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A NOTE ON PRODUCTS OF FINITE-DIMENSIONAL QUADRATIC MATRICES*

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Abstract. Let F be a field, n be a positive integer, and $q(x) = (x-\lambda_1)(x-\lambda_2)$, where λ_1 and λ_2 are two nonzero elements in F. Denote by $\mathbb{M}_n(F)$ the ring of all $n \times n$ matrices over F. A matrix $A \in \mathbb{M}_n(F)$ is called quadratic with respect to $q(x)$ if $q(A) = 0$. In this paper, we investigate the question of when a matrix in $\mathbb{M}_n(F)$ can be expressed as a product of quadratic matrices with respect to $q(x)$. First, we prove that if F is a field with more than $n+1$ elements, $k \geq 0$ is an integer, and $A \in \mathbb{M}_n(F)$ has determinant $\lambda_1^{s+2n}\lambda_2^{t+2n}$, where $s, t \geq 0$ are integers such that $s+t = kn$, then A can be expressed as a product of $k+4$ quadratic matrices with respect to $q(x)$. In particular, if $\lambda_1 = 1$, $\lambda_2^r = 1$ for some integer $r \geq 2$, and $A \in \mathbb{M}_n(F)$ has a determinant that is a power of λ_2, then A can be expressed as a product of at most $2r$ quadratic matrices with respect to $q(x)$. As a corollary, we derive results on the factorization of matrices as products of certain special quadratic matrices.

Key words. General linear group, Involution, Product of matrices, Quadratic matrix, Unipotent matrix of index 2.

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1. Introduction and main results. Throughout this paper, unless otherwise stated, we always assume that F is an arbitrary field, n is a positive integer, and $q(x) = (x-\lambda_1)(x-\lambda_2)$, where λ_1 and λ_2 are two nonzero elements in F. We do not require λ_1 to be distinct from λ_2. We denote

- $\mathbb{M}_n(F)$ as the ring of all $n \times n$ square matrices over F,
- $\mathbb{GL}_n(F)$ as the group of all invertible matrices in $\mathbb{M}_n(F)$,
- $\mathbb{SL}_n(F)$ as the subgroup of $\mathbb{GL}_n(F)$ consisting of all matrices with determinant equal to 1,
- $\pm\mathbb{SL}_n(F)$ as the subgroup of $\mathbb{GL}_n(F)$ consisting of all matrices with determinant equal to 1 or -1,
- $\det(A)$ as the determinant of a matrix A in $\mathbb{M}_n(F)$,
- I_n as the identity element of $\mathbb{M}_n(F)$.

A matrix A in $\mathbb{M}_n(F)$ is called quadratic with respect to $q(x)$ if $q(A) = 0$. For example, an involutory matrix (resp., a unipotent matrix of index 2, a skew-involutory matrix) is a quadratic matrix with respect to $x^2 - 1$ (resp., $(x-1)^2$, x^2+1). The properties of quadratic matrices and detailed information can be found in [1]. In this paper, we are interested in the question of when a matrix in $\mathbb{GL}_n(F)$ can be expressed as a product of k quadratic matrices with respect to $q(x)$, where k is a given positive integer. This problem has a long history and has drawn significant attention from mathematicians.

Let us begin with the products of involutions. In [16], A. R. Sampson proved that if A is a matrix in $\pm\mathbb{SL}_n(\mathbb{R})$, then A can be expressed as a product of involutions. However, he did not specify the minimal

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number of involutions required in each such factorization. Later, in 1976, W. H. Gustafson, P. R. Halmos, and H. Radjavi extended Sampson's result to an arbitrary field F and highlighted that every matrix in $\pm\mathbb{SL}_n(F)$ can be expressed as a product of at most four involutions (see [7, 8]). On the products of unipotent matrices of index 2, J. H. Wang and P. Y. Wu showed in [18] that every matrix in $\mathbb{SL}_n(\mathbb{C})$ can be expressed as a product of at most four unipotent matrices of index 2. A slight extension of this result was given by A. Lev in [11]. In a recent paper [15], C. de Seguins Pazzis demonstrated that the results in [18, 11] hold for an arbitrary field. The products of two or three involutions and unipotent matrices of index 2 have also been extensively studied in various works (e.g., see [4, 6, 9, 15, 19]).

In recent years, there has been growing interest in the broader problem of products of generalized quadratic matrices (see [5, 12, 13, 14]). Most recently, in [2], M. H. Bien, M. Ramezan-Nassab, and L. Q. Truong showed that if F is an algebraically closed field of characteristic not equal to 2 and $\lambda_1\lambda_2 = 1$, then every matrix in $\mathbb{SL}_n(F)$ can be expressed as a product of at most four quadratic matrices with respect to $q(x)$. The products of generalized upper triangular quadratic matrices have also been studied in [3]. Motivated by these works, we continue the investigation of products of generalized quadratic matrices in $\mathbb{GL}_n(F)$. It is straightforward to observe that if A is a product of h quadratic matrices with respect to $q(x)$, then $\det(A) = \lambda_1^s\lambda_2^t$ for some integers $s, t \geq 0$ such that $s + t = hn$. However, the converse does not hold in general. For instance, if $n \geq 3$ and $\lambda \in \mathbb{C} \setminus \{\pm 1\}$ satisfies $\lambda^n = 1$, then by [18, Theorem 3.1], the matrix $\lambda \mathrm{I}_n$ cannot be expressed as a product of three quadratic matrices with respect to $(x-1)^2$. Moreover, a sufficient condition for a matrix in $\mathbb{SL}_n(F)$ to be expressible as a product of three quadratic matrices with respect to $(x-1)^2$ is given in [15, Theorem 1.6]. First, we provide a sufficient condition for a matrix to be expressible as a product of $h \geq 4$ quadratic matrices with respect to $q(x)$. Our first main result is stated as follows.

Theorem 1.1. *Let $k, s, t \geq 0$ be integers such that $s + t = kn$, and let $A \in \mathbb{M}_n(F)$ have determinant $\lambda_1^{s+2n}\lambda_2^{t+2n}$. If n is odd or F has more than $n + 1$ elements, then A can be expressed as a product of $k + 4$ quadratic matrices with respect to $q(x)$.*

By directly applying Theorem 1.1, we obtain the result of J. H. Wang and P. Y. Wu on the product of four unipotent matrices of index 2, as presented in [18]. Furthermore, Theorem 1.1 allows us to generalize a result from [2] to fields with sufficiently many elements that are not necessarily algebraically closed and have characteristic different from 2.

The remainder of this paper considers the case when $q(x)$ has one root equal to 1, and the other root has finite order in the multiplicative group $F \setminus \{0\}$. Without the loss of generality, we assume that $\lambda_1 = 1$ and $\lambda_2^r = 1$ for some integer $r \geq 2$. As mentioned earlier, this case is particularly interesting, and when $\lambda_2 = 1$ or $\lambda_2 = -1$, the problem is completely solved by W. H. Gustafson, P. R. Halmos, H. Radjavi in [8], and C. de Seguins Pazzis in [15]. In this case, it is obvious that if A is a product of quadratic matrices with respect to $q(x)$, then $\det(A)$ has the form λ_2^t for some integer t. The following theorem shows that any matrix A of such form is a product of at most $2r$ quadratic matrices with respect to $q(x)$.

Theorem 1.2. *Suppose $\lambda_1 = 1$ and $\lambda_2^r = 1$ for some integer $r \geq 2$. Let A be a matrix in $\mathbb{M}_n(F)$ such that $\det(A) = \lambda_2^t$ for some integer t. Then A can be expressed as a product of at most $2r$ quadratic matrices with respect to $q(x)$.*

2. Products of quadratic matrices.

Let λ be an element in F and m be a positive integer. The Jordan block of size m with eigenvalue λ in $\mathbb{M}_m(F)$ is denoted by $J_m(\lambda)$, that is,

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

In this section, we discuss some fundamental properties of products of quadratic matrices. Additionally, we outline preliminary results on the factorization of triangular matrices into products of quadratic matrices, with further details referenced from [3]. First, we can easily verify the following two minor remarks.

Remark 2.1. *Let $k \geq 1$ be an integer and A be a matrix in $\mathbb{M}_n(F)$. If A is a product of k quadratic matrices with respect to $q(x)$, then for every P in $\mathbb{GL}_n(F)$, $P^{-1}AP$ is also a product of k quadratic matrices with respect to $q(x)$.*

Let $n_1, n_2, \ldots, n_s$ be positive integers and let A_i, for $i \in \overline{1,s}$, be a matrix in $\mathbb{M}_{n_i}(F)$. Denote by $A_1 \oplus A_2 \oplus \ldots \oplus A_s$, or simply $\bigoplus_{i=1}^{s} A_i$, the block diagonal matrix

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix}.$$

Thus, $\bigoplus_{i=1}^{s} A_i$ is a matrix in $\mathbb{M}_n(F)$ where $n = n_1 + n_2 + \ldots + n_s$.

Remark 2.2. *Let $n_1, n_2, \ldots, n_s, k$ be positive integers. If A_i is a product of k quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_{n_i}(F)$ for every $i \in \overline{1,s}$, then $A_1 \oplus A_2 \oplus \ldots \oplus A_s$ is a product of k quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$, where $n = n_1 + n_2 + \ldots + n_s$.*

The important lemma below provides a sufficient condition for a matrix to be expressed as a product of two upper triangular quadratic matrices.

Lemma 2.3. [3, Lemma 2.4] *Let $A = (a_{ij})$ be an upper triangular matrix in $\mathbb{M}_n(F)$ that satisfies the following conditions:*

(1) $a_{ii} = \lambda_1\lambda_2$ *for all* $1 \leq i \leq n$,
(2) $a_{i,i+1} \neq 0$ *for all* $1 \leq i \leq n-1$.

Then A can be expressed as a product of two upper triangular quadratic matrices with respect to $q(x)$.

Corollary 2.4. *Let $A \in \mathbb{M}_n(F)$ be an (upper or lower) triangular matrix whose diagonal entries are equal to $\lambda_1\lambda_2$. Then A can be expressed as a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.*

Proof. Since A is a triangular matrix whose diagonal entries are equal to $\lambda_1\lambda_2$, there exists a matrix $P \in \mathbb{GL}_n(F)$ such that

$$P^{-1}AP = J_{n_1}(\lambda_1\lambda_2) \oplus J_{n_2}(\lambda_1\lambda_2) \oplus \ldots \oplus J_{n_s}(\lambda_1\lambda_2),$$

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where $n_1, n_2, \dots, n_s$ are positive integers such that $n_1 + n_2 + \cdots + n_s = n$. By Lemma 2.3, each $J_{n_i}(\lambda_1\lambda_2)$ can be expressed as a product of two quadratic matrices with respect to $q(x)$. By Remarks 2.2 and 2.1, it follows that A can be expressed as a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. $\square$

For convenience of presentation, we adopt the convention that the identity matrix is regarded as a product of zero quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. It is obvious that for every pair of integers $s, t \geq 0$ such that $s + t = n$, there exists a diagonal matrix D in $\mathbb{GL}_n(F)$ such that $\det(D) = \lambda_1^s\lambda_2^t$ and D is a quadratic matrix with respect to $q(x)$ in $\mathbb{GL}_n(F)$. For example,

$$D = \lambda_1 \mathrm{I}_s \oplus \lambda_2 \mathrm{I}_t.$$

Therefore, we can easily establish the following lemma, and its proof is omitted.

Lemma 2.5. *Let $k, s, t \geq 0$ be integers such that $s + t = kn$. Then there exists a diagonal matrix D in $\mathbb{GL}_n(F)$ satisfying the following conditions:*

(1) $\det(D) = \lambda_1^s\lambda_2^t$,
(2) *D can be expressed as a product of k quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.*

3. Factorizations of matrices. This section outlines the primary strategy for matrix factorizations. The proofs of Theorems 1.1 and 1.2 share a common structure. Specifically, we divide the proof into two cases: scalar matrices and nonscalar matrices. Recall that a matrix A in $\mathbb{M}_n(F)$ is called scalar if it can be written in the form $A = \lambda \mathrm{I}_n$ for some element λ in F. Otherwise, A is said to be nonscalar. For the case of nonscalar matrices, we borrow the following factorization theorem from the paper [17] by A. R. Sourour.

Theorem 3.1. [17, Theorem 1] *Let A be a nonscalar matrix in $\mathbb{GL}_n(F)$, and let b_i, c_i, for $i \in \overline{1, n}$, be elements in F such that $\prod_{i=1}^{n} b_i c_i = \det(A)$. Then there exist matrices B and C in $\mathbb{GL}_n(F)$ with eigenvalues $b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$, respectively, such that $A = BC$. Moreover, B and C can be chosen so that B is lower triangularizable and C is simultaneously upper triangularizable.*

In fact, there is relatively little information available for the case of scalar matrices. In this section, we present factorization theorems for scalar matrices (see Theorems 3.2 and 3.3), which serve as key tools in establishing the main results of this paper.

For each element μ in $F \setminus \{0\}$, denote

$$\mathcal{D}(\mu) = \begin{pmatrix} \lambda_1\lambda_2\mu & 0 \\ 0 & \lambda_1\lambda_2\mu^{-1} \end{pmatrix} \in \mathbb{GL}_2(F).$$

Theorem 3.2. *Suppose n is an odd natural number and A is a scalar matrix in $\mathbb{GL}_n(F)$ such that $\det(A) = \lambda_1^{2n}\lambda_2^{2n}$. Then A can be expressed as a product of two matrices B and C in $\mathbb{GL}_n(F)$, where*

$$B = \bigoplus_{i=1}^{\frac{n-1}{2}} \mathcal{D}(b_i) \oplus (\lambda_1\lambda_2),$$

$$C = (\lambda_1\lambda_2) \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \mathcal{D}(c_i),$$

with $b_i, c_i \in F \setminus \{0\}$ *for all* $i \in \overline{1, \frac{n-1}{2}}$. *Moreover, if* F *has characteristic different from* 2*, the elements* b_i *and* c_i *can be chosen such that* $b_i, c_i \neq -1$ *for all* $i \in \overline{1, \frac{n-1}{2}}$.

Proof. Since A is a scalar matrix in $\mathbb{GL}_n(F)$, whose determinant is $\lambda_1^{2n}\lambda_2^{2n}$, we can write A as $A = \mu \mathrm{I}_n$, where μ is an element of $F \setminus \{0\}$ such that $\mu^n = \lambda_1^{2n}\lambda_2^{2n}$. Let $b_i = (\lambda_1\lambda_2)^{-4i+2}\mu^{2i-1}$ and $c_i = (\lambda_1\lambda_2)^{-4i}\mu^{2i}$ for all $i \in \overline{1, \frac{n-1}{2}}$. Define

$$B = \bigoplus_{i=1}^{\frac{n-1}{2}} \mathcal{D}(b_i) \oplus (\lambda_1\lambda_2) \quad \text{and} \quad C = (\lambda_1\lambda_2) \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \mathcal{D}(c_i).$$

Since $\mu^n = \lambda_1^{2n}\lambda_2^{2n}$, it is straightforward to verify that $A = BC$.

Suppose F has characteristic different from 2. We will prove that $b_i, c_i \neq -1$ for all $i \in \overline{1, \frac{n-1}{2}}$, that is, $(\lambda_1\lambda_2)^{-2i}\mu^i \neq -1$ for all $i \in \overline{1, n-1}$. Indeed, suppose there exists an $i \in \overline{1, n-1}$ such that $(\lambda_1\lambda_2)^{-2i}\mu^i = -1$. Since n is odd, we have $(\lambda_1\lambda_2)^{-2ni}\mu^{ni} = -1$. However, since $\mu^n = \lambda_1^{2n}\lambda_2^{2n}$, we obtain $-1 = 1$, which is a contradiction. ∎

Theorem 3.3. *Suppose* n *is an even natural number,* F *has more than* $n+1$ *elements, and* A *is a scalar matrix in* $\mathbb{GL}_n(F)$ *such that* $\det(A) = \lambda_1^{2n}\lambda_2^{2n}$. *Then* A *can be expressed as a product of two matrices* B *and* C *in* $\mathbb{GL}_n(F)$*, where* B *and* C *are similar to*

$$B' = \bigoplus_{i=1}^{\frac{n}{2}} \mathcal{D}(b_i) \quad \textit{and} \quad C' = \bigoplus_{i=1}^{\frac{n}{2}} \mathcal{D}(c_i),$$

with $b_i, c_i \in F \setminus \{0, -1\}$ *for all* $i \in \overline{1, \frac{n}{2}}$*, respectively.*

Proof. We write A as $A = \mu \mathrm{I}_n$, where μ is an element of $F \setminus \{0\}$ such that $\mu^n = \lambda_1^{2n}\lambda_2^{2n}$. Since F has more than $n+1$ elements, we can choose an element α in $F \setminus \{0\}$ such that $\alpha \neq -(\lambda_1\lambda_2)^{4i-2}\mu^{-2i+1}$ and $\alpha \neq -(\lambda_1\lambda_2)^{4i-4}\mu^{-2i+2}$ for all $i \in \overline{1, \frac{n}{2}}$. Set

$$B = \bigoplus_{i=1}^{\frac{n}{2}} (\lambda_1\lambda_2)^{-4i+2}\mu^{2i-1}\mathcal{D}(\alpha),$$

and

$$C = \bigoplus_{i=1}^{\frac{n}{2}} (\lambda_1\lambda_2)^{4i-4}\mu^{-2i+2}\mathcal{D}\left(\alpha^{-1}\right).$$

By a simple calculation, it can be verified that $A = BC$. We now proceed to define $b_i = (\lambda_1\lambda_2)^{-4i+2}\mu^{2i-1}\alpha$ and $c_i = (\lambda_1\lambda_2)^{4i-4}\mu^{-2i+2}\alpha^{-1}$ for all $i \in \overline{1, \frac{n}{2}}$. Since $\mu^n = \lambda_1^{2n}\lambda_2^{2n}$, it is straightforward to see that B and C are similar to

$$B' = \bigoplus_{i=1}^{\frac{n}{2}} \mathcal{D}(b_i) \quad \text{and} \quad C' = \bigoplus_{i=1}^{\frac{n}{2}} \mathcal{D}(c_i),$$

respectively. Finally, by the choice of α, it is obvious that $b_i, c_i \notin \{0, -1\}$ for all $i \in \overline{1, \frac{n}{2}}$. Thus, we have completed the proof of this theorem. ∎

4. Proofs of the main results.

From the results presented in Sections 2 and 3, we are now prepared to establish the two main results of this paper. For clarity, this section is divided into two subsections.

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4.1. Proof of Theorem 1.1. The following lemma shows the factorization of a matrix of the form $\mathcal{D}(\mu)$, for $\mu \in F \setminus \{0\}$, as a product of quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$. It is used very effectively in the proof for the case of scalar matrices.

Lemma 4.1. *Let μ be an element of $F \setminus \{0\}$. The matrix $\mathcal{D}(\mu)$ can be expressed as a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$ if $\mu \neq -1$ or if the characteristic of F is 2.*

Proof. If $\mu \neq -1$, we define

$$A_1 = \begin{pmatrix} \dfrac{(\lambda_1+\lambda_2)\mu}{1+\mu} & -\mu \\ \dfrac{\lambda_1\lambda_2}{\mu} - \dfrac{(\lambda_1+\lambda_2)^2}{(1+\mu)^2} & \dfrac{\lambda_1+\lambda_2}{1+\mu} \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \dfrac{(\lambda_1+\lambda_2)\mu}{1+\mu} & 1 \\ \dfrac{(\lambda_1+\lambda_2)^2\mu}{(1+\mu)^2} - \lambda_1\lambda_2 & \dfrac{\lambda_1+\lambda_2}{1+\mu} \end{pmatrix}.$$

A simple calculation shows that both A_1 and A_2 have trace $\lambda_1+\lambda_2$ and determinant $\lambda_1\lambda_2$. Therefore, $q(A_1) = q(A_2) = 0$, so A_1 and A_2 are quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$. Moreover, it is easy to see that $\mathcal{D}(\mu) = A_1A_2$, so $\mathcal{D}(\mu)$ can be expressed as a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$.

If $\mu = -1$ and the characteristic of F is 2, then $\mu = 1$. In this case, it is trivial that $\mathcal{D}(1)$ is a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$. □

Note that for $\lambda_1 = \lambda_2 = 1$, according to [10, Lemma 2.8], the matrix $\mathcal{D}(-1)$ cannot be expressed as a product of two quadratic matrices with respect to $(x-1)^2$ in $\mathbb{GL}_2(\mathbb{C})$. Therefore, in the general case, we must assume $\mu \neq -1$. We now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. According to Lemma 2.5, there exists a diagonal matrix D in $\mathbb{GL}_n(F)$ such that $\det(D) = \lambda_1^s\lambda_2^t$, and D can be expressed as a product of k quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. It is obvious that $D^{-1}A$ is a matrix in $\mathbb{GL}_n(F)$ with determinant $\lambda_1^{2n}\lambda_2^{2n}$. We will prove that $D^{-1}A$ can be expressed as a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$ by considering the following two cases.

Case 1. $D^{-1}A$ is nonscalar. It follows from Theorem 3.1 that there exists a matrix P in $\mathbb{GL}_n(F)$ such that $P^{-1}(D^{-1}A)P$ can be written as the product

$$P^{-1}(D^{-1}A)P = \underbrace{\begin{pmatrix} \lambda_1\lambda_2 & & & \\ & \lambda_1\lambda_2 & & \\ & & \ddots & \\ * & & & \lambda_1\lambda_2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \lambda_1\lambda_2 & & & * \\ & \lambda_1\lambda_2 & & \\ & & \ddots & \\ & & & \lambda_1\lambda_2 \end{pmatrix}}_{C}.$$

Since B and C are triangular matrices whose diagonal entries are equal to $\lambda_1\lambda_2$, it follows from Corollary 2.4 that they can be expressed as products of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Finally, by Remark 2.1, $D^{-1}A$ is a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

Case 2. $D^{-1}A$ is scalar. In this case, we further consider two subcases based on the parity of n.

Subcase 2.1. n is odd. According to Theorem 3.2, $D^{-1}A$ can be expressed as a product of two matrices B and C in $\mathbb{GL}_n(F)$, where

$$B=\bigoplus_{i=1}^{\frac{n-1}{2}}\mathcal{D}(b_i)\oplus(\lambda_1\lambda_2)\quad\text{and}\quad C=(\lambda_1\lambda_2)\oplus\bigoplus_{i=1}^{\frac{n-1}{2}}\mathcal{D}(c_i),$$

with $b_i,c_i\in F\setminus\{0\}$ for all $i\in\overline{1,\frac{n-1}{2}}$. Moreover, if F has characteristic different from 2, the elements b_i and c_i can be chosen such that $b_i,c_i\neq-1$ for all $i\in\overline{1,\frac{n-1}{2}}$. Therefore, by applying Lemma 4.1, $\mathcal{D}(b_i)$ and $\mathcal{D}(c_i)$ are products of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$ for all $i\in\overline{1,\frac{n-1}{2}}$. From Remark 2.2, it follows that B and C are products of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Thus, $D^{-1}A$ can be expressed as a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

Subcase 2.2. n is even. From Theorem 3.3, it follows that $D^{-1}A$ can be expressed as a product of two matrices B and C in $\mathbb{GL}_n(F)$, where B and C are similar to

$$B'=\bigoplus_{i=1}^{\frac{n}{2}}\mathcal{D}(b_i)\quad\text{and}\quad C'=\bigoplus_{i=1}^{\frac{n}{2}}\mathcal{D}(c_i),$$

with $b_i,c_i\in F\setminus\{0,-1\}$ for all $i\in\overline{1,\frac{n}{2}}$, respectively. As stated in Lemma 4.1, $\mathcal{D}(b_i)$ and $\mathcal{D}(c_i)$ are products of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_2(F)$ for all $i\in\overline{1,\frac{n}{2}}$. Therefore, it is easy to see that B and C are products of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$ by Remarks 2.2 and 2.1. Hence, $D^{-1}A$ can be expressed as a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

In conclusion, from both cases above, we deduce that $D^{-1}A$ can be expressed as a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Thus, $A=D(D^{-1}A)$ can be expressed as a product of $k+4$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. ◻

Next, we discuss some direct consequences of Theorem 1.1. First, we apply Theorem 1.1 to the case where F is the field of complex numbers $\mathbb{C}$, $\lambda_1=1$, $\lambda_2=1$, and $k=0$. From this, we easily obtain [18, Theorem 3.5] on the product of unipotent matrices of index 2.

COROLLARY 4.2. [18, Theorem 3.5] *Every matrix in $\mathbb{SL}_n(\mathbb{C})$ can be expressed as a product of four unipotent matrices of index 2 in $\mathbb{GL}_n(\mathbb{C})$.*

Also from Theorem 1.1, we establish that [2, Corollary 1.5] remains valid for any field with sufficiently many elements, not necessarily algebraically closed, and of characteristic different from 2.

COROLLARY 4.3. *Suppose $\lambda_1\lambda_2=1$ and A is a matrix in $\mathbb{SL}_n(F)$. If n is odd or F has more than $n+1$ elements, then A can be expressed as a product of four quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.*

4.2. Proof of Theorem 1.2.

In this subsection, we always assume that $\lambda_1=1$ and $\lambda_2^r=1$ for some integer $r\geq 2$. Since the identity matrix is a quadratic matrix with respect to $q(x)$, it is simple to prove the following remark.

REMARK 4.4. *Let n_i,k_i, for $i\in\overline{1,s}$, be positive integers. If A_i is a product of k_i quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_{n_i}(F)$ for all $i\in\overline{1,s}$, then $A_1\oplus A_2\oplus\ldots\oplus A_s$ is a product of k quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$, where $k=\max\{k_1,k_2,\ldots,k_s\}$ and $n=n_1+n_2+\cdots+n_s$.*

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Theorem 4.5. *If A is a nonscalar matrix in $\mathbb{M}_n(F)$ such that $\det(A) = \lambda_2^t$ for some integer t, then A can be expressed as a product of at most $r + 2$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.*

Proof. From Theorem 3.1, it follows that there exists a matrix $P \in \mathbb{GL}_n(F)$ such that $P^{-1}AP$ can be expressed as the product

$$P^{-1}AP = \underbrace{\begin{pmatrix} \lambda_2 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ * & & & \lambda_2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \lambda_2 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_2^{t-2n+1} \end{pmatrix}}_{C}.$$

We now prove that C can be expressed as a product of at most r quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

Case 1. $\lambda_2^{t-2n+1} = \lambda_2$. By Corollary 2.4, C is a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Since $r \geq 2$, C can be expressed as a product of at most r quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

Case 2. $\lambda_2^{t-2n+1} \neq \lambda_2$. In this case, C is similar to

$$\left(\lambda_2^{t-2n+1}\right) \oplus J_{n_1}(\lambda_2) \oplus J_{n_2}(\lambda_2) \oplus \ldots \oplus J_{n_s}(\lambda_2),$$

where $n_1, n_2, \ldots, n_s$ are positive integers and $n_1 + n_2 + \ldots + n_s = n - 1$. Since $\lambda_2^r = 1$, (λ_2^{t-2n+1}) can be expressed as a product of at most r quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_1(F)$. By Corollary 2.4, $J_{n_1}(\lambda_2) \oplus J_{n_2}(\lambda_2) \oplus \ldots \oplus J_{n_s}(\lambda_2)$ is a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_{n-1}(F)$. From Remark 4.4, it follows that C can be expressed as a product of at most $\max\{r, 2\} = r$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$.

Moreover, by Corollary 2.4, B is a product of two quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Thus, $P^{-1}AP$ is a product of $r + 2$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Finally, Remark 2.1 implies that the same conclusion holds for A. ☐

As stated in Theorem 4.5, we have proven Theorem 1.2 for the case of nonscalar matrices. Therefore, it remains to consider scalar matrices.

Proof of Theorem 1.2. Suppose that $A = \mu \mathrm{I}_n$, where μ is an element of $F \setminus \{0\}$ such that $\mu^n = \lambda_2^t$. We now consider the following two cases.

Case 1. $\lambda_2 = 1$ or $\lambda_2 = -1$. In this case, we can choose $r = 2$. If $\lambda_2 = 1$, then $\det(A) = 1$. According to [15, Theorem 1.5], A can be expressed as a product of four unipotent matrices of index 2. Therefore, A can be expressed as a product of $2r$ quadratic matrices with respect to $q(x) = (x-1)^2$. If $\lambda_2 = -1$, then $\det(A) = \pm 1$. From [8, Theorem], it follows that A can be expressed as a product of four involutions. Therefore, A can be expressed as a product of $2r$ quadratic matrices with respect to $q(x) = (x-1)(x+1)$.

Case 2. $\lambda_2 \neq \pm 1$. In this case, it is trivial that $\lambda_2^2 \neq 1$ and $r \geq 3$. Then, we can express A as the product

$$A = \underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \lambda_2^{r-2} \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \ddots & & \\ & & & \mu & \\ & & & & \mu\lambda_2^2 \end{pmatrix}}_{C}.$$

According to Remark 4.4, it is easy to see that B can be expressed as a product of $\max\{1, r-2\} = r-2$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Since $\lambda_2^2 \neq 1$, the matrix C is nonscalar. Moreover, since $\det(C) = \mu^n \lambda_2^2 = \lambda_2^{t+2}$, it follows from Theorem 4.5 that C can be expressed as a product of at most $r+2$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. Hence, $A = BC$ can be expressed as a product of at most $2r$ quadratic matrices with respect to $q(x)$ in $\mathbb{GL}_n(F)$. $\square$

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