Sigma \Lambda_{12}^{(2)}$, $M_{21}^{(2)} = F P_Y^\perp$ with F being arbitrary, $K_{22}^{(2)}, M_{22}^{(2)}$ are two arbitrary symmetric matrices, $\Lambda_{11}^{(2)} = V_1^T \Lambda V_1$, $\Lambda_{12}^{(2)} = V_1^T \Lambda V_2$.

Proof. Substitute (2.5) into (2.29). \square

Before we consider Problem 1.2 under the condition (2.28), we give some notation.

$$\begin{pmatrix} K_{11}^{(a)} & (K_{21}^{(a)})^T & M_{11}^{(a)} & (M_{21}^{(a)})^T \\ K_{21}^{(a)} & K_{22}^{(a)} & M_{21}^{(a)} & M_{22}^{(a)} \end{pmatrix} = U^T \begin{pmatrix} K_2^{(a)} & M_2^{(a)} \end{pmatrix} \text{diag}(U, U),$$

$$\tilde{\Lambda}_{11} = \begin{pmatrix} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} & I \end{pmatrix}, \quad \widetilde{K M}_{11} = \begin{pmatrix} K_{11}^{(a)} & M_{11}^{(a)} \end{pmatrix}, \quad \widetilde{K M}_{21} = \begin{pmatrix} K_{21}^{(a)} & M_{21}^{(a)} \end{pmatrix}.$$

THEOREM 2.9. *If $K_0 X_1 = M_0 X_1 \Lambda$ and $K_1 = 0, M_1 = 0$, Problem 1.2 has a solution and*

$$(K_2, M_2) = U \begin{pmatrix} M_{11}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} & (M_{21}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1})^T & M_{11}^{(2)} & (F P_Y^\perp)^T \\ M_{21}^{(2)} \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} & K_{22}^{(a)} & F P_Y^\perp & M_{22}^{(a)} \end{pmatrix} \text{diag}(U, U)^T,$$

in which $M_{11}^{(2)} = P_Y^\perp H P_Y^\perp$ with H being a symmetric solution of $\|P_Y^\perp H P_Y^\perp \tilde{\Lambda}_{11} - \widetilde{KM}_{11}\| = \min$; $P_Y^\perp H P_Y^\perp \Sigma \Lambda_{11}^{(2)} \Sigma^{-1} = \Sigma^{-1} (\Lambda_{11}^{(2)})^T \Sigma P_Y^\perp H P_Y^\perp$; and $F = \widetilde{KM}_{21} (P_Y^\perp \tilde{\Lambda}_{11})^\dagger + W P_{P_Y^\perp \tilde{\Lambda}_{11}}^\perp$ with W being arbitrary.

Proof. It is clear that $\|K - K_a\|^2 + \|M - M_a\|^2 = \min$ if and only if

$$(2.30) \quad \|K_2 - K_2^{(a)}\|_F^2 + \|M_2 - M_2^{(a)}\|_F^2 = \min.$$

Equation (2.30) holds if and only if

$$\begin{aligned} \|M_{11}^{(2)} \tilde{\Lambda}_{11} - \widetilde{KM}_{11}\|^2 &= \min, & \|M_{21}^{(2)} \tilde{\Lambda}_{11} - \widetilde{KM}_{21}\|^2 &= \min, \\ \|K_{22}^{(2)} - K_{22}^{(a)}\|_F^2 &= \min, & \|M_{22}^{(2)} - M_{22}^{(a)}\|_F^2 &= \min. \quad \square \end{aligned}$$

REMARK 2.10. For details about solving $\min_{H \in \mathbb{SR}^{n \times n}} \|P_Y^\perp H P_Y^\perp \tilde{\Lambda}_{11} - \widetilde{KM}_{11}\|$, see Theorem 3.3.2 in [18].

3. A special case. In this section, we consider one simple but not easy case of Problem 1.1 and Problem 1.2.

PROBLEM 3.1. Given p eigenpairs by (X, Λ) , where $X \in \mathbb{R}^{n \times p}$ and $\Lambda \in \mathbb{R}^{p \times p}$ is a block diagonal matrix with 2-by-2 blocks or single real eigenvalues on its main diagonal, and $K_0 \in \mathbb{SR}^{r \times r}$. Find a real matrix $K \in \mathbb{SR}^{n \times n}$ such that

$$(3.1) \quad KX = X\Lambda, \quad K([1, r]) = K_0,$$

where $K([1, r])$ is the $r \times r$ leading principal submatrix of K .

PROBLEM 3.2. Given $K_a \in \mathbb{SR}^{n \times n}$ with $K_a([1, r]) = K_0$, find $\hat{K} \in S_E$ such that

$$\|K_a - \hat{K}\| = \inf_{K \in S_E} \|K_a - K\|,$$

where S_E is the solution set of Problem 3.1.

Let $X \in \mathbb{R}^{n \times p}$ and $K \in \mathbb{SR}^{n \times n}$ have the partitions as in (2.1) and the singular value decomposition of X_2 be given as in (2.5).

THEOREM 3.3. Suppose that $K_0 \in \mathbb{SR}^{r \times r}$, $X \in \mathbb{R}^{n \times p}$, and $\Lambda \in \mathbb{R}^{p \times p}$ are given as in Problem 3.1. Then Problem 3.1 is solvable if and only if

$$(3.2) \quad (K_0 X_1 - X_1 \Lambda) P_{X_2^T}^\perp = 0, \quad P_{X_2^T}^\perp (\Lambda^T X_2^T - X_1^T (X_1 \Lambda - K_0 X_1) X_2^\dagger) = 0,$$

and

$$(3.3) \quad X^T X \Lambda = \Lambda^T X^T X$$

hold.

If (3.2) and (3.3) hold, then the general solution of Problem 3.1 is

$$(3.4) \quad K = \begin{pmatrix} K_0 & (X_1\Lambda - K_0X_1)X_2^\dagger + P_{(X_1P_{X_2^T}^\perp)}^\perp LP_{X_2}^\perp \\ K_1^T & BX_2^\dagger + (BX_2^\dagger)^T P_{X_2}^\perp + P_{X_2}^\perp HP_{X_2}^\perp \end{pmatrix},$$

where $B = X_2\Lambda - (X_2^\dagger)^T(X_1\Lambda - K_0X_1)^T X_1 - P_{X_2}^\perp L^T P_{(X_1P_{X_2^T}^\perp)}^\perp X_1$, both $L \in \mathbb{R}^{r \times (n-r)}$ and $H \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}$ are arbitrary.

Proof. Applying (2.1), (3.1) is equivalent to the following two equations

$$(3.5) \quad K_1 X_2 = X_1\Lambda - K_0 X_1,$$

$$(3.6) \quad K_1^T X_1 = X_2\Lambda - K_2 X_2.$$

By Lemma 2.2, (3.5) has a solution K_1 if and only if

$$(X_1\Lambda - K_0 X_1)P_{X_2^T}^\perp = 0,$$

in which case,

$$(3.7) \quad K_1 = (X_1\Lambda - K_0 X_1)X_2^\dagger + Y P_{X_2}^\perp,$$

where $Y \in \mathbb{R}^{r \times (n-r)}$ is arbitrary. Substitute (3.7) into (3.6),

$$(3.8) \quad K_2 X_2 = X_2\Lambda - K_1^T X_1 = X_2\Lambda - (X_2^\dagger)^T(X_1\Lambda - K_0 X_1)^T X_1 - P_{X_2}^\perp Y^T X_1.$$

By Lemma 2.1, there exists a symmetric matrix K_2 satisfying (3.8) if and only if

$$(3.9) \quad (X_2\Lambda - K_1^T X_1)P_{X_2^T}^\perp = 0,$$

$$(3.10) \quad P_{X_2}(X_2\Lambda - K_1^T X_1)X_2^\dagger \in \mathbb{S}\mathbb{R}^{q \times (n-r)}.$$

Let $G = P_{X_2^T}^\perp(\Lambda^T X_2^T - X_1^T(X_1\Lambda - K_0 X_1)X_2^\dagger)$. Since $P_{X_2^T}^\perp(P_{X_2^T} X_1^T)(P_{X_2^T} X_1^T)^\dagger = 0$,

$$(3.11) \quad P_{X_2^T} X_1^T (P_{X_2^T} X_1^T)^\dagger G (P_{X_2}^\perp)^\dagger P_{X_2}^\perp \equiv 0.$$

Substitute (3.7) into (3.9), to get

$$(3.12) \quad P_{X_2^T}^\perp X_1^T Y P_{X_2}^\perp = G.$$

By (3.11) and Lemma 2.2, (3.12) holds if and only if $G = 0$, which is exactly the second equation in (3.2), and in this case

$$(3.13) \quad Y = L - (P_{X_2^T}^\perp X_1^T)^\dagger (P_{X_2^T}^\perp X_1^T) L P_{X_2}^\perp,$$

where $L \in \mathbb{R}^{r \times (n-r)}$ is arbitrary. Substituting (3.13) into (3.7), we have

$$(3.14) \quad K_1 = (X_1\Lambda - K_0 X_1)X_2^\dagger + P_{(X_1P_{X_2^T}^\perp)}^\perp L P_{X_2}^\perp.$$

Now we substitute (3.14) into (3.10), to get

$$\begin{aligned} P_{X_2}(X_2\Lambda - K_1^T X_1)X_2^\dagger &= (X_2\Lambda - X_2X_2^\dagger K_1^T X_1)X_2^\dagger \\ &= X_2\Lambda X_2^\dagger - (X_2^\dagger)^T \Lambda^T X_1^T X_1 X_2^\dagger + (X_1 X_2^\dagger)^T K_0^T X_1 X_2^\dagger \\ &= (X_2^\dagger)^T (X_2^T X_2 \Lambda - \Lambda^T X_1^T X_1) X_2^\dagger + (X_1 X_2^\dagger)^T K_0 X_1 X_2^\dagger. \end{aligned}$$

So we can see that the condition (3.10) is equivalent to (3.3). At this point, we have found the necessary and sufficient condition for the solvability of Problem 3.1. Next we prove that (3.4) is the general solution of Problem 3.1 if (3.2) and (3.3) hold. If (3.2) and (3.3) hold, we have proved that K_1 has the expression as in (3.14); what is left to prove is $K_2 = BX_2^\dagger + (BX_2^\dagger)^T P_{X_2}^\perp + P_{X_2}^\perp H P_{X_2}^\perp$. Indeed, if (3.2) and (3.3) hold, then (3.9) and (3.10) are satisfied. Applying Lemma 2.1 and substituting (3.13) into (3.8), (3.8) has a symmetric solution $K_2 = BX_2^\dagger + (BX_2^\dagger)^T P_{X_2}^\perp + P_{X_2}^\perp H P_{X_2}^\perp$. \square

If X_2 is square and nonsingular, then $P_{X_2}^\perp = 0$ and $P_{X_2^T}^\perp = 0$ and $B = X_2\Lambda - (X_2^{-1})^T(X_1\Lambda - K_0X_1)^T X_1$. So we have

COROLLARY 3.4. *Suppose that $K_0 \in \mathbb{S}\mathbb{R}^{r \times r}$, $X \in \mathbb{R}^{n \times p}$, and $\Lambda \in \mathbb{R}^{p \times p}$ is a block diagonal matrix. If X_2 is square and nonsingular, then Problem 3.1 has a unique solution*

$$K = \begin{pmatrix} K_0 & (X_1\Lambda - K_0X_1)X_2^{-1} \\ K_1^T & X_2\Lambda X_2^{-1} - (X_2^{-1})^T(X_1\Lambda - K_0X_1)^T X_1 X_2^{-1} \end{pmatrix}$$

if and only if $X^T X \Lambda = \Lambda X^T X$.

Now we present a general solution of Problem 3.2 assuming Problem 3.1 is solvable.

THEOREM 3.5. *For given matrices $K_a \in \mathbb{R}^{n \times n}$ with $K_a([1, r]) = K_0$ as in (2.25). If (3.2) and (3.3) hold, then Problem 3.2 has a unique solution*

$$\hat{K} = \begin{pmatrix} K_0 & (X_1\Lambda - K_0X_1)X_2^\dagger + \tilde{L} \\ K_1^T & BX_2^\dagger + (BX_2^\dagger)^T P_{X_2}^\perp + P_{X_2}^\perp K_2^{(a)} P_{X_2}^\perp \end{pmatrix},$$

where $B = X_2\Lambda X_2^\dagger - (X_2^\dagger)^T(X_1\Lambda - K_0X_1)^T X_1 X_2^\dagger - \tilde{L}^T X_1 X_2^\dagger$, \tilde{L} satisfies

$$\begin{aligned} \text{vec}(\tilde{L}) &= [(X_1 X_2^\dagger)^T \otimes (X_1 X_2^\dagger) Q + 2I \otimes (I + (X_1 X_2^\dagger)(X_1 X_2^\dagger)^T)]^{-1} \\ (3.15) \quad &\text{vec}(2\tilde{K}_1 - (X_1 X_2^\dagger)(\tilde{K}_2 + \tilde{K}_2^T)), \end{aligned}$$

Q is a permutation matrix such that $\text{vec}(\tilde{L}^T) = Q \text{vec}(\tilde{L})$ and $\tilde{K}_1 = K_1^{(a)} - (X_1\Lambda - K_0X_1)X_2^\dagger$.

Proof. Let $\tilde{L} = P_{(X_1 P_{X_2}^\perp)}^\perp L P_{X_2}^\perp$. From Theorem 3.3,

$$\|K^{(a)} - K\|^2 = 2\|K_1^{(a)} - K_1\|^2 + \|K_2^{(a)} - K_2\|^2$$

$$\begin{aligned}
 &= 2\|K_1^{(a)} - (X_1\Lambda - K_0X_1)X_2^\dagger - \tilde{L}\|^2 \\
 &\quad + \|K_2^{(a)} - X_2\Lambda X_2^\dagger + (X_2^\dagger)^T(X_1\Lambda - K_0X_1)^T X_1X_2^\dagger \\
 &\quad + \tilde{L}^T X_1X_2^\dagger + (X_1X_2^\dagger)^T \tilde{L} - P_{X_2}^\perp H P_{X_2}^\perp\|^2.
 \end{aligned}$$

By Lemma 2.5,

$$\begin{aligned}
 \min_{H=H^T} \|K^{(a)} - K\|^2 &= 2\|K_1^{(a)} - (X_1\Lambda - K_0X_1)X_2^\dagger - \tilde{L}\|^2 \\
 &\quad + \|K_2^{(a)} - X_2\Lambda X_2^\dagger + (X_2^\dagger)^T(X_1\Lambda - K_0X_1)^T X_1X_2^\dagger \\
 &\quad + \tilde{L}^T X_1X_2^\dagger + (X_1X_2^\dagger)^T \tilde{L} - P_{X_2}^\perp K_2^{(a)} P_{X_2}^\perp\|^2,
 \end{aligned}$$

and the minimum is attained if and only if $P_{X_2}^\perp(H - K_2^{(a)})P_{X_2}^\perp = 0$. Let $\tilde{K}_1 = K_1^{(a)} - (X_1\Lambda - K_0X_1)X_2^\dagger$, $\tilde{K}_2 = K_2^{(a)} - P_{X_2}^\perp K_2^{(a)} P_{X_2}^\perp - X_2\Lambda X_2^\dagger + (X_2^\dagger)^T(X_1\Lambda - K_0X_1)^T X_1X_2^\dagger$, and

$$f(\tilde{L}) = 2\|\tilde{K}_1 - \tilde{L}\|^2 + \|\tilde{K}_2 + \tilde{L}^T X_1X_2^\dagger + (X_1X_2^\dagger)^T \tilde{L}\|^2.$$

Then

$$\begin{aligned}
 f(\tilde{L}) &= 2\text{tr}(\tilde{K}_1^T \tilde{K}_1) + 2\text{tr}(\tilde{L}^T \tilde{L}) + \text{tr}(\tilde{K}_2^T \tilde{K}_2) + 2\text{tr}(\tilde{K}_2^T (X_1X_2^\dagger)^T \tilde{L}) \\
 &\quad + 2\text{tr}(\tilde{K}_2 (X_1X_2^\dagger)^T \tilde{L}) + 2\text{tr}(X_1X_2^\dagger (X_1X_2^\dagger)^T \tilde{L} \tilde{L}^T) \\
 &\quad + 2\text{tr}((X_1X_2^\dagger)^T \tilde{L} (X_1X_2^\dagger)^T \tilde{L}) - 4\text{tr}(\tilde{K}_1^T \tilde{L}).
 \end{aligned}$$

Consequently, to get the minimal value of $f(\tilde{L})$, we should find its minimal point. As

$$\frac{\partial f(\tilde{L})}{\partial \tilde{L}} = 2(X_1X_2^\dagger)^T \tilde{L}^T (X_1X_2^\dagger) + 4(I + (X_1X_2^\dagger)(X_1X_2^\dagger)^T) \tilde{L} - 4\tilde{K}_1 + 2(X_1X_2^\dagger)(\tilde{K}_2 + \tilde{K}_2^T),$$

$\frac{\partial f(\tilde{L})}{\partial \tilde{L}} = 0$ is equivalent to

$$(3.16) \quad \Phi \text{vec}(\tilde{L}) = \text{vec}(2\tilde{K}_1 - (X_1X_2^\dagger)(\tilde{K}_2 + \tilde{K}_2^T)),$$

where $\Phi = (X_1X_2^\dagger)^T \otimes (X_1X_2^\dagger)Q + 2I \otimes (I + (X_1X_2^\dagger)(X_1X_2^\dagger)^T)$, and Q is a permutation matrix such that $\text{vec}(\tilde{L}^T) = Q\text{vec}(\tilde{L})$. Since Φ is nonsingular, (3.16) has a unique solution (3.15) and $f(\tilde{L})$ has a unique minimal value point \tilde{L} . \square

4. Conclusions. Motivated by Y. Yuan and H. Dai [21], this paper is concerned with the symmetric inverse generalized eigenvalue problem and the problem of the optimal approximation for a given symmetric matrix pencil under a given spectral constraint and a symmetric submatrix pencil constraint. By using the Moore-Penrose generalized inverse and the singular value decomposition, we first present the solvability condition and the expression for the solution of these two problems. For the

case that the mass matrix is unit, there is a formula of the general solution which is inexpensive to compute and can be used routinely in practice.

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