Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

INVERSE INTERVAL MATRIX: A SURVEY*

JIRI ROHN† AND RAENA FARHADSEFAT‡

Abstract. Results on the inverse interval matrix, both theoretical and computational, are surveyed. Described are, among others, formulae for the inverse interval matrix, NP-hardness of its computation, various classes of interval matrices for which the inverse can be given explicitly, and closed-form formulae for an enclosure of the inverse.

Key words. Interval matrix, Inverse interval matrix, NP-hardness, Enclosure, Unit midpoint, Inverse sign stability, Nonnegative invertibility, Absolute value equation, Algorithm.

AMS subject classifications. 15A09, 65G20, 65G40.

1. Introduction. In our recent paper [20], we presented a survey of forty necessary and sufficient conditions for regularity of interval matrices. It is now followed by a survey of properties of the inverse interval matrix which is closely related to the previous topic because the inverse interval matrix is only defined for regular interval matrices.

After some preliminaries in Sections 2 and 3, the inverse interval matrix is defined in Section 4. Next we introduce matrices B_y defined for each ± 1-vector y and demonstrate their use for inverse matrix representation (Theorem 7.1) and for establishing finite formulae for the inverse interval matrix (Theorem 8.1). Then we present Coxson's result [3] showing that computing the inverse interval matrix is NP-hard. In Section 10 we show that for an interval matrix with unit midpoint the inverse interval matrix can be given explicitly by simple formulae (Theorem 10.2). Explicit formulae for an enclosure of the inverse of a strongly regular interval matrix are given in Section 11. In the next four sections, we give explicit formulae for the interval inverse of interval matrices that are either inverse sign stable (Section 12), or are of inverse sign pattern (Section 13), or are nonnegative invertible (Section 14), or have uniform width (Section 15). In the last Section 16, we describe available software

*Received by the editors on December 17, 2009. Accepted for publication on July 2, 2011. Handling Editor: Bryan Shader.

†Institute of Computer Science, Czech Academy of Sciences, Prague, and School of Business Administration, Anglo-American University, Prague, Czech Republic (rohn@cs.cas.cz). This author's work was supported by the Czech Republic Grant Agency under grants 201/09/1957 and 201/08/J020, and by the Institutional Research Plan AV0Z10300504.

‡Faculty of Basis Sciences, Hamadan Branch, Islamic Azad University, Hamadan, Iran (rftmath@gmail.com).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

for computing the inverse interval matrix or its enclosure. The Appendix contains a MATLAB-like description of an algorithm for solving an absolute value equation which is used in Section 6 for computation of the matrices B_y.

2. Notation. We use the following notation. A_{ij} denotes the ijth entry, $A_{i\bullet}$ the ith row and $A_{\bullet j}$ the jth column of a matrix A. Matrix inequalities, as $A \le B$ or $A < B$, are understood componentwise. $A \circ B$ denotes the Hadamard (entrywise) product of $A, B \in \mathbb{R}^{m\times n}$, i.e., $(A \circ B)_{ij} = A_{ij}B_{ij}$ for each i, j. The minimum (or maximum) matrix of a compact (in particular, finite) set of matrices X is defined componentwise, i.e.,

$$(\min\{\, A \mid A \in X \,\})_{ij} = \min\{\, A_{ij} \mid A \in X \,\},$$

$$(\max\{\, A \mid A \in X \,\})_{ij} = \max\{\, A_{ij} \mid A \in X \,\}$$

for each i, j. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. For each matrix A we define its sign matrix $\mathrm{sgn}(A)$ by

$$(\mathrm{sgn}(A))_{ij} = \begin{cases} 1 & \text{if } A_{ij} \ge 0, \\ -1 & \text{if } A_{ij} < 0 \end{cases}$$

for each i, j. The same notation also applies to vectors that are considered one-column matrices. I is the unit matrix, e_j is the jth column of I, $e = (1, \dots, 1)^T$ is the vector of all ones, and $E = ee^T$ is the matrix of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \mathrm{diag}\,(y_1, \dots, y_n) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix},$$

and $\varrho(A)$ is the spectral radius of A.

3. Interval matrices. Given two $n \times n$ matrices A_c and Δ, $\Delta \ge 0$, the set of matrices

$$\mathbf{A} = \{A \mid |A - A_c| \le \Delta\}$$

is called a (square) interval matrix with midpoint matrix A_c and radius matrix Δ. Since the inequality $|A - A_c| \le \Delta$ is equivalent to $A_c - \Delta \le A \le A_c + \Delta$, we can also write

$$\mathbf{A} = \{A \mid \underline{A} \le A \le \overline{A}\} = [\underline{A}, \overline{A}],$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

where $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$ are called the bounds of $\mathbf{A}$.

Given an $n \times n$ interval matrix $\mathbf{A}$, we define matrices

$$A_{yz} = A_c - T_y \Delta T_z \tag{3.1}$$

for each $y, z \in Y_n$. The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases} \qquad (i, j = 1, \ldots, n),$$

so that $A_{yz} \in \mathbf{A}$ for each $y, z \in Y_n$. Since the cardinality of Y_n is 2^n, the cardinality of the set of matrices $\{A_{yz} \mid y, z \in Y_n\}$ is at most 2^{2n}.

4. Definition of the inverse interval matrix. A square interval matrix $\mathbf{A}$ is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and it is said to be *singular* otherwise (i.e., if it contains a singular matrix). In particular, an interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ with

$$\varrho(|A_c^{-1}|\Delta) < 1 \tag{4.1}$$

is regular (Beeck [1]); interval matrices satisfying (4.1) are called *strongly regular*. The inverse interval matrix is defined only for regular interval matrices.

Definition 4.1. For a regular interval matrix $\mathbf{A}$ we define its inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ by

$$\underline{B} = \min\{ A^{-1} \mid A \in \mathbf{A} \},$$
$$\overline{B} = \max\{ A^{-1} \mid A \in \mathbf{A} \}$$

(componentwise).

Comment. Thus, $\mathbf{A}^{-1}$ is the narrowest interval matrix enclosing the set of matrices $\{ A^{-1} \mid A \in \mathbf{A} \}$. Instead of "inverse interval matrix", we sometimes simply say "interval inverse".

5. The matrices B_y. First we show that regularity of an $n \times n$ interval matrix implies the existence of 2^n uniquely determined matrices.

Theorem 5.1. [15, Thm. 5.1, (A3)] *For a square interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is regular,*
(ii) for each $y \in Y_n$ the matrix equation

$$A_c B - T_y \Delta |B| = I \tag{5.1}$$

has a unique matrix solution B_y,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

(iii) for each $y \in Y_n$ the matrix equation (5.1) has a solution.

The main message here is the implication "(i)⇒(ii)"; (iii) is added for completeness.

It is useful to formulate the equation (5.1) columnwise.

THEOREM 5.2. *Let $\mathbf{A}$ be regular. Then for each $y \in Y_n$ and for each $j \in \{1,\dots,n\}$ we have*

$$(B_y)_{\bullet j} = x_{yj},$$

where x_{yj} is the unique solution of the equation

$$A_c x - T_y \Delta |x| = e_j. \tag{5.2}$$

This theorem forms the basis of an algorithm for computing the B_y's presented in the next section. We have still another expression for the jth column of B_y by means of the matrices A_{yz} introduced in (3.1).

THEOREM 5.3. *Let $\mathbf{A}$ be regular. Then for each $y \in Y_n$ and for each $j \in \{1,\dots,n\}$ we have*

$$(B_y)_{\bullet j} = \left(A_{yz(j)}^{-1}\right)_{\bullet j}, \tag{5.3}$$

where

$$z(j) = \operatorname{sgn}((B_y)_{\bullet j}).$$

Since $z(j)$ depends on j, we cannot generally state that $B_y = A_{yz}^{-1}$ for some z. It may even be that $B_y^{-1} \notin \mathbf{A}$. As a consequence of (5.3), we obtain that

$$(B_y)_{ij} = \left(A_{yz(j)}^{-1}\right)_{ij} \tag{5.4}$$

for each y, i, j. Of course, (5.3) and (5.4) cannot be directly used for computation of $(B_y)_{\bullet j}$ since they contain $z(j)$, the sign vector of the result.

6. Computation of the B_y's. Theorem 5.2 shows us a way how to compute the matrix B_y column-by-column provided we are able to solve an equation of the type

$$Ax + B|x| = b, \tag{6.1}$$

called an *absolute value equation.* This can be done by a finite algorithm **signaccord** from [21] whose detailed MATLAB-like description is given in the Appendix. Its syntax is

$$[x, S, flag] = \textbf{signaccord}(A, B, b),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

where A, B, b is the data of (6.1), x is a solution of (6.1) (if found), S is a singular matrix in the interval matrix $[A - |B|, A + |B|]$ (if found), and $flag$ is a verbal description of the output ($'solution'$ or $'singular'$). The behavior of the algorithm is described in Theorem 17.1. Its important feature is that for a regular interval matrix $[A - |B|, A + |B|]$ it always finds a solution to (6.1) (in infinite precision arithmetic), which in this case is unique [21]. As reported in [21], the algorithm takes on average about $0.11 \cdot n$ steps (passes through the **while** loop), where n is the matrix size.

Solving the equations (5.2) for $j = 1, \ldots, n$, we obtain an algorithm (Fig. 6.1) for computing the matrix B_y for a given y.

function $B_y =$ **bymatrix** $(\mathbf{A}, y)$
for $j = 1 : n$
  $[x, S, flag] =$ **signaccord** $(A_c, -T_y\Delta, e_j)$;
  if $flag = {'singular'}$, $B_y = [\,]$; **return**
  end
  $(B_y)_{\bullet j} = x$;
end

FIG. 6.1. *An algorithm for computing B_y.*

The following theorem (unpublished) follows directly from Theorems 5.2 and 17.1.

THEOREM 6.1. *For each square interval matrix $\mathbf{A}$ and for each $y \in Y_n$ the algorithm (Fig. 6.1) either finds a matrix B_y satisfying (5.1), or issues an empty matrix B_y in which case $\mathbf{A}$ is singular.*

It should be noted that success in computation of a *single* matrix B_y does not guarantee regularity; it is the existence of solutions of *all* the equations (5.1), $y \in Y_n$ that implies regularity of $\mathbf{A}$ (Theorem 5.1, (iii)).

7. Inverse matrix representation theorem. The following theorem, which is of independent interest, brings us closer to the formulae for the inverse interval matrix to be given in the next section.

THEOREM 7.1. [15, Thm. 6.1] *Let $\mathbf{A}$ be regular. Then for each $A \in \mathbf{A}$ there exist nonnegative diagonal matrices L_y, $y \in Y_n$, satisfying $\sum_{y \in Y_n} L_y = I$ such that*

$$A^{-1} = \sum_{y \in Y_n} B_y L_y \tag{7.1}$$

holds.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

The formula (7.1) implies that for each i, j we have

$$(A^{-1})_{ij} = \sum_{y \in Y_n} (B_y)_{ij} (L_y)_{jj}, \tag{7.2}$$

where all the $(L_y)_{jj}$'s are nonnegative and $\sum_{y \in Y_n} (L_y)_{jj} = I_{jj} = 1$. Hence, $(A^{-1})_{ij}$ is a convex combination of the values $(B_y)_{ij}$ over all $y \in Y_n$.

Using the formula (5.3), we can reformulate the representation theorem in terms of the matrices A_{yz} defined in (3.1).

THEOREM 7.2. [18, Thm. 1.1] *Let* $\mathbf{A}$ *be regular. Then for each* $A \in \mathbf{A}$ *there exist nonnegative diagonal matrices* L_{yz}, $y, z \in Y_n$, *satisfying* $\sum_{y,z \in Y_n} L_{yz} = I$ *such that*

$$A^{-1} = \sum_{y,z \in Y_n} A_{yz}^{-1} L_{yz} \tag{7.3}$$

holds.

Hence, $(A^{-1})_{ij}$ is a convex combination of the values $(A_{yz}^{-1})_{ij}$ over all $y, z \in Y_n$. The expansion (7.3) is perhaps more clear than (7.1) because it employs explicitly expressed matrices A_{yz}^{-1} instead of rather obscure matrices B_y, but the number of matrices A_{yz}^{-1} is 2^{2n} compared to "only" 2^n matrices B_y.

8. Formulae for the inverse interval matrix. Finally, using (7.2) and (5.4), we obtain the following simply formulated, but important result.

THEOREM 8.1. [15, Thm. 6.2] *Let* $\mathbf{A}$ *be regular. Then its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = \min_{y \in Y_n} B_y,$$

$$\overline{B} = \max_{y \in Y_n} B_y.$$

Similarly, from Theorem 7.2 we can derive an analogous result.

THEOREM 8.2. [18, (1.3), (1.4)] *Let* $\mathbf{A}$ *be regular. Then its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = \min_{y,z \in Y_n} A_{yz}^{-1},$$

$$\overline{B} = \max_{y,z \in Y_n} A_{yz}^{-1}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

The formulation of Theorem 8.2 is advantageous in that it leads us to some clues about matrices at which bounds of the inverse interval matrix are attained.

THEOREM 8.3. [18, Thm. 1.2] *Let* $\mathbf{A}$ *be regular and let* $i, j \in \{1, \dots, n\}$. *Then we have:*

(i) $\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some* $y, z \in Y_n$ *satisfying*

$$y^T \circ (A_{yz}^{-1})_{i\bullet} \le 0^T, \tag{8.1}$$

$$z \circ (A_{yz}^{-1})_{\bullet j} \ge 0, \tag{8.2}$$

(ii) $\overline{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some* $y, z \in Y_n$ *satisfying*

$$y^T \circ (A_{yz}^{-1})_{i\bullet} \ge 0^T,$$

$$z \circ (A_{yz}^{-1})_{\bullet j} \ge 0.$$

For instance, the Hadamard product inequalities (8.1), (8.2) are equivalent to

$$y_k (A_{yz}^{-1})_{ik} \le 0 \qquad (k = 1, \dots, n), \tag{8.3}$$

$$z_h (A_{yz}^{-1})_{hj} \ge 0 \qquad (h = 1, \dots, n).$$

Thus, if we know in advance that, e.g., $\underline{B}_{ik} > 0$, then $(A_{yz}^{-1})_{ik} > 0$ for each $y, z \in Y_n$ and (8.3) implies that $y_k = -1$; similarly, if $\overline{B}_{ik} < 0$, then (8.3) gives $y_k = 1$. Hence, preliminary knowledge of the signs of the bounds may lead us to a reduction, sometimes significant, of the number of matrices A_{yz} to be inverted. We shall explore these ideas further in Section 12.

9. NP-hardness. The formulae given for the inverse interval matrix in Theorems 8.1 and 8.2 are inherently exponential. The question whether essentially simpler formulae may be found was answered in the negative by Coxson [3] who proved that computation of the inverse interval matrix is NP-hard.

THEOREM 9.1. [3] *The following problem is NP-hard:*

Instance. *A strongly regular interval matrix* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *with symmetric rational* A_c *and* Δ.

Question. *Is* $\overline{B}_{11} \ge 1$, *where* $[\underline{B}, \overline{B}] = \mathbf{A}^{-1}$ *?*

Hence, if the famous conjecture "P≠NP" is true, then there does not exist a polynomial-time algorithm for computing the interval inverse. In view of this fact, in what follows we shall concentrate on special classes of interval matrices for which the inverse can be computed by simpler means.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

10. Inverse of an interval matrix with unit midpoint. The first such a class is formed by interval matrices with unit midpoint, i.e., of the form $\mathbf{A} = [I-\Delta, I+\Delta]$. Such matrices are regular if and only if $\varrho(\Delta) < 1$ holds [22, Prop. 4.1], which is equivalent to

$$M := (I - \Delta)^{-1} \geq 0. \tag{10.1}$$

Hence, we assume that $\varrho(\Delta) < 1$ throughout this section. The main point here consists in the fact all the matrices B_y, $y \in Y_n$ can be described explicitly. The following theorem gives a general matrix formula (10.2) as well as three different componentwise formulae (10.3), (10.4), and (10.5). We use $M = (m_{ij})$ given by (10.1) and $\mu = (\mu_j)$ defined by

$$\mu_j = \frac{m_{jj}}{2m_{jj} - 1} \qquad (j = 1, \ldots, n).$$

Theorem 10.1. [22, Thm. 4.2] *Let $\varrho(\Delta) < 1$. Then for each $y \in Y_n$ the unique solution of the matrix equation*[1]

$$B - T_y \Delta |B| = I$$

is given by

$$B_y = T_y M T_y + T_y(M - I)T_\mu(I - T_y), \tag{10.2}$$

i.e., componentwise

$$(B_y)_{ij} = y_i y_j m_{ij} + y_i(1 - y_j)(m_{ij} - I_{ij})\mu_j, \tag{10.3}$$

or

$$(B_y)_{ij} = \begin{cases} y_i m_{ij} & \text{if } y_j = 1, \\ y_i(2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases} \tag{10.4}$$

or

$$(B_y)_{ij} = \frac{(y_i + (1 - y_i)I_{ij})m_{ij}}{y_j + (1 - y_j)m_{jj}} \tag{10.5}$$

$(i, j = 1, \ldots, n)$.

Using Theorem 8.1, we obtain simple formulae for the interval inverse in this case.

[1] This is the equation (5.1) with $A_c = I$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

THEOREM 10.2. [22, Thm. 4.3] *Let* $\mathbf{A} = [I - \Delta, I + \Delta]$ *with* $\varrho(\Delta) < 1$. *Then the inverse interval matrix* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\begin{aligned} \underline{B} &= -M + T_\kappa, \\ \overline{B} &= M, \end{aligned} \tag{10.6}$$

where

$$\kappa_j = \frac{2m_{jj}^2}{2m_{jj} - 1} \qquad (j = 1, \ldots, n),$$

or componentwise

$$\underline{B}_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \mu_j & \text{if } i = j, \end{cases}$$
$$\overline{B}_{ij} = m_{ij}$$

$(i, j = 1, \ldots, n)$.

In particular, we have this consequence.

THEOREM 10.3. [22, Cor. 4.4] *If* $\varrho(\Delta) < 1$, *then the inverse interval matrix* $[I - \Delta, I + \Delta]^{-1} = [\underline{B}, \overline{B}]$ *satisfies*

$$\tfrac{1}{2} \leq \underline{B}_{jj} \leq 1 \leq \overline{B}_{jj}$$

for each j.

According to (10.6), $\overline{B} = (I - \Delta)^{-1}$. The last theorem of this section reveals at what matrices the entries of $\underline{B}$ are attained.

THEOREM 10.4. [22, Thm. 5.1] *For each* i, j *we have:*

(i) if $i \neq j$, *then*

$$\underline{B}_{ij} = (I - T_y \Delta T_y)^{-1}_{ij}$$

for each $y \in Y$ *satisfying* $y_i y_j = -1$,

(ii) if $i = j$, *then*

$$\underline{B}_{jj} = (I - T_y \Delta T_z)^{-1}_{jj}$$

for each $y \in Y$ *satisfying* $y_j = -1$ *and* $z = y + 2e_j$.

11. Enclosure of the inverse interval matrix. An interval matrix $\mathbf{C}$ is called an *enclosure* of $\mathbf{A}^{-1}$ if $\mathbf{A}^{-1} \subseteq \mathbf{C}$. Computation of an enclosure of the inverse of a strongly regular interval matrix can be performed in polynomial time, as shown in

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

the following theorem which is a follow-up of previous results by Hansen [8], Bliek [2] and Rohn [17] on interval linear equations.

THEOREM 11.1. [4, Thm. 2.40] *Let* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *be strongly regular. Then we have*

$$\mathbf{A}^{-1} \subseteq [\underline{\underline{B}}, \overline{\overline{B}}],$$

where

$$\begin{aligned}
M &= (I - |A_c^{-1}|\Delta)^{-1},\\
\mu &= (M_{11}, \ldots, M_{nn})^T,\\
T_\nu &= (2T_\mu - I)^{-1},\\
\underset{\sim}{B} &= -M|A_c^{-1}| + T_\mu(A_c^{-1} + |A_c^{-1}|),\\
\widetilde{B} &= \quad M|A_c^{-1}| + T_\mu(A_c^{-1} - |A_c^{-1}|),\\
\underline{\underline{B}} &= \min\{\underset{\sim}{B}, T_\nu \underset{\sim}{B}\},\\
\overline{\overline{B}} &= \max\{\widetilde{B}, T_\nu \widetilde{B}\}.
\end{aligned}$$

Other types of enclosures were studied by Hansen [6], Hansen and Smith [7], Herzberger and Bethke [11], and Herzberger [9], [10].

The preliminary knowledge of an enclosure may make computation of the interval inverse easier, see Theorem 12.2 below.

12. Inverse sign stability. Let Z be a matrix satisfying $|Z| = E$, i.e., a ± 1-matrix. We say that a regular interval matrix $\mathbf{A}$ is *inverse Z-stable* if

$$Z \circ A^{-1} > 0$$

holds for each $A \in \mathbf{A}$. This means that for each i, j, either $(A^{-1})_{ij} < 0$ for each $A \in \mathbf{A}$ (if $Z_{ij} = -1$), or $(A^{-1})_{ij} > 0$ for each $A \in \mathbf{A}$ (if $Z_{ij} = 1$). We say simply that $\mathbf{A}$ is inverse sign stable if it is inverse Z-stable for some Z.

We have the following finite characterization.

THEOREM 12.1. [18, Thm. 2.1] $\mathbf{A}$ *is inverse Z-stable if and only if each A_{yz} is nonsingular and*

$$Z \circ A_{yz}^{-1} > 0 \tag{12.1}$$

holds for each $y, z \in Y_n$.

Notice that regularity of $\mathbf{A}$ is not assumed; it follows from (12.1). The next theorem gives a sufficient inverse Z-stability condition verifiable in polynomial time.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

THEOREM 12.2. [Improved version of [18], Thm. 2.2] *If* $\mathbf{A}$ *is strongly regular and if*

$$\underline{\underline{B}} \circ \overline{\overline{B}} > 0$$

holds, where $\underline{\underline{B}}, \overline{\overline{B}}$ *are as in Theorem 11.1, then* $\mathbf{A}$ *is inverse* Z*-stable, where* $Z = \operatorname{sgn}(\underline{\underline{B}})$.

The main reason for introducing inverse Z-stable matrices is the following theorem which gives explicit componentwise formulae for entries of the bounds of the inverse interval matrix. It is an easy consequence of Theorem 8.3.

THEOREM 12.3. [18, Thm. 2.3] *Let* $\mathbf{A}$ *be inverse* Z*-stable. Then the bounds of its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *are given by the explicit formulae*

$$\begin{aligned} \underline{B}_{ij} &= (A^{-1}_{-y(i),z(j)})_{ij} \\ \overline{B}_{ij} &= (A^{-1}_{y(i)z(j)})_{ij} \qquad (i, j = 1, \ldots, n), \end{aligned}$$

where $y(i) = \operatorname{sgn}((Z_{i\bullet})^T)$ *and* $z(j) = \operatorname{sgn}(Z_{\bullet j})$ *for each* i, j.

13. Inverse sign pattern. Let $\mathbf{A}$ be regular. If there exist (fixed) $z, y \in Y_n$ such that

$$(zy^T) \circ A^{-1} \geq 0 \tag{13.1}$$

for each $A \in \mathbf{A}$, then $\mathbf{A}$ is said to be of the *inverse sign pattern* (z, y). In other words, for each i, j we have $z_i y_j (A^{-1})_{ij} \geq 0$ for each $A \in \mathbf{A}$, so that $z_i y_j$ prescribes the sign of $(A^{-1})_{ij}$. If strict inequality holds in (13.1), then $\mathbf{A}$ is inverse zy^T-stable. The property (13.1) can be succinctly reformulated as

$$T_z A^{-1} T_y \geq 0$$

for each $A \in \mathbf{A}$. It is a rather surprising fact that for both the characterization and the explicit form of interval inverse we need only two matrices in this case, namely A^{-1}_{yz} and $A^{-1}_{-y,z}$.

THEOREM 13.1. [15, Thm. 4.6] $\mathbf{A}$ *is of the inverse sign pattern* (z, y) *if and only if* A_{yz} *and* $A_{-y,z}$ *are nonsingular and*

$$T_z A^{-1}_{yz} T_y \geq 0, \tag{13.2}$$

$$T_z A^{-1}_{-y,z} T_y \geq 0 \tag{13.3}$$

hold[2].

[2] Which implicitly asserts that the two conditions (13.2) and (13.3) imply regularity of $\mathbf{A}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

The following theorem has not been published so far.

THEOREM 13.2. *If* $\mathbf{A}$ *is of the inverse sign pattern* (z, y)*, then its inverse interval matrix is given by*

$$\mathbf{A}^{-1} = [\min\{A_{yz}^{-1}, A_{-y,z}^{-1}\}, \max\{A_{yz}^{-1}, A_{-y,z}^{-1}\}]. \tag{13.4}$$

See Garloff [5] for the special case of $y = z = (1, -1, 1, -1, \dots, (-1)^{n-1})^T$.

14. Nonnegative invertibility. An interval matrix $\mathbf{A}$ is said to be *nonnegative invertible* if it is of the inverse sign pattern (e, e), i.e., if

$$A^{-1} \geq 0$$

holds for each $A \in \mathbf{A}$. As immediate consequences of Theorems 13.1 and 13.2, we obtain the following two results.

THEOREM 14.1. [12] $\mathbf{A}$ *is nonnegative invertible if and only if* $\underline{A}^{-1} \geq 0$ *and* $\overline{A}^{-1} \geq 0$.

THEOREM 14.2. [13] *If* $\mathbf{A} = [\underline{A}, \overline{A}]$ *is nonnegative invertible, then*

$$\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}]. \tag{14.1}$$

The last formula follows from the fact that $\underline{A}^{-1} - \overline{A}^{-1} = \underline{A}^{-1}(\overline{A} - \underline{A})\overline{A}^{-1} \geq 0$ which gives $\underline{A}^{-1} \geq \overline{A}^{-1}$, hence (13.4) implies (14.1). Finally, we have the following inverse expansion theorem.

THEOREM 14.3. [14, Thm. 2] *If* $\mathbf{A}$ *is inverse nonnegative, then for each* $A \in \mathbf{A}$ *there holds*

$$A^{-1} = \Big(\sum_{j=0}^{\infty}(\overline{A}^{-1}(\overline{A} - A))^j\Big)\overline{A}^{-1}.$$

15. Uniform width. An interval matrix $\mathbf{A}$ is said to be of *uniform width* if it is of the form

$$\mathbf{A} = [A_c - \alpha E, A_c + \alpha E] \tag{15.1}$$

for some $\alpha \geq 0$. For sufficiently small α, its inverse can be again expressed explicitly. Let us denote

$$c = |A_c^{-1}|e,$$
$$d = |A_c^{-1}|^T e.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

Theorem 15.1. [16, Thm. 2] *Let A_c be nonsingular and let $\alpha \geq 0$ satisfy*

$$\alpha(cd^T + \|c\|_1 |A_c^{-1}|) < |A_c^{-1}|. \tag{15.2}$$

Then for the interval inverse $[\underline{B}, \overline{B}]$ of (15.1) we have

$$\begin{aligned}
\underline{B}_{ij} &= (A_c^{-1})_{ij} - \frac{\alpha c_i d_j}{1 + \alpha z(j)^T A_c^{-1} y(i)},\\
\overline{B}_{ij} &= (A_c^{-1})_{ij} + \frac{\alpha c_i d_j}{1 - \alpha z(j)^T A_c^{-1} y(i)} \qquad (i, j = 1, \ldots, n),
\end{aligned}$$

where

$$\begin{aligned}
y(i) &= \operatorname{sgn}(((A_c^{-1})_{i\bullet})^T),\\
z(j) &= \operatorname{sgn}((A_c^{-1})_{\bullet j}).
\end{aligned}$$

The condition (15.2) provides for both strong regularity and inverse sign stability of $\mathbf{A}$.

16. Software. The freely available verification software package VERSOFT [26] written in INTLAB [23], [24], a toolbox of MATLAB, contains a file VERINVERSE.M [25] for computing a verified inverse of a square interval matrix. Its syntax is

```
[B,S] = verinverse(A),
```

where A is an interval matrix, B is its verified interval inverse (if found), and S is a very tight interval matrix which is a part of A and is verified to contain a singular matrix in A (if found). B and S are never assigned numerical values simultaneously; at least one of them is a matrix of *NaN*'s as the two options - regularity and singularity - exclude each other. The interval matrix B, if computed, is verified to contain the interval inverse of A and the overestimation is solely due to the outward rounding committed; in infinite precision arithmetic it would compute the exact interval inverse. It is based on a not-a-priori-exponential algorithm **hull** for solving interval linear equations described in [19]; its theoretical basis and implementation details have not been published. Nevertheless, the computation may occasionally last long as the problem is NP-hard (Theorem 9.1). In such cases we recommend computation of a polynomial-time enclosure described in Theorem 11.1. This enclosure has not been included into VERSOFT. INTLAB users may employ the function INV.M adapted for an interval argument by S.M. Rump [24].

17. Appendix: An algorithm for solving the absolute value equation. This appendix contains a MATLAB-like description of an algorithm for solving the absolute value equation accompanied by a finite termination theorem. Both these results were referred to in Section 6.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

THEOREM 17.1. [21, Thm. 3.1] *For each $A, B \in \mathbb{R}^{n\times n}$ and each $b \in \mathbb{R}^n$, the sign accord algorithm (Fig. 17.1) in a finite number of steps either finds a solution of the equation*

$$Ax + B|x| = b,$$

or states singularity of the interval matrix $[A-|B|, A+|B|]$ (and, in most cases, also finds a singular matrix $S \in [A-|B|, A+|B|]$).

function $[x, S, flag] =$ **signaccord** (A, B, b)
% Finds a solution to $Ax + B|x| = b$ or states
% singularity of $[A - |B|, A + |B|]$.
$x = [\,]$; $S = [\,]$; $flag = {'singular'}$;
if A is singular, $S = A$; **return**, **end**
$p = 0 \in \mathbb{R}^n$;
$z = \mathrm{sgn}(A^{-1}b)$;
if $A + BT_z$ is singular, $S = A + BT_z$; **return**, **end**
$x = (A + BT_z)^{-1}b$;
$C = -(A + BT_z)^{-1}B$;
while $z_jx_j < 0$ for some j
    $k = \min\{j \mid z_jx_j < 0\}$;
    if $1 + 2z_kC_{kk} \le 0$
        $S = A + B(T_z + (1/C_{kk})e_ke_k^T)$;
        $x = [\,]$;
        return
    end
    $p_k = p_k + 1$;
    if $\log_2 p_k > n - k$, $x = [\,]$; **return**, **end**
    $z_k = -z_k$;
    $\alpha = 2z_k/(1 - 2z_kC_{kk})$;
    $x = x + \alpha x_kC_{\bullet k}$;
    $C = C + \alpha C_{\bullet k}C_{k\bullet}$;
end
$flag = {'solution'}$;

FIG. 17.1. *The sign accord algorithm [21].*

Acknowledgment. The authors thank the referee for helpful suggestions that resulted in essential improvement of the text of the paper.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

ELA

REFERENCES

[1] H. Beeck. Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen. In K. Nickel, editor, *Interval Mathematics*, Lecture Notes in Computer Science, Springer-Verlag, Berlin, 29:150–159, 1975.

[2] C. Bliek. *Computer Methods for Design Automation*. PhD Thesis, Massachusetts Institute of Technology, Cambridge, MA, July 1992.

[3] G.E. Coxson. Computing exact bounds on elements of an inverse interval matrix is NP-hard. *Reliable Computing*, 5:137–142, 1999.

[4] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann. *Linear Optimization Problems with Inexact Data*. Springer-Verlag, New York, 2006.

[5] J. Garloff. Totally nonnegative interval matrices. In K. Nickel, editor, *Interval Mathematics 1980*, Academic Press, New York, 317–327, 1980. .

[6] E. Hansen. Interval arithmetic in matrix computations, Part I. *SIAM Journal on Numerical Analysis*, 2:308–320, 1965.

[7] E. Hansen and R. Smith. Interval arithmetic in matrix computations, Part II. *SIAM Journal on Numerical Analysis*, 4:1–9, 1967.

[8] E.R. Hansen. Bounding the solution of interval linear equations. *SIAM Journal on Numerical Analysis*, 29:1493–1503, 1992.

[9] J. Herzberger. Bemerkungen zur Konvergenz eines Iterationsverfahrens zur Einschließung der Inversen einer Intervallmatrix. (Remarks on the convergence of an iteration method for the inclusion of the inverse of an interval matrix.) *Zeitschrift für Angewandte Mathematik und Mechanik*, 70(6):555–557, 1990.

[10] J. Herzberger. On the efficiency of an iterative method for enclosing the inverse of an interval matrix. (Über die Wirksamkeit eines Iterationsverfahrens zur Einschließung der Inversen einer Intervallmatrix.) *Zeitschrift für Angewandte Mathematik und Mechanik*, 70(10):470–472, 1990.

[11] J. Herzberger and D. Bethke. On two algorithms for bounding the inverse of an interval matrix. *Interval Computations*, 1:44–53, 1991.

[12] J. Kuttler. A fourth-order finite-difference approximation for the fixed membrane eigenproblem. *Mathematics of Computation*, 25:237–256, 1971.

[13] A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, 1990.

[14] J. Rohn. Inverse-positive interval matrices. *Zeitschrift für Angewandte Mathematik und Mechanik*, 67:T492–T493, 1987.

[15] J. Rohn. Systems of linear interval equations. *Linear Algebra and its Applications*, 126:39–78, 1989.

[16] J. Rohn. Sensitivity of a system of linear equations to fixed-point data rounding. Unpublished manuscript, 1990. Available at http://www.cs.cas.cz/ rohn/publist/sensitivity.doc.

[17] J. Rohn. Cheap and tight bounds: The recent result by E. Hansen can be made more efficient. *Interval Computations*, 4:13–21, 1993.

[18] J. Rohn. Inverse interval matrix. *SIAM Journal on Numerical Analysis*, 30:864–870, 1993.

[19] J. Rohn. A handbook of results on interval linear problems, 2005. Internet text available at http://www.cs.cas.cz/rohn/handbook.

[20] J. Rohn. Forty necessary and sufficient conditions for regularity of interval matrices: A survey. *Electronic Journal of Linear Algebra*, 18:500–512, 2009.

[21] J. Rohn. An algorithm for solving the absolute value equation. *Electronic Journal of Linear Algebra*, 18:589–599, 2009.

[22] J. Rohn. Explicit inverse of an interval matrix with unit midpoint. *Electronic Journal of Linear Algebra*, 22:138–150, 2011.

[23] S.M. Rump. INTLAB - INTerval LABoratory. In T. Csendes, editor, *Developments in Reliable*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 704-719, July 2011

Computing, Kluwer Academic Publishers, Dordrecht, 77–104, 1999.

[24] S.M. Rump. INTLAB, 2009. Available at http://www.ti3.tu-harburg.de/rump/intlab.

[25] VERINVERSE: Verified inverse interval matrix, 2007. Available at http://www.cs.cas.cz/rohn/matlab/verinverse.html.

[26] VERSOFT: Verification software in MATLAB/INTLAB, 2009. Available at http://www.cs.cas.cz/rohn/matlab.