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ON SINGULAR DISTANCE MATRICES OF UNICYCLIC GRAPHS*

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Abstract. A unicyclic graph is a connected graph having exactly one cycle. It is known that the distance matrix $D(G)$ of a unicyclic graph G is nonsingular if and only if the cycle in G is of odd length. Motivated by the inverse formula for a nonsingular $D(G)$, in this paper, we establish an explicit Moore–Penrose inverse formula for the distance matrix of a unicyclic graph with even cycle. This formula is expressed as the sum of a symmetric Laplacian-like matrix and a rank one matrix. As consequences, we study the existence of an eigenvalue of $D(G)$ and deduce a known formula for the inertia of $D(G)$ when the cycle in G is of even length.

Key words. Distance matrix, Unicyclic graph, Moore–Penrose inverse, Eigenvalue.

AMS subject classifications. 05C12, 05C50, 15A09.

1. Introduction. Let G be a simple connected graph with the vertex set $V_G = \{v_1, v_2, \dots, v_n\}$. The *distance* between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of a shortest path between v_i and v_j. The *distance matrix* $D(G)$ of G is an $n \times n$ symmetric matrix whose (i, j)-th entry is $d(v_i, v_j)$. This matrix has been extensively studied in the literature and has applications in chemistry, communication networks, psychology, etc., see [2, 4, 23] and the references therein.

Beginning with the seminal work of Graham and Pollak [10] and Graham and Lovász [9], the problem of finding the determinant, eigenvalues, inverse (if it exists), and inertia of $D(G)$ has received much attention in the literature. Let us present some results that are pertinent to our discussion. It has been proved in [10] that the determinant of $D(T_n)$ is $(-1)^{n-1}(n-1)2^{n-1}$, where T_n is a tree of order n. This shows that $D(T_n)$ is nonsingular when $n > 1$. Also, the inertia (see definition in Section 4) of $D(T_n)$ is derived in [10]. These results have been extended to the distance matrix of a weighted tree [5]. In a recent paper [8], the authors studied certain generalized distance matrices of directed trees with edgeweights in a unital commutative ring. Among other results, they derived the determinants of these matrices that generalize the determinant formulae for the distance matrices of trees and weighted trees proved in [10] and [5], respectively.

The inverse of $D(T_n)$, denoted by $D(T_n)^{-1}$, is derived in [9], which is completely determined by the degrees of the vertices of T_n and adjacencies between vertices of T_n. To state the precise result, we need the notion of the Laplacian matrix of a graph. We recall this next. The *Laplacian matrix* $L(G) := (l_{ij})$ of G is an $n \times n$ symmetric matrix such that $l_{ii} = \deg(v_i)$ and the off-diagonal entry l_{ij} is either -1 or 0 depending on whether the vertices v_i and v_j are adjacent or not adjacent in G, where $\deg(v_i)$ is the degree of the vertex v_i. Clearly, the row sums of $L(G)$ are equal to zero. The inverse formula for $D(T_n)$ is given by

$$D(T_n)^{-1} = \frac{-1}{2} L(T_n) + \frac{1}{2(n-1)} \tau\tau', \tag{1.1}$$

where τ is the column vector of order n whose i-th component is $2 - \deg(v_i)$.

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Motivated by this, similar inverse formulae have been obtained for distance matrices of weighted trees [5], block graphs [6], cycle-clique graphs with odd cycles [14], biblock graphs [15], and cactoid graphs [13]. In all the cases, $L(T_n)$ and τ in (1.1) are replaced by appropriate Laplacian-like matrices and column vectors, respectively. A square matrix L is said to be *Laplacian-like* if its row sums and column sums are equal to zero [28]. This generalizes the notion of the Laplacian matrix of a graph.

We now focus on the Moore–Penrose inverses of the matrices associated with graphs. An $n \times m$ matrix X is said to be the *Moore–Penrose inverse* of an $m \times n$ matrix A, if it satisfies the conditions $XAX = X$, $AXA = A$, and AX and XA are symmetric. It is well known that the Moore–Penrose inverse of A, denoted by $A^{\dagger}$, always exists and is unique. Moreover, if A is nonsingular, then $A^{\dagger} = A^{-1}$. For applications of Moore–Penrose inverses in systems of equations, least-square solutions, mathematical programming problems, etc., see [7]. Several researchers have investigated the problem of finding the Moore–Penrose inverses of matrices associated with graphs, including the distance matrix [3, 19], adjacency matrix [20, 22], incidence matrix [12], Laplacian matrix [21], and signless Laplacian matrix [1, 11].

Analogous to the inverse formula given in (1.1), the Moore–Penrose inverses of the singular distance matrices of wheel graphs and helm graphs have been established in [3] and [19], respectively. That is, for each of the graphs mentioned above, there exist a Laplacian-like matrix L, a nonzero scalar α, and a column vector $\mathbf{w}$ such that

$$D(G)^{\dagger} = \frac{-1}{2} L + \alpha \mathbf{w}\mathbf{w}'. \tag{1.2}$$

Before proceeding further, we would like to mention the significance of the formula (1.2) proved in [18] for a general real square matrix M. Under an assumption, the invertibility of M is characterized in terms of the cofactor of any element of L, and if M is invertible then a formula to find the determinant of M is provided (Theorems 2.5 and 2.6, [18]).

For a general graph G, finding an elegant formula for $D(G)^{\dagger}$ is a challenging problem, and it remains open. So, it would be natural to restrict our attention to certain graph classes. In this paper, we consider the class of unicyclic graphs and derive the Moore–Penrose inverses of singular distance matrices of graphs in this class (Theorem 3.4). As far as we know, this is the first attempt to provide the Moore–Penrose inverse formulae, similar to (1.2), for the singular distance matrices of all graphs in a class.

Recall that a connected graph is *unicyclic* if it has exactly one cycle. Several studies have been conducted on different aspects of this graph. For instance, Bapat et al. [5] gave the determinants and the inertias of distance matrices of unicyclic graphs. Following this, Zhang [27] obtained the determinants and the inertias of distance matrices of the line graphs of unicyclic graphs and using these results, he also identified the graphs attaining the extreme distance energies among these graphs. Hessert and Mallik derived the Moore–Penrose inverse for the incidence matrix of a unicyclic graph [12]. The problem of determining the graphs attaining extremal Wiener index among unicyclic graphs with different graph parameters has been studied in [24], [26] and references therein. Hou et al. [14] provided the inverse of the nonsingular distance matrix of a cycle-clique graph which includes a unicyclic graph with the cycle of odd length.

We next state some applications of the Moore–Penrose inverse formula derived in this paper. Let $\mathbb{R}^p$ denote the space of all real column vectors with p coordinates. An $n \times n$ real symmetric matrix $A = (a_{ij})$ is said to be an *Euclidean distance matrix*, abbreviated as EDM, if there exist vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^p$ such that $a_{ij} = \|\mathbf{a}_i - \mathbf{a}_j\|^2$, for $i, j = 1, 2, \ldots, n$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^p$ and p need not be equal to n. It has been shown that the distance matrices of trees, block graphs, cycles, wheel graphs, and

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odd-unicyclic graphs are EDMs (see [16, 17, 18]). Using the formula given in Theorem 3.4, we show that a singular $D(G)$ of a unicyclic graph G is an EDM (Theorem 4.3). As mentioned earlier, the inertia of a singular $D(G)$ is derived in [5]. It is shown here that this result can be deduced from Theorem 4.3.

A vertex v of G is called *pendant* if $\deg(v)$ is equal to 1 and is said to be *quasipendant* if there is a pendant vertex u in G such that u is adjacent to v. We use the notations $P(G)$ and $Q(G)$ to denote the number of pendant and quasipendant vertices in G, respectively. For the distance matrix of a tree T_n, it has been shown that -2 is an eigenvalue with multiplicity at least $P(T_n) - Q(T_n) - 1$, see [4]. We prove that an analogous result holds for the distance matrix of a unicyclic graph G using the formula given for $D(G)^\dagger$, see Theorem 4.8.

The paper is organized as follows. In the following section, we collect certain definitions, notations, and results, which will be used in the sequel. In Section 3, we obtain the Moore–Penrose inverse formula for the distance matrix of a unicyclic graph with an even cycle. This formula is given as the sum of a constant multiple of a symmetric Laplacian-like matrix and a rank one matrix. Section 4 deals with some consequences of the formula presented in this paper.

2. Preliminaries. In this section, we fix some notations and recall a few known results that are required in subsequent sections. Throughout this paper, we denote vectors by lowercase boldface letters. We use the notations $\mathbf{0}$, $\mathbf{e}$, and $\mathbf{e}_i$, respectively, to represent the column vectors whose entries are all zero, all one, and 1 in i-th coordinate and 0 elsewhere. As usual, O is the zero matrix, I is the identity matrix, and J is the all ones matrix. For a matrix M, we use the notations M_{i*}, M_{*j}, $(M)_{ij}$ and M' to denote the i-th row, j-th column, (i,j)-th entry, and transpose of M, respectively.

We next discuss the notion and some results on circulant matrices which will be used frequently. For a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)' \in \mathbb{R}^n$, the *circulant matrix*, denoted by $\mathrm{Circ}(\mathbf{x}')$, is given by

$$\mathrm{Circ}(\mathbf{x}') = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_n & x_1 & x_2 & \dots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \dots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \dots & x_1 \end{bmatrix}.$$

Let $a, b \in \mathbb{R}$. Suppose that $A = \mathrm{Circ}(\mathbf{x}')$ and $B = \mathrm{Circ}(\mathbf{y}')$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$(i)\ AB = BA, \quad (ii)\ AB = \mathrm{Circ}(\mathbf{x}'B) \quad \text{and} \quad (iii)\ \mathrm{Circ}(a\mathbf{x}' + b\mathbf{y}') = aA + bB. \tag{2.1}$$

The next result presents the eigenvalues of a circulant matrix.

Lemma 2.1 ([25]). *Let C be the circulant matrix generated by a vector $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$. Then the eigenvalues of C are $f(\omega_j)$, $j = 0, 1, \dots, n-1$ where $\omega_j = \exp\left(\frac{i2\pi j}{n}\right) = \cos(\frac{2\pi j}{n}) + i\sin(\frac{2\pi j}{n})$ and $f(x) = c_1 + c_2 x + \dots + c_n x^{n-1}$.*

We wish to point out that a circulant matrix need not be symmetric in general. However, it is symmetric if the defining vector satisfies a symmetric property which we define below. Let $n \geq 3$. We say that the vector $\mathbf{x}$ follows symmetry in its last $n-1$ coordinates if $x_i = x_{n+2-i}$ for all $i = 2, 3, \dots, n$. In particular, if n is even, then $\mathbf{x} = (x_1, x_2, x_3, \dots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}}, \dots, x_3, x_2)$. It is shown that if $\mathbf{x}$ follows symmetry in its last $n-1$ coordinates, then $\mathrm{Circ}(\mathbf{x}')$ is a symmetric matrix (see Theorem 10 in [17]). The converse is also true.

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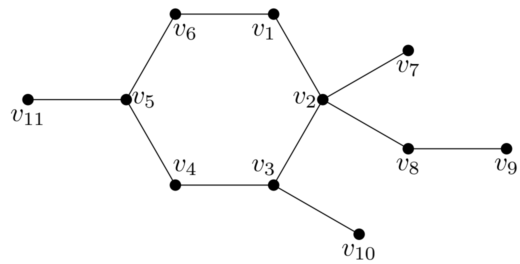

Figure 1: An example of a unicyclic graph.

Next, we present certain graph-theoretic notions and terminologies. We use the notation $v_i \sim v_j$ if the vertices v_i and v_j in G are adjacent. A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Let v be a vertex of G. The *degree* of v, denoted by $\deg(v)$, is the number of vertices in G adjacent to v, and the *block degree* $\widehat{d_G}(v)$ of v is the number of blocks of G containing v.

Let $G(m,l)$ denote a unicyclic graph of order $m+l$ consisting of the unique cycle C_m on m vertices and l vertices which do not lie in C_m. Note that there is a one-to-one correspondence between these l vertices and edges of $G(m,l)$, which are not present in C_m. We say that $G(m,l)$ is odd-unicyclic (even-unicyclic) if m is odd (respectively, even). Recall that the vertex set of $G(m,l)$ is $\{v_1, v_2, \dots, v_n\}$. Then $n = m + l$. Without loss of generality, assume that the first m vertices are in C_m and are labeled as follows: the vertex v_i is adjacent to the vertices v_{i-1} and v_{i+1} (the indices are taken modulo m). A simple example of a unicyclic graph $G(6,5)$ on 11 vertices with the cycle C_6 is depicted in Fig. 1.

The main objective of this article is to obtain the Moore–Penrose inverse formula for the distance matrix of an even-unicyclic graph. In the following, we state a key result which is used to derive the aforementioned formula. This result provides a necessary and sufficient conditions for a symmetric matrix to have its Moore–Penrose inverse as the sum of a symmetric Laplacian-like matrix and a rank one matrix.

THEOREM 2.2 ([19]). *Let M be a symmetric matrix of order n such that $M\mathbf{x} = \mathbf{e}$ has a solution. Suppose that α is a nonzero real number, L is a symmetric Laplacian-like matrix, and $\mathbf{w} \in \mathbb{R}^n$ satisfying $\mathbf{e}'\mathbf{w} = 1$. Then $M^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ if and only if $M\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ and the matrix $V = LM + 2I - 2\mathbf{w}\mathbf{e}'$ is a symmetric matrix satisfying $MV = O$ and $V(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$.*

3. Moore–Penrose inverse of the distance matrix of an even-unicyclic graph. It has been shown that the distance matrix of a unicyclic graph G is nonsingular if and only if the cycle in G is of odd length [5, 14]. Motivated by the inverse formulae for the distance matrices of trees and block graphs, Hou et al. [14] obtained the inverse formula, similar to (1.2), for the distance matrix of an odd-unicyclic graph. In fact, the result is proved for the more general case of a cycle-clique graph with odd cycles [14]. Inspired by this, in this section, we obtain the Moore–Penrose inverse of the distance matrix of an even-unicyclic graph in the form similar to (1.2).

We first establish the Moore–Penrose inverse of the distance matrix of the cycle C_{2k} where $k \geq 2$. To derive the formula, we define a scalar $\widetilde{\alpha} \in \mathbb{R}$, a $2k$-dimensional vector $\widetilde{\mathbf{w}}$ and a $2k \times 2k$ matrix $\mathcal{L}$ as follows:

$$\widetilde{\alpha} = \frac{2}{k}, \widetilde{\mathbf{w}} = \frac{1}{2k}\mathbf{e}, \text{ and } \mathcal{L} = \frac{1}{4}\mathrm{Circ}(\widetilde{\mathbf{z}}'), \tag{3.1}$$

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where $\widetilde{\mathbf{z}} = 2(\mathbf{e}_1 - \mathbf{e}_{k+1}) - (\mathbf{e}_2 + \mathbf{e}_{2k}) + (\mathbf{e}_k + \mathbf{e}_{k+2})$. Since the vector $\widetilde{\mathbf{z}}$ follows symmetry in its last $2k-1$ coordinates, the matrix $\mathcal{L}$ is symmetric.

THEOREM 3.1. *Let $D(C_{2k})$ be the distance matrix of the cycle C_{2k}, where $k \geq 2$. Let $\widetilde{\alpha}, \widetilde{\mathbf{w}}$ and $\mathcal{L}$ be defined as in (3.1). Then $D(C_{2k})^{\dagger} = -\frac{1}{2}\mathcal{L} + \widetilde{\alpha}\ \widetilde{\mathbf{w}}\widetilde{\mathbf{w}}'$.*

Proof. We prove the result by using Theorem 2.2. Let $\mathcal{D} = D(C_{2k})$. Then $\mathcal{D} = \mathrm{Circ}(\mathbf{d}')$, where $\mathbf{d} = (0,1,2,\ldots,k-1,k,k-1,\ldots,2,1)'$. Since $\widetilde{\mathbf{w}} = \frac{1}{2k}\mathbf{e} \in \mathbb{R}^{2k}$, we have $\mathbf{e}'\widetilde{\mathbf{w}} = 1$ and $\mathcal{D}\widetilde{\mathbf{w}} = (1/\widetilde{\alpha})\mathbf{e}$. Let $\widetilde{V} = \mathcal{L}\mathcal{D} + 2I - 2\widetilde{\mathbf{w}}\mathbf{e}'$. We claim that $\widetilde{V}$ is a symmetric circulant matrix. First, let us compute $\mathcal{L}\mathcal{D}$. Since both $\mathcal{L}$ and $\mathcal{D}$ are circulant matrices generated by $\frac{1}{4}\widetilde{\mathbf{z}}$ and $\mathbf{d}$, respectively, we have $\mathcal{L}\mathcal{D} = \frac{1}{4}\mathrm{Circ}(\widetilde{\mathbf{z}}'\mathcal{D})$. Note that $\mathcal{D}_{i*} + \mathcal{D}_{(k+i)*} = k\mathbf{e}'$ for any $i = 1,2,\ldots,k$. This implies that $\mathcal{D}_{i*} - \mathcal{D}_{(k+i)*} = 2\mathcal{D}_{i*} - k\mathbf{e}'$. Now,

$$\begin{aligned}
\widetilde{\mathbf{z}}'\mathcal{D} &= 2(\mathcal{D}_{1*} - \mathcal{D}_{(k+1)*}) - (\mathcal{D}_{2*} + \mathcal{D}_{(2k)*}) + (\mathcal{D}_{k*} + \mathcal{D}_{(k+2)*})\\
&= 2(\mathcal{D}_{1*} - \mathcal{D}_{(k+1)*}) - (\mathcal{D}_{2*} - \mathcal{D}_{(k+2)*}) + (\mathcal{D}_{k*} - \mathcal{D}_{(2k)*})\\
&= 2(2\mathcal{D}_{1*} - k\mathbf{e}') - (2\mathcal{D}_{2*} - k\mathbf{e}') + (2\mathcal{D}_{k*} - k\mathbf{e}')\\
&= 4(\mathbf{e}'_{k+1} - \mathbf{e}'_1).
\end{aligned}$$

Thus, $\widetilde{V} = \mathrm{Circ}(\mathbf{e}'_{k+1} - \mathbf{e}'_1) + 2\mathrm{Circ}(\mathbf{e}'_1) - \frac{1}{k}\mathbf{e}\mathbf{e}' = \mathrm{Circ}(\mathbf{e}'_1 + \mathbf{e}'_{k+1}) - \frac{1}{k}\mathrm{Circ}(\mathbf{e}')$ which is a symmetric matrix. Furthermore, by (2.1), it follows that

$$\begin{aligned}
\mathcal{D}\widetilde{V} &= \widetilde{V}\mathcal{D}\\
&= \mathrm{Circ}((\mathbf{e}'_1 + \mathbf{e}'_{k+1})\mathcal{D}) - \frac{1}{k}\mathrm{Circ}(\mathbf{e}'\mathcal{D})\\
&= \mathrm{Circ}(\mathcal{D}_{1*} + \mathcal{D}_{(k+1)*}) - \frac{1}{k}\mathrm{Circ}(k^2\mathbf{e}')\\
&= O.
\end{aligned}$$

Using the facts that $\mathcal{L}_{i*} = -\mathcal{L}_{(i+k)*}$ for $i = 1,2,\ldots,k$ and $\mathcal{L}\mathbf{e} = \mathbf{0}$, we deduce that $\widetilde{V}\mathcal{L} = O$ and $\widetilde{V}\widetilde{\mathbf{w}} = \frac{1}{2k}\widetilde{V}\mathbf{e} = \mathbf{0}$. This completes the proof. ∎

We now proceed to find the Moore–Penrose inverse of the distance matrix of an even-unicyclic graph. Recall that a unicyclic graph is denoted by $G(m,l)$ or simply by G when m and l are not required to be specified. Throughout this section, we assume that $m = 2k$ where $k \geq 2$. We say that a vertex of $G(m,l)$ is adjacent with the cycle C_m if it is adjacent to a vertex in C_m. Among the l vertices of $G(m,l)$, which are not in C_m, it is assumed that l_1 vertices are adjacent with C_m. In order to obtain the Moore–Penrose inverse formula similar to (1.2), we need to find a scalar α, a vector $\mathbf{w}$, and a Laplacian-like matrix L.

The scalar α and the vector $\boldsymbol{w}$: For each vertex v_j in the cycle C_m of G, we define $p_j = \widehat{d}_G(v_j) - 1$ where $j = 1,2,\ldots,m$. We would like to mention that, in this paper, p_j's are defined only for the vertices in C_m even though it can be computed for any vertex in G. Further, for each $i \in \{1,2,\ldots,m\}$, it is easy to see that there exists a unique number $j_i \in \{1,2,\ldots,m\}$ such that $i + k \equiv j_i \pmod{m}$ where j_i is taken to be m if $i + k \equiv 0 \pmod{m}$. Now, define an n-dimensional column vector $\mathbf{w}$ and a real scalar α as follows:

$$\alpha = \frac{4}{m+2l} \text{ and } \mathbf{w} = \frac{1}{2m}(w_1, w_2, \ldots, w_n)', \text{ where} \tag{3.2}$$

$$w_i = \begin{cases} (2 - l_1) - (p_i - p_{j_i})k & \text{if } 1 \leq i \leq m,\\ (2 - \widehat{d}_G(v_i))m & \text{if } m+1 \leq i \leq n. \end{cases}$$

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As a numerical illustration, consider the graph given in Figure 1. We have $m = 6$, $l = 5$, and $l_1 = 4$. Note that $p_1 = 0$, $p_2 = 2$, $p_3 = 1$, $p_4 = 0$, $p_5 = 1$, and $p_6 = 0$. Hence,

$$\alpha = \frac{1}{4} \text{ and } \mathbf{w} = \frac{1}{12}(-2, -5, -5, -2, 1, 1, 6, 0, 6, 6, 6)'. \tag{3.3}$$

In the following, we show that the sum of the components of the above defined vector is 1.

LEMMA 3.2. *Let* $\mathbf{w} \in \mathbb{R}^n$ *be the vector given in* (3.2). *Then* $\mathbf{e}'\mathbf{w} = 1$.

Proof. Since $\sum_{i=1}^{m}(p_i - p_{j_i}) = 0$, we have

$$\begin{aligned}\sum_{i=1}^{n} w_i &= (2 - l_1)m - k\sum_{i=1}^{m}(p_i - p_{j_i}) + m\sum_{i=m+1}^{n}(2 - \widehat{d}_G(v_i)) \\ &= 2m - l_1 m + m\sum_{i=m+1}^{n}(2 - \widehat{d}_G(v_i)).\end{aligned}$$

To prove the claim, it is enough to show that $\sum_{i=m+1}^{n}(2 - \widehat{d}_G(v_i)) = l_1$. We prove this by induction on l. It is easy to verify that the result is true for $l = 1$. Assume that the identity is true for $l - 1$. Let $G(m, l)$ be a unicyclic graph with $l \geq 2$. The vertices of G are relabeled so that it has a pendant vertex v_n. Let $\bar{G}$ be a graph obtained from G by deleting the pendant vertex v_n. By the induction hypothesis, $\sum_{i=m+1}^{n-1}(2 - \widehat{d}_{\bar{G}}(v_i)) = \bar{l}_1$, where $\bar{l}_1$ is the number of vertices in $\bar{G}$, which are not in the cycle C_m and adjacent with C_m. There are two cases depending on whether or not the vertex v_n is adjacent with C_m.

Case (i): The vertex v_n is adjacent with C_m.

Note that in this case, $\widehat{d}_G(v_i) = \widehat{d}_{\bar{G}}(v_i)$ for $i = m+1, \ldots, n-1$ and $l_1 = \bar{l}_1 + 1$. Hence,

$$\sum_{i=m+1}^{n}(2 - \widehat{d}_G(v_i)) = \sum_{i=m+1}^{n-1}(2 - \widehat{d}_{\bar{G}}(v_i)) + (2 - \widehat{d}_G(v_n)) = \bar{l}_1 + 1 = l_1.$$

Case (ii): The vertex v_n is not adjacent with C_m. Without loss of generality, assume that $v_n \sim v_{n-1}$.

In this case, $\widehat{d}_G(v_i) = \widehat{d}_{\bar{G}}(v_i)$ for $i = m+1, \ldots, n-2$ and $\widehat{d}_G(v_{n-1}) = \widehat{d}_{\bar{G}}(v_{n-1}) + 1$. We have

$$\sum_{i=m+1}^{n}(2 - \widehat{d}_G(v_i)) = \sum_{i=m+1}^{n-1}(2 - \widehat{d}_{\bar{G}}(v_i)) - 1 + (2 - \widehat{d}_G(v_n)) = \bar{l}_1 = l_1.$$

Hence the result. ∎

The Laplacian-like matrix L: Corresponding to each vertex v_i of G which is not in C_m, we define the following m-dimensional column vectors $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})'$, for $i = m+1, m+2, \ldots, n$. If v_i is not adjacent with the cycle C_m, then $\mathbf{x}_i$ is taken to be $\mathbf{0}$. If v_i is adjacent with C_m, say $v_i \sim v_q$ for some $q \in \{1, 2, \ldots, m\}$, then

$$x_{ij} = \frac{1}{m}\begin{cases} -(k+1) & \text{if } j = q, \\ k - 1 & \text{if } j = q_0, \\ -1 & \text{otherwise,} \end{cases} \tag{3.4}$$

where $q + k \equiv q_0 \pmod{m}$ and q_0 is taken to be m if $q = k$. In this case, we can write $\mathbf{x}_i = \frac{1}{m}(k\mathbf{e}_{q_0} - k\mathbf{e}_q - \mathbf{e})$. Now define an $n \times n$ matrix L partitioned in the following form

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}' & L_{22} \end{bmatrix}, \tag{3.5}$$

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where

$$L_{11} = \mathcal{L} + \sum_{i=m+1}^{n} \mathbf{x}_i \mathbf{x}_i',\ L_{12} = \begin{bmatrix} \mathbf{x}_{m+1} & \cdots & \mathbf{x}_n \end{bmatrix},$$

L_{22} is the principal submatrix of the Laplacian matrix of G obtained by deleting its first m rows and columns, and $\mathcal{L}$ is the matrix given in (3.1). Since $L(G)$ is symmetric, L_{22} is a symmetric matrix of order $n-m$.

To illustrate the above construction, we consider again the graph in Figure 1. The vertices v_7 and v_8 are adjacent to v_2, which is in the cycle C_6. Therefore, we have $\mathbf{x}_7 = \mathbf{x}_8 = \frac{1}{6}(-1,-4,-1,-1,2,-1)'$. Similarly, it is easy to see that $\mathbf{x}_{10} = \frac{1}{6}(-1,-1,-4,-1,-1,2)'$ and $\mathbf{x}_{11} = \frac{1}{6}(-1,2,-1,-1,-4,-1)'$. Since v_9 is not adjacent with C_6, we have $\mathbf{x}_9 = \mathbf{0}$. Thus, the matrix

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}' & L_{22} \end{bmatrix}, \tag{3.6}$$

where the matrices L_{11}, L_{12}, and L_{22} are given by

$$\begin{bmatrix} \frac{11}{18} & -\frac{1}{18} & \frac{4}{9} & -\frac{7}{18} & \frac{5}{18} & -\frac{2}{9} \\ -\frac{1}{18} & \frac{55}{36} & \frac{1}{36} & \frac{4}{9} & -\frac{41}{36} & \frac{13}{36} \\ \frac{4}{9} & \frac{1}{36} & \frac{37}{36} & -\frac{1}{18} & \frac{13}{36} & -\frac{23}{36} \\ -\frac{7}{18} & \frac{4}{9} & -\frac{1}{18} & \frac{11}{18} & -\frac{2}{9} & \frac{5}{18} \\ \frac{5}{18} & -\frac{41}{36} & \frac{13}{36} & -\frac{2}{9} & \frac{43}{36} & -\frac{11}{36} \\ -\frac{2}{9} & \frac{13}{36} & -\frac{23}{36} & \frac{5}{18} & -\frac{11}{36} & \frac{25}{36} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{2}{3} & -\frac{2}{3} & 0 & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{6} & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{3} & -\frac{1}{6} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively.

In the following result, we will show that the constructed matrix L in (3.5) is symmetric and $L\mathbf{e} = \mathbf{0}$.

Lemma 3.3. *Let L be defined as in (3.5). Then L is a symmetric Laplacian-like matrix.*

Proof. From the proof of Theorem 3.1, we see that the matrix $\mathcal{L}$ is symmetric and $\mathcal{L}\mathbf{e} = \mathbf{0}$. Since L_{22} is symmetric, we have L is symmetric. We claim that $L\mathbf{e} = \mathbf{0}$. Let $L = (l_{ij})$ and let $m+1 \leq i \leq n$. Then $l_{ii} = \deg(v_i)$ and the vertex v_i is adjacent with at most one vertex in C_m because G is unicyclic. If v_i is adjacent with C_m, then the vector $\mathbf{x}_i \neq \mathbf{0}$ and $\mathbf{x}_i'\mathbf{e} = -1$. Also, -1 occurs exactly $\deg(v_i) - 1$ times in the corresponding row of L_{22}. This implies that the sum of the entries in the i-th row of L is equal to zero. On the other hand, if v_i is not adjacent with C_m, then $\mathbf{x}_i = \mathbf{0}$ and $\mathbf{x}_i'\mathbf{e} = 0$. In this case, v_i is adjacent with vertices of G which are not in C_m and hence the corresponding row sums of L_{22} and L are equal to zero. Therefore, $L_{12}'\mathbf{e} + L_{22}\mathbf{e} = \mathbf{0}$. Moreover, $\sum_{i=m+1}^{n} \mathbf{x}_i\mathbf{x}_i'\mathbf{e} + \sum_{i=m+1}^{n} \mathbf{x}_i = \mathbf{0}$. Since $\mathcal{L}\mathbf{e} = \mathbf{0}$, we have

$$L\mathbf{e} = \begin{bmatrix} L_{11}\mathbf{e} + L_{12}\mathbf{e} \\ L_{12}'\mathbf{e} + L_{22}\mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathcal{L}\mathbf{e} + \sum_{i=m+1}^{n} \mathbf{x}_i\mathbf{x}_i'\mathbf{e} + \sum_{i=m+1}^{n} \mathbf{x}_i \\ \mathbf{0} \end{bmatrix} = \mathbf{0}.$$

This completes the proof. ∎

Relationships between the parameters of $D(G)$ and $D(\bar{G})$:

Throughout this section, we assume that v_n is a pendant vertex of $G(m,l)$, and $\bar{G}$ is the subgraph of G obtained by deleting the vertex v_n from G. Then $\bar{G} = \bar{G}(m,l-1)$ is a unicyclic graph of order $n-1$ having $l-1$ number of edges of $\bar{G}$ which are not in the cycle C_m. Corresponding to G, we defined α, $\mathbf{w}$, and L in (3.2)

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and (3.5). We call these the *parameter* of $D(G)$. In the proofs of the forthcoming lemmas, we use induction on l. So, we need to explore certain relationships between the parameters of $D(G) = (d_{ij})$ and $D(\bar{G}) = (\bar{d}_{ij})$, which will be used frequently. For the sake of distinction, the parameters of $D(\bar{G})$ are denoted by $\bar{\alpha}$, $\bar{\mathbf{w}}$, and $\bar{L}$. When $l \geq 2$, we have to derive the parameters relationships in two distinct cases depending on whether or not the pendant vertex v_n is adjacent with the cycle C_m. Without loss of generality, we assume that either the vertex v_n is adjacent to the vertex $v_m \in C_m$ or $v_n \sim v_{n-1}$ and $v_{n-1} \notin C_m$.

1. *The relation between* α *and* $\bar{\alpha}$: Since α is given in terms of m and l, by (3.2), we have $\frac{1}{\alpha} = \frac{m+2l}{4} = \frac{1}{\bar{\alpha}} + \frac{1}{2}$.
2. *The vectors* $\mathbf{x}_i$ *and* $\bar{\mathbf{x}}_i$: Let $m+1 \leq i \leq n-1$. The vector $\mathbf{x}_i$ given in (3.4) is defined by the adjacency of the vertex v_i with C_m, which is not disturbed by the deletion of the vertex v_n from G. So, in both cases, $\mathbf{x}_i = \bar{\mathbf{x}}_i$ where $\bar{\mathbf{x}}_i$'s are vectors as defined in (3.4) with respect to $\bar{G}$. We now write the vector $\mathbf{x}_n$. If $v_n \sim v_{n-1}$, then $\mathbf{x}_n = \mathbf{0}$. On the other hand, if $v_n \sim v_m$, then $\mathbf{x}_n = \frac{1}{m}(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e})$.
3. *The vector* $\mathbf{w}$ *in terms of* $\bar{\mathbf{w}}$: Note that w_i is defined in terms of the block degrees of the vertices.
   (a) In the case of $v_n \sim v_m$, we have $l_1 = \bar{l}_1 + 1$. Also, $p_j = \bar{p}_j$ for $1 \leq j < m$ and $p_m = \bar{p}_m + 1$ where $\bar{p}_j = \widehat{d}_{\bar{G}}(v_j) - 1$. Therefore,
   $$w_i = \begin{cases} \bar{w}_i - 1 & 1 \leq i \leq m, i \neq k, m, \\ \bar{w}_k - 1 + k & i = k, \\ \bar{w}_m - 1 - k & i = m, \\ \bar{w}_i & m+1 \leq i \leq n-1, \\ m & i = n. \end{cases}$$
   Thus, $\mathbf{w} = \frac{1}{2m}(w_1, w_2, \ldots, w_n)' = \frac{1}{2m}(\bar{w}_1, \bar{w}_2, \ldots \bar{w}_{n-1}, m)' + \frac{1}{2}(\mathbf{x}_n', \mathbf{0}')'$ and hence
   $$\mathbf{w} = \begin{bmatrix} \bar{\mathbf{w}} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{y}_n \\ 1 \end{bmatrix} \text{ where } \mathbf{y}_n = \begin{bmatrix} \mathbf{x}_n \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}.$$
   (b) When $v_n \sim v_{n-1}$, it is easy to see that $l_1 = \bar{l}_1$ and $p_j = \bar{p}_j$ for $1 \leq i \leq m$. Hence,
   $$w_i = \begin{cases} \bar{w}_i & 1 \leq i \leq n-2, \\ \bar{w}_i - m & i = n-1, \\ m & i = n. \end{cases}$$
   This implies that
   $$\mathbf{w} = \begin{bmatrix} \bar{\mathbf{w}} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\mathbf{e}_{n-1} \\ 1 \end{bmatrix}.$$
4. *Expressing* D *in terms of* $\bar{D}$: We denote $D(\bar{G})$ by $\bar{D}$. Clearly, $d(u,v) = \bar{d}(u,v)$ for any two vertices u and v in $\bar{G}$.
   (a) Consider the case $v_n \sim v_m$. Let $1 \leq i \leq n-1$. Since G is unicyclic and $\deg(v_n) = 1$, any path in G between v_i and v_n passes through v_m. Therefore, $d(v_i, v_n) = 1 + \bar{d}(v_i, v_m)$ and
   $$D = \begin{bmatrix} \bar{D} & \bar{D}_{*m} + \mathbf{e} \\ \bar{D}_{m*} + \mathbf{e}' & 0 \end{bmatrix}.$$
   (b) If $v_n \sim v_{n-1}$, then by an argument similar to the previous case, we have $d(v_i, v_n) = 1 + \bar{d}(v_i, v_{n-1})$ and
   $$D = \begin{bmatrix} \bar{D} & \bar{D}_{*(n-1)} + \mathbf{e} \\ \bar{D}_{(n-1)*} + \mathbf{e}' & 0 \end{bmatrix}.$$

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5. *The relationship between L and $\bar{L}$*: Let $\bar{L} = \begin{bmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}'_{12} & \bar{L}_{22} \end{bmatrix}$, where $\bar{L}_{11}$ is an $m \times m$ matrix. Since $\mathbf{x}_i = \bar{\mathbf{x}}_i$ for $m+1 \le i \le n-1$, $L_{11} = \bar{L}_{11} + \mathbf{x}_n \mathbf{x}'_n$.
   (a) For the case $v_n \sim v_m$, it is easy to see that $L_{12} = \begin{bmatrix} \bar{L}_{12} & \mathbf{x}_n \end{bmatrix}$ and $L_{22} = \begin{bmatrix} \bar{L}_{22} & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix}$. This implies that
   $$L = \begin{bmatrix} \bar{L} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{y}_n \mathbf{y}'_n & \mathbf{y}_n \\ \mathbf{y}'_n & 1 \end{bmatrix},$$
   where $\mathbf{y}_n = \begin{bmatrix} \mathbf{x}_n \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}$ and $\bar{L}$ is an $(n-1) \times (n-1)$ matrix.
   (b) In the case of $v_n \sim v_{n-1}$, we have $\mathbf{x}_n = \mathbf{0}$ and the $(n-1)$-th diagonal entry of L is equal to the $(n-1)$-th diagonal entry of $\bar{L} + 1$. Since $\deg(v_n) = 1$, we have
   $$L = \begin{bmatrix} \bar{L} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{n-1} \mathbf{e}'_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{e}'_{n-1} & 1 \end{bmatrix}.$$

To express D, $\mathbf{w}$, and L in terms of $\bar{D}$, $\bar{\mathbf{w}}$, and $\bar{L}$, respectively, in a unified manner, we will introduce the following additional notations:

$$a := \begin{cases} m & \text{if } v_n \sim v_m, \\ n-1 & \text{if } v_n \sim v_{n-1}, \end{cases} \tag{3.7}$$

$$\mathbf{y} := \begin{cases} \mathbf{y}_n & \text{if } v_n \sim v_m, \\ -\mathbf{e}_{n-1} & \text{if } v_n \sim v_{n-1}. \end{cases} \tag{3.8}$$

Then

$$D = \begin{bmatrix} \bar{D} & \bar{D}_{*a} + \mathbf{e} \\ \bar{D}_{a*} + \mathbf{e}' & 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} \bar{\mathbf{w}} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} \text{ and } L = \begin{bmatrix} \bar{L} + \mathbf{y}\mathbf{y}' & \mathbf{y} \\ \mathbf{y}' & 1 \end{bmatrix}. \tag{3.9}$$

If $l = 1$, then $\bar{G} = C_m$. Hence, $\bar{\mathbf{w}} = \widetilde{\mathbf{w}} = \frac{1}{m}\mathbf{e}$, $\bar{L} = \mathcal{L}$, and $\bar{D} = \mathcal{D}$, where $\mathcal{D}$ is the distance matrix of C_m and $\mathcal{L}$ is given in (3.1). In this case, $a = m$ and $\mathbf{y} = \mathbf{x}_{m+1}$. Thus, we have

$$D = \begin{bmatrix} \mathcal{D} & \mathcal{D}_{*m} + \mathbf{e} \\ \mathcal{D}_{m*} + \mathbf{e}' & 0 \end{bmatrix}, \mathbf{w} = \frac{1}{2m} \begin{bmatrix} \mathbf{e} + k\mathbf{e}_k - k\mathbf{e}_m \\ m \end{bmatrix} \text{ and } L = \begin{bmatrix} \mathcal{L} + \mathbf{x}_{m+1}\mathbf{x}'_{m+1} & \mathbf{x}_{m+1} \\ \mathbf{x}'_{m+1} & 1 \end{bmatrix}. \tag{3.10}$$

We now state the main result of this section which gives the Moore–Penrose inverse formula for the distance matrix of an even-unicyclic graph.

THEOREM 3.4. *Let G be an even-unicyclic graph. Let α and $\mathbf{w}$ be given by (3.2) and L be defined as in (3.5). Then*
$$D(G)^\dagger = -\frac{1}{2}L + \alpha \mathbf{w}\mathbf{w}'.$$

The proof of this result is based on the following lemmas which provide certain identities involving D, L, and $\mathbf{w}$. In these lemmas, we continue to use the notations introduced above.

LEMMA 3.5. *Let $\mathcal{D}$ be the distance matrix of C_m. Let $\mathbf{x}_i$ be defined as in (3.4) where $m+1 \le i \le n$. Suppose that the vertex v_i of a unicyclic graph $G(m,l)$ is adjacent to a vertex v_q in C_m. Then $\mathcal{D}\mathbf{x}_i = -\mathcal{D}_{*q}$.*

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Proof. We have, from (3.4), $\mathbf{x}_i = \frac{1}{m}(k\mathbf{e}_{q_0} - k\mathbf{e}_q - \mathbf{e})$ where $q + k \equiv q_0 \pmod m$. Then $\mathcal{D}\mathbf{x}_i = \frac{1}{m}[k\mathcal{D}_{*q_0} - k\mathcal{D}_{*q} - \mathcal{D}\mathbf{e}]$. From the proof of Theorem 3.1, we have $\mathcal{D}_{*q} - \mathcal{D}_{*q_0} = 2\mathcal{D}_{*q} - k\mathbf{e}$. Using this identity together with the fact that $\mathcal{D}\mathbf{e} = k^2\mathbf{e}$, the result follows. □

Lemma 3.6. *Let a and $\mathbf{y}$ be defined as in (3.7) and (3.8), respectively. Then $\bar{D}\mathbf{y} = -\bar{D}_{*a}$.*

Proof. It is easy to verify the identity for the case $v_n \sim v_{n-1}$. Consider the case $v_n \sim v_m$. Then $a = m$ and $\mathbf{y} = \mathbf{y}_n = \begin{bmatrix}\mathbf{x}_n' & \mathbf{0}'\end{bmatrix}'$, where $\mathbf{x}_n = \frac{1}{m}(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e})$. Let $\bar{D}\mathbf{y} = (z_1, z_2, \ldots, z_{n-1})'$, where $z_i = \bar{D}_{i*}\mathbf{y}$. To find z_i, it is enough to consider the first m entries of $\bar{D}_{i*}$, because the last $l-1$ coordinates of $\mathbf{y}$ are zero. Then $z_i = (1/m)[k\bar{d}(v_i, v_k) - k\bar{d}(v_i, v_m) - \sum_{j=1}^{m} \bar{d}(v_i, v_j)]$. Let $i \in \{1, 2, \ldots, m\}$. Since $k = \bar{d}(v_k, v_m) = \bar{d}(v_k, v_i) + \bar{d}(v_i, v_m)$, we have $z_i = (1/m)[k(k - \bar{d}(v_i, v_m)) - k\bar{d}(v_i, v_m) - k^2] = -\bar{d}(v_i, v_m)$.

Now let $m+1 \le i \le n-1$. Since $\bar{G}$ is unicyclic, there exists a vertex $\widehat{v_i}$ in C_m such that $\bar{d}(v_i, \widehat{v_i}) \le \bar{d}(v_i, v_j)$ for all $j = 1, 2, \ldots, m$. This implies that $\bar{d}(v_i, v_j) = \bar{d}(v_i, \widehat{v_i}) + \bar{d}(\widehat{v_i}, v_j)$. By using this identity, we get,

$$\begin{aligned} z_i &= \frac{1}{m}[k(\bar{d}(\widehat{v_i}, v_k) - \bar{d}(\widehat{v_i}, v_m)) - m\bar{d}(v_i, \widehat{v_i}) - \sum_{j=1}^{m} \bar{d}(\widehat{v_i}, v_j)] \\ &= \frac{1}{m}[k(k - \bar{d}(\widehat{v_i}, v_m)) - k\bar{d}(\widehat{v_i}, v_m) - m\bar{d}(v_i, \widehat{v_i}) - k^2] \\ &= \frac{1}{m}[-2k\bar{d}(\widehat{v_i}, v_m) - m\bar{d}(v_i, \widehat{v_i})] \\ &= -\bar{d}(v_i, v_m). \end{aligned}$$

Thus, $\bar{D}\mathbf{y} = -\bar{D}_{*a}$. □

Lemma 3.7. *Suppose that $V = LD + 2I - 2\mathbf{w}\mathbf{e}'$. Then $V = \begin{bmatrix}\widetilde{V} & O \\ O & O\end{bmatrix}$, where $\widetilde{V}$ is an $m \times m$ matrix defined by $\widetilde{V} = Circ(\mathbf{e}_1' + \mathbf{e}_{k+1}') - \frac{1}{k}J$.*

Proof. We prove the result by induction on l. If $l = 1$, then from (3.10), we have $\mathbf{w} = \frac{1}{2m}\begin{bmatrix}\mathbf{e} + k\mathbf{e}_k - k\mathbf{e}_m \\ m\end{bmatrix}$ which can be written as $\mathbf{w} = \frac{1}{2m}\begin{bmatrix}2\mathbf{e} + m\mathbf{x}_{m+1} \\ m\end{bmatrix}$, where $\mathbf{x}_{m+1} = \frac{1}{m}(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e})$. In this case, the matrices D and L are given in (3.10). We partition the matrix V as $\begin{bmatrix}V_{11} & V_{12} \\ V_{21} & V_{22}\end{bmatrix}$ with V_{11} is an $m \times m$ matrix. Then

$$V_{11} = \mathcal{L}\mathcal{D} + \mathbf{x}_{m+1}(\mathbf{x}_{m+1}'\mathcal{D} + \mathcal{D}_{m*} + \mathbf{e}') + 2I - \frac{2}{m}\mathbf{e}\mathbf{e}' - \mathbf{x}_{m+1}\mathbf{e}'.$$

From the proof of Theorem 3.1, we have $\mathcal{L}\mathcal{D} + 2I - \frac{2}{m}\mathbf{e}\mathbf{e}' = \widetilde{V}$. Using this result and Lemma 3.5, we conclude that $V_{11} = \widetilde{V}$.

Since $\mathcal{D}\mathbf{x}_{m+1} = -\mathcal{D}_{*m}$, we obtain $\mathcal{D}_{m*}\mathbf{x}_{m+1} = -(\mathcal{D})_{mm} = 0$. From the proof of Theorem 3.1, we have $\mathcal{L}\mathcal{D} = \mathrm{Circ}(\mathbf{e}_{k+1}' - \mathbf{e}_1')$. Therefore, $\mathcal{L}\mathcal{D}_{*m} = \mathbf{e}_k - \mathbf{e}_m$. Note that $\mathcal{L}\mathbf{e} = \mathbf{0}$ and $\mathbf{x}_{m+1}'\mathbf{e} = -1$. Using all these facts, we see that $V_{12} = \mathcal{L}\mathcal{D}_{*m} - 2\mathbf{x}_{m+1} - \frac{2}{m}\mathbf{e} = \mathbf{0}$. Similarly, by using Lemma 3.5 together with $\mathbf{x}_{m+1}'\mathbf{e} = -1$, it is easy to verify that $V_{21} = \mathbf{0}'$ and $V_{22} = 0$.

By the induction hypothesis,

$$\bar{V} = \bar{L}\bar{D} + 2I - 2\bar{\mathbf{w}}\mathbf{e}' = \begin{bmatrix}\widetilde{V} & O \\ O & O\end{bmatrix}. \tag{3.11}$$

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From (3.9), we have

$$V = \begin{bmatrix} \bar{L} + \mathbf{y}\mathbf{y}' & \mathbf{y} \\ \mathbf{y}' & 1 \end{bmatrix} \begin{bmatrix} \bar{D} & \bar{D}_{*a} + \mathbf{e} \\ \bar{D}_{a*} + \mathbf{e}' & 0 \end{bmatrix} + \begin{bmatrix} 2I & \mathbf{0} \\ \mathbf{0}' & 2 \end{bmatrix} - 2 \begin{bmatrix} \bar{\mathbf{w}} + \frac{1}{2}\mathbf{y} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \mathbf{e}' & 1 \end{bmatrix}.$$

Partition V as $\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$, where V_{11} is an $(n-1)\times(n-1)$ matrix. Then

$$V_{11} = \bar{L}\bar{D} + \mathbf{y}(\mathbf{y}'\bar{D} + \bar{D}_{a*} + \mathbf{e}') + 2I - 2\bar{\mathbf{w}}\mathbf{e}' - \mathbf{y}\mathbf{e}'.$$

By the induction hypothesis and by Lemma 3.6, we have $V_{11} = \bar{V}$. Again by Lemma 3.6, $\bar{D}_{a*}\mathbf{y} = -(\bar{D})_{aa} = 0$. Using this fact and $\mathbf{y}'\mathbf{e} = -1$, it follows that $V_{21} = \mathbf{0}'$ and $V_{22} = 0$.

To complete the proof, it suffices to show that $V_{12} = \bar{L}\bar{D}_{*a} - 2\mathbf{y} - 2\bar{\mathbf{w}} = \mathbf{0}$. From (3.11), we have $\bar{V}_{*a} = \bar{L}\bar{D}_{*a} + 2\mathbf{e}_a - 2\bar{\mathbf{w}}$. We need to consider two possible cases for a. If $a = n-1$, then $\mathbf{y} = -\mathbf{e}_{n-1}$ and $(n-1)$-th column of $\bar{V}$ is zero. Hence, $V_{12} = \mathbf{0}$. On the other hand, if $a = m$, then $\mathbf{y} = \mathbf{y}_n = \begin{bmatrix} \mathbf{x}_n \\ \mathbf{0} \end{bmatrix}$ and $\bar{V}_{*m} = \begin{bmatrix} \widetilde{V}_{*m} \\ \mathbf{0} \end{bmatrix}$. Since $\widetilde{V} = \mathrm{Circ}(\mathbf{e}_1' + \mathbf{e}_{k+1}' - \frac{1}{k}\mathbf{e}')$, we have $\widetilde{V}_{*m} = \mathbf{e}_m + \mathbf{e}_k - \frac{1}{k}\mathbf{e}$. Note that $\mathbf{x}_n = (1/m)(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e})$. Using these facts, we obtain $\bar{V}_{*a} - 2\mathbf{e}_a - 2\mathbf{y} = \mathbf{0}$. This implies that $\bar{L}\bar{D}_{*a} - 2\bar{\mathbf{w}} = 2\mathbf{y}$ and hence $V_{12} = \mathbf{0}$. ∎

Lemma 3.8. *Let V be defined as in Lemma 3.7. Then the following hold:*

(i) $DV = O$;
(ii) $VL = O$;
(iii) $V\mathbf{w} = \mathbf{0}$.

Proof. If $l = 1$, then from (3.10), it follows that

$$DV = \begin{bmatrix} \mathcal{D}\widetilde{V} & \mathbf{0} \\ \mathcal{D}_{m*}\widetilde{V} + \mathbf{e}'\widetilde{V} & 0 \end{bmatrix}, VL = \begin{bmatrix} \widetilde{V}\mathcal{L} + \widetilde{V}\mathbf{x}_{m+1}\mathbf{x}_{m+1}' & \widetilde{V}\mathbf{x}_{m+1} \\ \mathbf{0}' & 0 \end{bmatrix} \text{ and } V\mathbf{w} = \frac{1}{2m}\begin{bmatrix} 2\widetilde{V}\mathbf{e} + m\widetilde{V}\mathbf{x}_{m+1} \\ 0 \end{bmatrix}.$$

From the proof of Theorem 3.1, we have $\mathcal{D}\widetilde{V} = O$, $\widetilde{V}\mathcal{L} = O$, and $\widetilde{V}\mathbf{e} = \mathbf{0}$. Thus, $DV = O$. Now

$$\begin{aligned} \widetilde{V}\mathbf{x}_{m+1} &= \frac{1}{m}(\mathrm{Circ}(\mathbf{e}_1') + \mathrm{Circ}(\mathbf{e}_{k+1}') - \frac{1}{k}J)(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e}) \\ &= \frac{1}{m}[(k\mathbf{e}_k - k\mathbf{e}_m - \mathbf{e}) + (k\mathbf{e}_m - k\mathbf{e}_k - \mathbf{e}) - \frac{1}{k}(-m\mathbf{e})] \\ &= \mathbf{0}. \end{aligned}$$

This implies that $VL = O$ and $V\mathbf{w} = \mathbf{0}$. Assume, by induction, that the lemma holds for $l-1$. That is,

$$\bar{D}\bar{V} = O,\ \bar{V}\bar{L} = O \text{ and } \bar{V}\bar{\mathbf{w}} = \mathbf{0}, \tag{3.12}$$

where $\bar{V} = \begin{bmatrix} \widetilde{V} & O \\ O & O \end{bmatrix}$ is a matrix of order $n-1$. From the proof of Lemma 3.7, we have $V = \begin{bmatrix} \bar{V} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix}$. Hence, by (3.9), we obtain

$$DV = \begin{bmatrix} \bar{D}\bar{V} & \mathbf{0} \\ \bar{D}_{a*}\bar{V} + \mathbf{e}'\bar{V} & 0 \end{bmatrix}, VL = \begin{bmatrix} \bar{V}\bar{L} + \bar{V}\mathbf{y}\mathbf{y}' & \bar{V}\mathbf{y} \\ \mathbf{0}' & 0 \end{bmatrix} \text{ and } V\mathbf{w} = \begin{bmatrix} \bar{V}\bar{\mathbf{w}} + \frac{1}{2}\bar{V}\mathbf{y} \\ 0 \end{bmatrix}.$$

Since $\widetilde{V}\mathbf{e} = \mathbf{0}$, we must have $\bar{V}\mathbf{e} = \mathbf{0}$. Using this and (3.12), we conclude that $DV = O$. Finally, it remains to show that $\bar{V}\mathbf{y} = \mathbf{0}$. First, consider the case $v_n \sim v_{n-1}$. Then $\mathbf{y} = -\mathbf{e}_{n-1}$. Since $(n-1)$-th column of $\bar{V}$ is zero, $\bar{V}\mathbf{y} = \mathbf{0}$. If $v_n \sim v_m$, then $\mathbf{y} = \begin{bmatrix} \mathbf{x}_n' & \mathbf{0}' \end{bmatrix}'$. By the same argument as for the case $l = 1$, we deduce that $\widetilde{V}\mathbf{x}_n = \mathbf{0}$ which implies that $\bar{V}\mathbf{y} = \mathbf{0}$. ∎

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Proof of Theorem 3.4. From Lemma 3.8, we have $V(-\frac{1}{2}L+\alpha\mathbf{w}\mathbf{w}') = O$. To complete the proof, from Theorem 2.2, it suffices to show that $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$. We prove this equality by induction on l. If $l = 1$, then $\alpha = \frac{4}{m+2}$ and from (3.10), we have

$$D\mathbf{w} = \frac{1}{2m}\begin{bmatrix} \mathcal{D}\mathbf{e} + k\mathcal{D}_{*k} - k\mathcal{D}_{*m} + m\mathcal{D}_{*m} + m\mathbf{e} \\ \mathcal{D}_{m*}\mathbf{e} + k\mathcal{D}_{m*}\mathbf{e}_k - k\mathcal{D}_{m*}\mathbf{e}_m + \mathbf{e}'\mathbf{e} \end{bmatrix}.$$

Since $\mathcal{D}\mathbf{e} = k^2\mathbf{e}$ and $\mathcal{D}_{*k} + \mathcal{D}_{*m} = k\mathbf{e}$, we have

$$D\mathbf{w} = \frac{1}{2m}\begin{bmatrix} (2k^2+m)\mathbf{e} \\ 2k^2+m \end{bmatrix} = \frac{k+1}{2}\mathbf{e} = \frac{1}{\alpha}\mathbf{e}.$$

Assume that the identity holds for $l-1$. From (3.9), we have

$$D\mathbf{w} = \begin{bmatrix} \bar{D} & \bar{D}_{*a} + \mathbf{e} \\ \bar{D}_{a*} + \mathbf{e}' & 0 \end{bmatrix}\left(\begin{bmatrix} \bar{\mathbf{w}} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \bar{D}\bar{\mathbf{w}} \\ \bar{D}_{a*}\bar{\mathbf{w}} + \mathbf{e}'\bar{\mathbf{w}} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \bar{D}\mathbf{y} + \bar{D}_{*a} + \mathbf{e} \\ \bar{D}_{a*}\mathbf{y} + \mathbf{e}'\mathbf{y} \end{bmatrix}.$$

Note that $\mathbf{e}'\mathbf{y} = -1$. By Lemmas 3.2 and 3.6, we have $\mathbf{e}'\bar{\mathbf{w}} = 1$, $\bar{D}\mathbf{y} + \bar{D}_{*a} = \mathbf{0}$ and $\bar{D}_{a*}\mathbf{y} = -(\bar{D})_{aa} = 0$. By the induction assumption, $D\mathbf{w} = (\frac{1}{\bar{\alpha}} + \frac{1}{2})\begin{bmatrix}\mathbf{e}' & 1\end{bmatrix}' = \frac{1}{\alpha}\mathbf{e}$. ∎

The following example illustrates Theorem 3.4.

EXAMPLE 3.9. *Consider the unicyclic graph G given in Figure 1. Corresponding to this graph, the scalar α and the vector $\mathbf{w}$ are given in (3.3). By Theorem 3.4, $D(G)^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ where the associated Laplacian-like matrix L is given in (3.6). That is,*

$$D(G)^\dagger = \begin{bmatrix}
-\frac{43}{144} & \frac{13}{288} & -\frac{59}{288} & \frac{29}{144} & -\frac{41}{288} & \frac{31}{288} & \frac{1}{16} & \frac{1}{12} & -\frac{1}{48} & \frac{1}{16} & \frac{1}{16} \\
\frac{13}{288} & -\frac{415}{576} & \frac{17}{576} & -\frac{59}{288} & \frac{323}{576} & -\frac{109}{576} & \frac{9}{32} & \frac{1}{3} & -\frac{5}{96} & \frac{1}{32} & -\frac{7}{32} \\
-\frac{59}{288} & \frac{17}{576} & -\frac{271}{576} & \frac{13}{288} & -\frac{109}{576} & \frac{179}{576} & \frac{1}{32} & \frac{1}{12} & -\frac{5}{96} & \frac{9}{32} & \frac{1}{32} \\
\frac{29}{144} & -\frac{59}{288} & \frac{13}{288} & -\frac{43}{144} & \frac{31}{288} & -\frac{41}{288} & \frac{1}{16} & \frac{1}{12} & -\frac{1}{48} & \frac{1}{16} & \frac{1}{16} \\
-\frac{41}{288} & \frac{323}{576} & -\frac{109}{576} & \frac{31}{288} & -\frac{343}{576} & \frac{89}{576} & -\frac{5}{32} & -\frac{1}{6} & \frac{1}{96} & \frac{3}{32} & \frac{11}{32} \\
\frac{31}{288} & -\frac{109}{576} & \frac{179}{576} & -\frac{41}{288} & \frac{89}{576} & -\frac{199}{576} & \frac{3}{32} & \frac{1}{12} & \frac{1}{96} & -\frac{5}{32} & \frac{3}{32} \\
\frac{1}{16} & \frac{9}{32} & \frac{1}{32} & \frac{1}{16} & -\frac{5}{32} & \frac{3}{32} & -\frac{7}{16} & 0 & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\
\frac{1}{12} & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} & \frac{1}{12} & 0 & -1 & \frac{1}{2} & 0 & 0 \\
-\frac{1}{48} & -\frac{5}{96} & -\frac{5}{96} & -\frac{1}{48} & \frac{1}{96} & \frac{1}{96} & \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} & \frac{1}{16} & \frac{1}{16} \\
\frac{1}{16} & \frac{1}{32} & \frac{9}{32} & \frac{1}{16} & \frac{3}{32} & -\frac{5}{32} & \frac{1}{16} & 0 & \frac{1}{16} & -\frac{7}{16} & \frac{1}{16} \\
\frac{1}{16} & -\frac{7}{32} & \frac{1}{32} & \frac{1}{16} & \frac{11}{32} & \frac{3}{32} & \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{16} & -\frac{7}{16}
\end{bmatrix}.$$

4. Some consequences of the inverse formula for the distance matrix of a unicyclic graph. In this section, we discuss two consequences of the formula given for $D(G)^\dagger$, where G is a unicyclic graph. Using Theorem 3.4, we first show that a singular $D(G)$ is an Euclidean distance matrix (Theorem 4.3). It is proved by showing that the Laplacian-like matrix given in (3.5) is positive semidefinite. For a real symmetric matrix A, the *inertia* is defined by $\mathrm{In}(A) = (n_+(A), n_0(A), n_-(A))$, where $n_+(A)$, $n_0(A)$ and $n_-(A)$, respectively, denote the number of positive, zero, and negative eigenvalues of A. Using the fact that a singular $D(G)$ is an EDM, we recover the inertia of a singular $D(G)$, which was obtained by different

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means in [5]. The second result is to study the existence of an eigenvalue of $D(G)$. As mentioned in the introduction, -2 is an eigenvalue of $D(T_n)$ with multiplicity greater than or equal to $P(T_n) - Q(T_n) - 1$, see [4]. Motivated by this, we obtain an analogue for $D(G)$ in Theorem 4.8.

An $n \times n$ real matrix A is said to be *positive semidefinite* (*positive definite*) if $\mathbf{x}'A\mathbf{x} \geq 0$ (respectively, $\mathbf{x}'A\mathbf{x} > 0$) for all nonzero $\mathbf{x} \in \mathbb{R}^n$. We abbreviate the positive semidefinite as PSD. We now state some results on Schur complement, which are used to prove our next lemma. Let C be a nonsingular matrix and A be a square matrix. Suppose that $M = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ is a square partitioned symmetric matrix. The *Schur complement* of C in M is $A - BC^{-1}B'$. It is known that if C is positive definite then M is PSD if and only if the Schur complement of C in M is PSD [25]. Also, $\text{rank}(M) = \text{rank}(C) + \text{rank}(A - BC^{-1}B')$.

We first show that the constructed Laplacian-like matrix L in the previous section is PSD.

Lemma 4.1. *Let $G(m, l)$ be a unicyclic graph where $m = 2k$. Let L be defined as in (3.5). Then L is a positive semidefinite matrix and rank $(L) = k + l$.*

Proof. We first prove that the symmetric matrix $\mathcal{L}$ given in (3.1) is PSD with $\text{rank}(\mathcal{L}) = k$. Recall that $\mathcal{L}$ is a circulant matrix generated by $\frac{1}{4}\widetilde{\mathbf{z}}$ where $\widetilde{\mathbf{z}} = 2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_k - 2\mathbf{e}_{k+1} + \mathbf{e}_{k+2} - \mathbf{e}_{2k}$. By using Lemma 2.1, the eigenvalues of $\mathcal{L}$ are $\frac{1}{4}f(\omega_j)$, where $f(x) = 2 - x + x^{k-1} - 2x^k + x^{k+1} - x^{m-1}$ and $\omega_j = \exp\left(\frac{i2\pi j}{m}\right)$, $0 \leq j \leq m-1$. Since $m = 2k$, we have $\omega_j^k = \exp\left(\frac{i2k\pi j}{m}\right) = \exp(i\pi j)$. Note that $\omega_j^{m-1} = \frac{1}{\omega_j}$ and $\omega_j + \frac{1}{\omega_j} = 2\cos(\frac{2\pi j}{m})$. Now

$$\begin{aligned} f(\omega_j) &= 2 - \omega_j + \omega_j^{k-1} - 2\omega_j^k + \omega_j^{k+1} - \omega_j^{m-1} \\ &= 2 - 2\omega_j^k + \omega_j^k\left(\omega_j + \frac{1}{\omega_j}\right) - \left(\omega_j + \frac{1}{\omega_j}\right) \\ &= 2 - 2\exp(i\pi j) + 2\exp(i\pi j)\cos\left(\frac{2\pi j}{m}\right) - 2\cos\left(\frac{2\pi j}{m}\right) \\ &= \begin{cases} 4 - 4\cos(\frac{2\pi j}{m}) & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases} \end{aligned}$$

Thus, for any odd integer j, we have $f(\omega_j) > 0$. Therefore, all the eigenvalues of $\mathcal{L}$ are nonnegative. Hence, $\mathcal{L}$ is PSD, and the rank of $\mathcal{L}$ is k. We now proceed by induction on l. If $l = 1$, then from (3.10), we have $L = \begin{bmatrix} \mathcal{L} + \mathbf{x}_{m+1}\mathbf{x}'_{m+1} & \mathbf{x}_{m+1} \\ \mathbf{x}'_{m+1} & 1 \end{bmatrix}$. Then the Schur complement of 1 in L is $\mathcal{L}$. Therefore, the matrix L is PSD and $\text{rank}(L) = 1 + \text{rank}(\mathcal{L}) = k + 1$. Assume that the result is true for $l - 1$. To prove for l, we recall the matrix L given in (3.9) in terms of $\bar{L}$. As $L = \begin{bmatrix} \bar{L} + \mathbf{y}\mathbf{y}' & \mathbf{y} \\ \mathbf{y}' & 1 \end{bmatrix}$, the Schur complement of 1 in L is $\bar{L}$. By the induction assumptions, the result follows. ∎

Next, we show that the distance matrix of an even-unicyclic graph is an EDM. In order to prove the result, let us recall a characterization of an EDM from [18].

Theorem 4.2. *[18] Let A be an $n \times n$ nonzero symmetric matrix with zero diagonal entries. Suppose that $A\mathbf{x} = \mathbf{e}$ has a solution and there exists a Laplacian-like matrix L of order n such that $A^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ for some $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^n$. Then A is an EDM if and only if L is a positive semidefinite matrix. Furthermore, rank $(A) =$ rank $(L) + 1$.*

In the following theorem, we get the desired result.

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Theorem 4.3. *Let $G(m,l)$ be a unicyclic graph where $m = 2k$. Then the distance matrix $D(G)$ is an Euclidean distance matrix and its rank is $k+l+1$.*

Proof. The result follows from Lemma 4.1 and Theorems 3.4 and 4.2. ◻

Remark 4.4. *It is proved that $(1,k-1,k+l)$ is the inertia of the distance matrix of a unicyclic graph $G(m,l)$ where $m = 2k$ (Theorem 3.7 in [5]). This result can be obtained as a consequence of the above theorem. Since $D(G)$ is an EDM, it has a unique positive eigenvalue (see [18]). As $D(G)$ is symmetric, $n_-(D(G)) = rank\ (D(G)) - n_+(D(G)) = k+l$. Hence, $In(D(G)) = (1,k-1,k+l)$.*

To establish another main result of this section, in the next two lemmas, we show that 1 is an eigenvalue of the Laplacian-like matrix associated with $D(G)$. We need to consider the even-unicyclic and odd-unicyclic cases separately because the associated Laplacian-like matrices are different.

For a quasipendant vertex v, let X_v denote the set of all pendant vertices in G which are adjacent to v. Then, $X_v \cap X_w = \phi$ if $v \neq w$ and $P(G) = \sum_v |X_v|$ where v runs over all quasipendant vertices of G and $|X_v|$ is the cardinality of X_v. For a square matrix A, the *multiplicity* of an eigenvalue γ refers to the number of times γ appears as a root of the characteristic equation of A. Note that the multiplicity of an eigenvalue γ is always greater than or equal to the number of linearly independent eigenvectors of A corresponding to γ.

Lemma 4.5. *Let $D(G)$ be the distance matrix of an even-unicyclic graph $G(m,l)$ and L be the associated Laplacian-like matrix of $D(G)$ given in (3.5). Then 1 is an eigenvalue of L with multiplicity at least $P(G)-Q(G)$.*

Proof. Let v be a quasipendant vertex of G. Suppose that u and w are two pendant vertices of G, which are adjacent to v. Then u and w are not in C_m. Without loss of generality, we assume that $u = v_{m+i}$ and $w = v_{m+j}$ where $i < j$. There are two possibilities depending on whether or not the vertex v lies in the cycle C_m. In both cases, by (3.4) and (3.5), we have $\mathbf{x}_{m+i} = \mathbf{x}_{m+j}$ and $L_{22}(\mathbf{e}_i - \mathbf{e}_j) = \mathbf{e}_i - \mathbf{e}_j$. Therefore,

$$L\begin{bmatrix}\mathbf{0}\\ \mathbf{e}_i-\mathbf{e}_j\end{bmatrix} = L_{*(m+i)} - L_{*(m+j)} = \begin{bmatrix}\mathbf{x}_{m+i}-\mathbf{x}_{m+j}\\ \mathbf{e}_i-\mathbf{e}_j\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{e}_i-\mathbf{e}_j\end{bmatrix}.$$

This implies that $\begin{bmatrix}\mathbf{0}' & (\mathbf{e}_i-\mathbf{e}_j)'\end{bmatrix}'$ is an eigenvector of L corresponding to the eigenvalue 1. Proceeding in this way, we generate $|X_v|-1$ linearly independent eigenvectors of L corresponding to 1 with respect to the quasipendant vertex v. Hence, we obtain $\sum_v(|X_v|-1) = P(G)-Q(G)$ linearly independent eigenvectors of L corresponding to 1. This completes the proof. ◻

Note that Lemma 4.5 gives a lower bound for the multiplicity of an eigenvalue 1. If $P(G)-Q(G) = 0$, then we cannot conclude that 1 is not an eigenvalue of L. For instance, consider the graph given in Figure 1. Then $P(G) = Q(G) = 4$. But 1 is an eigenvalue of L with $\begin{bmatrix}1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0\end{bmatrix}'$ as a corresponding eigenvector.

We now consider the odd-unicyclic graph. Let m be an odd integer and G be a unicyclic graph with the cycle C_m. Recall that the distance matrix $D(G)$ is nonsingular [5], and its inverse formula is obtained in [14]. We now recollect the Laplacian-like matrix, associated with the inverse formula for $D(G)$, from [14]. Note that a block of G is either C_m or K_2 (= the complete graph on 2 vertices). Suppose that $H_0, H_1, \ldots, H_l$ are the blocks of G. Without loss of generality, assume that $H_0 = C_m$ and $H_j = K_2$ for $1 \leq j \leq l$.

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For the cycle on $m = 2k + 1$ vertices, the Laplacian-like matrix $L(H_0)$ of order m is given by

$$(L(H_0))_{rs} = \begin{cases} 2 - \frac{4}{m} & \text{if } v_r = v_s, \\ 1 - \frac{4}{m} & \text{if } s = r + k \text{ or } s = r + k + 1 \text{ (taking modulo } 2k + 1\text{)}, \\ -\frac{4}{m} & \text{otherwise.} \end{cases}$$

If $1 \le j \le l$, then the Laplacian-like matrix $L(H_j) = \frac{1}{2}L(K_2)$, where $L(K_2)$ is the Laplacian matrix of K_2.

For each $j \in \{0, 1, \ldots, l\}$, construct a graph G_j with $V(G_j) = V(G)$ and $E(G_j) = E(H_j)$ and an $n \times n$ matrix $\widetilde{\mathcal{L}}_j$ as follows:

$$(\widetilde{\mathcal{L}}_j)_{rs} = \begin{cases} (L(H_j))_{rs} & \text{if } v_r, v_s \in V(H_j), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, G_j has isolated vertices and containing H_j as a subgraph. It has been proved that

$$D(G)^{-1} = -\frac{1}{2}L + \alpha \mathbf{w}\mathbf{w}' \text{ where } L = 2\sum_{j=0}^{l} \widetilde{\mathcal{L}}_j, \tag{4.1}$$

for some nonzero scalar α and $\mathbf{w} \in \mathbb{R}^n$, see [14, 18] for more details.

Lemma 4.6. *Let $D(G)$ be the distance matrix of an odd-unicyclic graph G and L be the associated Laplacian-like matrix of $D(G)$ given in (4.1). Then 1 is an eigenvalue of L with multiplicity at least $P(G) - Q(G)$.*

Proof. Note that each edge of G belongs to exactly one of its blocks. From the construction of the matrix L, it is easy to see that if v_s is a pendant vertex of G, then the corresponding column L_{*s} of L has exactly two nonzero entries. To be precise, we have

$$(L)_{rs} = \begin{cases} 1 & \text{if } v_r = v_s, \\ -1 & \text{if } v_r \sim v_s, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose v_{m+i} and v_{m+j} are two distinct pendant vertices adjacent with a quasipendant vertex v. Then we have

$$L(\mathbf{e}_{m+i} - \mathbf{e}_{m+j}) = L_{*(m+i)} - L_{*(m+j)} = \mathbf{e}_{m+i} - \mathbf{e}_{m+j}.$$

By the same argument used in the proof of Lemma 4.5, the desired conclusion follows. ∎

The next result presents an interlacing property between the distance matrix $D(G)$ of a unicyclic graph G and its associated Laplacian-like matrix L.

Remark 4.7. *Let G be a unicyclic graph of order n. It is shown that $D(G)$ is an EDM if the cycle length is odd (Theorem 4.3 in [18]). Using this result and Theorem 4.3, we conclude that $D(G)$ is an EDM. Therefore, by Theorem 4.2, L is a positive semidefinite matrix. Hence all the eigenvalues of L are nonnegative. As $D(G)$ is an EDM, it has only one positive eigenvalue. Suppose that rank $(L) = s$. Then from Theorem 4.2, we have rank $(D(G)) = s + 1$. Therefore, $D(G)$ has s negative eigenvalues and 0 is an eigenvalue of $D(G)$ with multiplicity $n - (s + 1)$. Let $\gamma_1 > 0 = \gamma_{s+2} = \cdots = \gamma_n > \gamma_2 \ge \gamma_3 \ge \cdots \ge \gamma_{s+1}$ and $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_s > 0 = \beta_{s+1} = \cdots = \beta_n$ be the eigenvalues of $D(G)$ and L, respectively. From Theorem 5.2 in [18], we obtain the interlacing property between the eigenvalues of $D(G)$ and L. That is,*

$$\gamma_1 > -\frac{2}{\beta_1} \ge \gamma_2 \ge -\frac{2}{\beta_2} \ge \gamma_3 \ge \cdots \ge \gamma_s \ge -\frac{2}{\beta_s} \ge \gamma_{s+1}.$$

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Using the interlacing result, in the following, we establish the existence of the eigenvalue −2 of $D(G)$.

Theorem 4.8. *Let G be a unicyclic graph and $D(G)$ be the distance matrix of G. Then −2 is an eigenvalue of $D(G)$ with multiplicity at least $P(G) - Q(G) - 1$.*

Proof. By combining Lemmas 4.5 and 4.6 together with Remark 4.7, we obtain the required result. ∎

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