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ANOTHER LOOK AT THE EIGENVALUES OF FUNCTIONS OF A PAIR OF ORTHOGONAL PROJECTORS*

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Abstract. The paper is concerned with eigenvalues of functions of a pair of orthogonal projectors, i.e., Hermitian idempotent matrices. By utilizing an approach based on a joint decomposition of the pair emerging from the spectral theorem, further insight into the topic is provided, supplementing the results already available in the literature. The research reveals several new facts, leading to the conclusion that the approach exploited offers a handy tool to cope with problems which require knowledge of eigenvalues of various functions of orthogonal projectors. Among the results established are characterizations of numbers of different eigenvalues of selected functions expressed in terms of ranks of the matrices involved in the joint decomposition. Related results concerned with square roots of functions determined by a pair of orthogonal projectors are provided as well.

Key words. Hermitian idempotent matrix, Matrix diagonalization, Partitioned matrix, Matrix root, Fundamental subspaces, Rank, Partial order.

AMS subject classifications. 15A18, 15A20, 65F15.

1. Introduction. The focal aim of the paper is to characterize numbers of different eigenvalues of selected functions of a pair of orthogonal projectors (i.e., Hermitian idempotent matrices), by exploiting an approach which enables to extend the considerations to other functions of the pair. The problem of determining eigenvalues of different functions of two orthogonal projectors was already considered in the literature, and various characterizations were established. This kind of considerations can be traced back at least to [1], where some general results on possible eigenvalues of the product and difference of two orthogonal projectors were obtained. Nevertheless, there are still relevant problems that were so far not investigated, indicating that further research is needed. The analysis carried out in the present article is intended to provide further insight into the topic. The characterizations derived in what follows describe the eigenvalue distribution for functions determined by two orthogonal projectors in terms of ranks of the matrices involved in a joint decomposition of the pair, which emerges from the spectral theorem. An approach based on the decomposition enables to characterize numbers of different eigenvalues in an applicable and informative way, and, importantly, its applicability extends to a comprehensive number of functions of a pair of orthogonal projectors.

It is noteworthy that the problem of characterizing numbers of different eigenvalues deserves attention not only as an interesting algebraical issue, as necessity to acquire such information occurs in various research areas of applied origin. For example, eigenvalues of a sum of two orthogonal projectors play a role in the logic of quantum mechanics [21, Theorem 3] and in the domain decomposition methods [18, Section 6]. The distribution of eigenvalues of a sum of two random orthogonal projectors was analyzed in [17], also indicating

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links of the problem with investigations carried out in physics. Further references to topics within physics, which are related to the considerations of the present paper, are pointed out in the next section.

In what follows we introduce notation and the joint decomposition of a pair of orthogonal projectors, which provides a handy tool to perform subsequent calculations. Section 2 is devoted to the eigenvalues of selected functions of a pair of orthogonal projectors, whereas Section 3 provides examples demonstrating applicability of the results derived in the preceding section. The final Section 4 contains results dealing with square roots of selected functions of a pair of orthogonal projectors, providing evidence that the joint decomposition of the pair can be beneficially utilized also to cope with problems that require knowledge of the roots.

The symbols $\mathbb{C}_{m,n}$ and $\mathbb{R}_{m,n}$ will stand for the sets of $m \times n$ complex and real matrices, respectively. By $\mathbf{M}^*$, $\mathbf{M}^\dagger$, $\mathcal{R}(\mathbf{M})$, $\mathcal{N}(\mathbf{M})$, $\mathrm{rk}(\mathbf{M})$ we will mean, correspondingly, conjugate transpose, Moore–Penrose inverse, column space (range), null space, and rank of $\mathbf{M} \in \mathbb{C}_{m,n}$. Whenever $\mathbf{M}$ is of real entries, we will use $\mathbf{M}'$ to denote its transpose, and when $\mathbf{M}$ is square, by $\mathrm{tr}(\mathbf{M})$ we will mean its trace. The symbol $\mathbf{I}_n$ will stand for the identity matrix of order n, whereas $\mathcal{S}$ will denote the subset of real numbers specified by $\mathcal{S} = (0,1) \cup (1,2)$.

The following lemma, being a consequence of the spectral theorem, recalls the joint decomposition of a pair of orthogonal projectors.

LEMMA 1.1. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be orthogonal projectors and let* $\mathrm{rk}(\mathbf{P}) = r$. *Then there exists a unitary* $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that*

$$\mathbf{P} = \mathbf{U} \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \quad \text{and} \quad \mathbf{Q} = \mathbf{U} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} \mathbf{U}^*, \tag{1.1}$$

with $\mathbf{B} \in \mathbb{C}_{r,n-r}$ *and Hermitian* $\mathbf{A} \in \mathbb{C}_{r,r}$, $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$.

It seems that the decomposition recalled in Lemma 1.1 was first used in [26], and then utilized in [27]. Afterwards, the decomposition was exploited in a number of papers, for example in [3]-[5], [7].

Three particular versions of the joint decomposition (1.1) are obtained when:

$\mathbf{P} = \mathbf{0}$, in which case the matrices $\mathbf{A}$ and $\mathbf{B}$ are absent,

$\mathbf{P} = \mathbf{I}_n$, in which case the matrices $\mathbf{B}$ and $\mathbf{D}$ are absent, and

$\mathbf{P} = \mathbf{Q}$, in which case $\mathbf{A} = \mathbf{I}_r$, $\mathbf{B} = \mathbf{0}$, and $\mathbf{D} = \mathbf{0}$.

Nevertheless, in general, the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ enjoy several useful properties; for a collection of them see any of [3]-[5], [7]. For example, it was indicated in [7, p. 249] that the eigenvalues of $\mathbf{A}$ and $\mathbf{D}$, both being Hermitian nonnegative definite contractions, belong to the set $[0,1]$. Furthermore, it was shown that the number of eigenvalues of $\mathbf{A}$ equal to one is $\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$, whereas the number of eigenvalues of $\mathbf{D}$ equal to one is $\mathrm{rk}(\mathbf{D}) - \mathrm{rk}(\mathbf{B})$. Another fact stated in [7, Lemma 1] is that each of $\mathbf{A}$ and $\mathbf{D}$ has $\mathrm{rk}(\mathbf{B})$ eigenvalues belonging to the set $(0,1)$, and that the sum of those $2\mathrm{rk}(\mathbf{B})$ eigenvalues equals $\mathrm{rk}(\mathbf{B})$.

From among further known properties of the decomposition (1.1), we recall five rank identities, which will be used in subsequent derivations. It was shown in [5, Theorem 1] that

$$\mathrm{rk}(\mathbf{Q}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D}), \tag{1.2}$$

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whereas, from [5, Lemma 6] it is known that

$$\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{QP}) = \mathrm{rk}(\mathbf{A}), \quad \mathrm{rk}(\mathbf{P}+\mathbf{Q}) = r + \mathrm{rk}(\mathbf{D}), \tag{1.3}$$

$$\mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}), \quad \mathrm{rk}(\mathbf{PQ}-\mathbf{QP}) = 2\mathrm{rk}(\mathbf{B}). \tag{1.4}$$

For a review of computational methods utilized to determine the spectral decomposition and problems the methods face, as well as for a list of common applications of the spectral theorem see e.g., [30].

2. Eigenvalues of functions of a pair of orthogonal projectors. The following theorem was established in [5, Theorem 6]. An alternative statement of this result was given in [5, Theorem 8].

THEOREM 2.1. *Let the orthogonal projectors* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be partitioned as in* (1.1)*. Then:*

(i) *eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *belong to the set* $[0,2]$*,*

(ii) $\mathbf{P}+\mathbf{Q}$ *has* $\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$ *eigenvalues equal to* 2*,*

(iii) $\mathbf{P}+\mathbf{Q}$ *has* $r - \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D})$ *eigenvalues equal to* 1*,*

(iv) $\mathbf{P}+\mathbf{Q}$ *has* $n - r - \mathrm{rk}(\mathbf{D})$ *eigenvalues equal to* 0*,*

(v) $\mathbf{P}+\mathbf{Q}$ *has* $2\mathrm{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $\mathcal{S}$*.*

Observe that Theorem 2.1(iv) provides a direct way to verify whether $\mathbf{P}+\mathbf{Q}$ is singular or not, which does not even require knowledge of an expression for the sum. The criterion asserts that $\mathbf{P}+\mathbf{Q}$ is nonsingular if and only if $\mathrm{rk}(\mathbf{D}) = n - r$.

By utilizing the facts available in the literature, Theorem 2.1 can be alternatively formulated in terms of dimensions of subspaces determined by the fundamental subspaces assigned to $\mathbf{P}$ and $\mathbf{Q}$.

COROLLARY 2.2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be orthogonal projectors. Then:*

(i) *eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *belong to the set* $[0,2]$*,*

(ii) $\mathbf{P}+\mathbf{Q}$ *has* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$ *eigenvalues equal to* 2*,*

(iii) $\mathbf{P}+\mathbf{Q}$ *has* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] + \dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$ *eigenvalues equal to* 1*,*

(iv) $\mathbf{P}+\mathbf{Q}$ *has* $\dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$ *eigenvalues equal to* 0*,*

(v) $\mathbf{P}+\mathbf{Q}$ *has* $\dim\{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})] \cap [\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})] \cap [\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})] \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}$ *eigenvalues belonging to the set* $\mathcal{S}$*.*

Proof. Points (ii) and (iv) of the corollary follow from [5, Lemma 8(i), (iv)], and point (iii) from [5, Theorem 4(iii)]. To establish point (v), observe that, in the light of (1.4),

$$\begin{aligned}2\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{PQ}-\mathbf{QP}) &= \dim \mathcal{R}(\mathbf{PQ}-\mathbf{QP}) \\ &= \dim\{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})] \cap [\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})] \cap [\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})] \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\},\end{aligned}$$

with the last equality being a consequence of the identities given in [4, pp. 526, 527]. ◻

The next theorem provides further input to Theorem 2.1, by splitting its point (v) into two more precise statements. Recall that when $\mathbf{P} = \mathbf{Q}$, then $\mathbf{A} = \mathbf{I}_r$, $\mathbf{B} = \mathbf{0}$, and $\mathbf{D} = \mathbf{0}$. In consequence, from Theorem 2.1

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we conclude that in this case $\mathbf{P}+\mathbf{Q}=2\mathbf{P}$ has r eigenvalues equal to two, $n-r$ zero eigenvalues, and has no eigenvalues belonging to the set $\mathcal{S}\cup\{1\}$. Hence, it is assumed in the next result that $\mathbf{P}\neq\mathbf{Q}$.

THEOREM 2.3. *Let the orthogonal projectors* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_{n,n}$, $\mathbf{P}\neq\mathbf{Q}$, *be partitioned as in* (1.1). *Then:*

(i) $\mathbf{P}+\mathbf{Q}$ *has* $\text{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $(0,1)$,

(ii) $\mathbf{P}+\mathbf{Q}$ *has* $\text{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $(1,2)$.

Proof. According to Theorem 2.1(v), the number of eigenvalues of $\mathbf{P}+\mathbf{Q}$ belonging to the set $\mathcal{S}$ equals $2\text{rk}(\mathbf{B})$. Hence, both statements of the theorem will be proved when we show that $\lambda\in\mathcal{S}$ is an eigenvalue of $\mathbf{P}+\mathbf{Q}$ if and only if $2-\lambda$ is an eigenvalue of $\mathbf{P}+\mathbf{Q}$ belonging to $\mathcal{S}$. To establish necessity, let us assume that $\lambda\in\mathcal{S}$ is such that $(\mathbf{P}+\mathbf{Q})\mathbf{x}=\lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}\in\mathbb{C}_{n,1}$. In consequence, $\mathbf{P}(\mathbf{P}+\mathbf{Q})\mathbf{x}=\lambda\mathbf{P}\mathbf{x}$ and $\mathbf{Q}(\mathbf{P}+\mathbf{Q})\mathbf{x}=\lambda\mathbf{Q}\mathbf{x}$, or, in other words,

$$\mathbf{P}\mathbf{Q}\mathbf{x}=(\lambda-1)\mathbf{P}\mathbf{x} \quad \text{and} \quad \mathbf{Q}\mathbf{P}\mathbf{x}=(\lambda-1)\mathbf{Q}\mathbf{x}. \tag{2.5}$$

Let vector $\mathbf{y}$ be given by $\mathbf{y}=(\mathbf{P}-\mathbf{Q})\mathbf{x}$. Then, on account of (2.5),

$$(\mathbf{P}+\mathbf{Q})\mathbf{y}=(\mathbf{P}+\mathbf{Q})(\mathbf{P}-\mathbf{Q})\mathbf{x}=\mathbf{P}\mathbf{x}-\mathbf{P}\mathbf{Q}\mathbf{x}+\mathbf{Q}\mathbf{P}\mathbf{x}-\mathbf{Q}\mathbf{x}=(2-\lambda)\mathbf{y},$$

which completes the "only if" part. The reverse implication needs no separate proof, as its validity is evident by the arguments exploited in the proof of necessity.

The established equivalence indicates that every eigenvalue of $\mathbf{P}+\mathbf{Q}$ in the set $(0,1)$ has its counterpart in the set $(1,2)$. This means that the number of eigenvalues in each set is the same and equals $\text{rk}(\mathbf{B})$. □

A direct consequence of Theorem 2.3 is that $\mathbf{P}+\mathbf{Q}$ can only have an even number of non-integer eigenvalues. These eigenvalues might be attributed a relevant meaning referring to the notion of the principal angles between subspaces assigned to the projectors, the fact asserted in [18, Theorem 2.18] or [13, Theorem 23]; alternatively see [9, Theorem 4.6]. Let us remark that another perspective to look at the eigenvalues of a sum of two orthogonal projectors was taken in [16], where the problem of constructing $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{P}+\mathbf{Q}$ has prescribed eigenvalues was considered.

Since the trace of a matrix coincides with the sum of its eigenvalues, from Theorems 2.1 and 2.3 we obtain

$$\text{tr}(\mathbf{P}+\mathbf{Q})=r+\text{rk}(\mathbf{A})-\text{rk}(\mathbf{B})+\text{rk}(\mathbf{D}). \tag{2.6}$$

Alternatively, the identity (2.6) could be established on account of (1.2) by utilizing the fact that $\text{tr}(\mathbf{P}+\mathbf{Q})=\text{tr}(\mathbf{P})+\text{tr}(\mathbf{Q})=r+\text{rk}(\mathbf{Q})$. Further implications of Theorem 2.1 are provided in the following corollary.

COROLLARY 2.4. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_{n,n}$ *be orthogonal projectors. Then:*

(i) *all eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *are equal to* 2 *if and only if* $\mathbf{P}=\mathbf{I}_n$ *and* $\mathbf{Q}=\mathbf{I}_n$,

(ii) *all eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *are equal to* 1 *if and only if* $\mathcal{R}(\mathbf{P})\overset{\perp}{\oplus}\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$,

(iii) *all eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *are equal to* 0 *if and only if* $\mathbf{P}=\mathbf{0}$ *and* $\mathbf{Q}=\mathbf{0}$,

(iv) *all eigenvalues of* $\mathbf{P}+\mathbf{Q}$ *belong to the set* $\mathcal{S}$ *if and only if* $\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$, $\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})=\mathbb{C}_{n,1}$, $\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$, $\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})=\mathbb{C}_{n,1}$.

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Proof. From Theorem 2.1(ii) it follows that all eigenvalues of $\mathbf{P}+\mathbf{Q}$ are equal to two if and only if $\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})=n$. This condition can equivalently be expressed as the conjunction $\mathrm{rk}(\mathbf{A})=n$ and $\mathrm{rk}(\mathbf{B})=0$; the latter of these identities results also from Theorem 2.1(v) as a consequence of the fact that when all eigenvalues are equal to two, then the number of non-integer eigenvalues is equal to zero. Now, since the only nonsingular projector is the identity matrix, we conclude that the conditions $\mathrm{rk}(\mathbf{A})=n$ and $\mathrm{rk}(\mathbf{B})=0$ are equivalent to $\mathbf{P}=\mathbf{I}_n$ and $\mathbf{Q}=\mathbf{I}_n$.

In view of Theorem 2.1(iii) it is seen that all eigenvalues of $\mathbf{P}+\mathbf{Q}$ are equal to one if and only if $r-\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})+\mathrm{rk}(\mathbf{D})=n$. This identity is satisfied if and only if $\mathrm{rk}(\mathbf{A})=0$, $\mathrm{rk}(\mathbf{B})=0$, and $\mathrm{rk}(\mathbf{D})=n-r$. On account of [5, Lemma 1(i)], it is clear that $\mathbf{A}=\mathbf{0}$ implies $\mathbf{B}=\mathbf{0}$, which means that the condition $\mathrm{rk}(\mathbf{B})=0$ is actually superfluous. By [5, Theorem 2(v)], the conjunction $\mathrm{rk}(\mathbf{A})=0$ and $\mathrm{rk}(\mathbf{D})=n-r$ is equivalent to $\mathcal{R}(\mathbf{P})\overset{\perp}{\oplus}\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$.

From Theorem 2.1(iv) we conclude that all eigenvalues of $\mathbf{P}+\mathbf{Q}$ are equal to zero if and only if $\mathrm{rk}(\mathbf{D})=-r$. This condition can only be satisfied when $r=0$ and $\mathbf{D}=\mathbf{0}$, which means that $\mathbf{P}$ and $\mathbf{Q}$ are both zero matrices.

Finally, from Theorem 2.1(v) we conclude that all eigenvalues of $\mathbf{P}+\mathbf{Q}$ belong to the set $\mathcal{S}$ if and only if $\mathrm{rk}(\mathbf{B})=\frac{n}{2}$. Hence, in light of [5, Theorem 9(ii)], it follows that every eigenvalue of $\mathbf{P}+\mathbf{Q}$ is in $\mathcal{S}$ if and only if the difference $\mathbf{PQ}-\mathbf{QP}$ has no zero eigenvalue in its spectrum. Now, the assertion is obtained on account of [4, Theorem 13]. □

Observe that all eigenvalues of $\mathbf{P}+\mathbf{Q}$ belong to $\mathcal{S}$ only if the projectors are of even order. It is also worth pointing out that $\mathbf{P}+\mathbf{Q}$ has no eigenvalues in $\mathcal{S}$ if and only if $\mathbf{B}=\mathbf{0}$, what is equivalent to $\mathbf{PQ}=\mathbf{QP}$; see [3, Lemma 2]. Commutativity of a pair of orthogonal projectors plays a distinguished role in several research areas, and there are several alternative characterizations of the property scattered over the literature; see [3] for over 40 conditions equivalent to $\mathbf{PQ}=\mathbf{QP}$ established by means of the decomposition (1.1). In [23] the commutativity was investigated in the context of a commensurability in quantum physics, and it was shown in [23, Theorem 1.29] that $\mathbf{PQ}=\mathbf{QP}\Leftrightarrow\mathcal{R}(\mathbf{P})=[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]+[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$.

It is noteworthy that the identities involving the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ occurring in the proof of Corollary 2.4 may be utilized to characterize a partial ordering (i.e., a binary relation which is reflexive, antisymmetric, and transitive) in the set of vectors of eigenvalues of $\mathbf{P}+\mathbf{Q}$. For this purpose let us construct vectors $\mathbf{v},\mathbf{w}\in\mathbb{R}_{n,1}$, $\mathbf{v}=\big(v_1,v_2,\ldots,v_n\big)'$ and $\mathbf{w}=\big(w_1,w_2,\ldots,w_n\big)'$ with entries arranged in a non-increasing order, i.e., $v_1\geqslant v_2\geqslant\cdots\geqslant v_n$ and $w_1\geqslant w_2\geqslant\cdots\geqslant w_n$. Then we claim that $\mathbf{v}$ is above $\mathbf{w}$ with respect to a partial ordering whenever

$$\sum_{i=1}^{k}v_i\geqslant\sum_{i=1}^{k}w_i,\quad k=1,2,\ldots,n. \tag{2.7}$$

Note that the condition (2.7) does not correspond to majorization, as it is not requested that for $k=n$ we have equality in (2.7) instead of inequality.

Let now $\mathbf{Q}$ be such that $\mathrm{rk}(\mathbf{A})=n$ and $\mathrm{rk}(\mathbf{B})=0$. Then the vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ is necessarily above—in the sense of the partial ordering specified in (2.7)—a vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ for any other orthogonal projector $\mathbf{Q}\in\mathbb{C}_{n,n}$. Moreover, when $\mathbf{Q}$ is such that $\mathrm{rk}(\mathbf{A})=0$ and $\mathrm{rk}(\mathbf{D})=n-r$, then the vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ is above any other vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ for any choice of $\mathbf{Q}$ for which $\mathrm{rk}(\mathbf{A})=\mathrm{rk}(\mathbf{B})$. Needless to say, when $r=0$ and $\mathbf{D}=\mathbf{0}$, then the vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ is necessarily below a vector of eigenvalues of $\mathbf{P}+\mathbf{Q}$ for any other orthogonal projector $\mathbf{Q}\in\mathbb{C}_{n,n}$.

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The literature indicates several links between the set-theoretical notion of a partial order and various research areas of applied origin, of which quantum logic is one; see e.g., [11, Chapter 2]. The reader interested in examples of an involvement of orthogonal projectors and partial orders in "laws of nature" is advised to browse through [24], and several facts on orthogonal projectors and their eigenvalues in the context of quantum mechanics are provided in [8].

Let us now consider eigenvalues of the difference $\mathbf{P}-\mathbf{Q}$. The next theorem improves [5, Theorem 5] in a similar way, in which Theorem 2.3 improves Theorem 2.1.

THEOREM 2.5. *Let the orthogonal projectors* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be partitioned as in* (1.1). *Then:*

(i) *eigenvalues of* $\mathbf{P}-\mathbf{Q}$ *belong to the set* $[-1,1]$,

(ii) $\mathbf{P}-\mathbf{Q}$ *has* $r-\mathrm{rk}(\mathbf{A}) = \dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$ *eigenvalues equal to* 1,

(iii) $\mathbf{P}-\mathbf{Q}$ *has* $n-r+\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})-\mathrm{rk}(\mathbf{D}) = \dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] + \dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$ *eigenvalues equal to* 0,

(iv) $\mathbf{P}-\mathbf{Q}$ *has* $\mathrm{rk}(\mathbf{D})-\mathrm{rk}(\mathbf{B}) = \dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$ *eigenvalues equal to* -1,

(v) $\mathbf{P}-\mathbf{Q}$ *has* $\mathrm{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $(0,1)$,

(vi) $\mathbf{P}-\mathbf{Q}$ *has* $\mathrm{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $(-1,0)$.

Proof. Points (i)-(iv) of the theorem were asserted in [5, Theorem 5], though the characterization referring to the subspace dimension in point (iii) was given in [5, Theorem 4(ii)]. The remaining two points are established on account of [5, Theorem 5] combined with the claim provided in [6, p. 163] that $\mathbf{P}-\mathbf{Q}$ has $r-\mathrm{rk}(\mathbf{A})+\mathrm{rk}(\mathbf{B})$ positive eigenvalues and $\mathrm{rk}(\mathbf{D})$ negative eigenvalues ∎

Note that in the light of Theorem 2.5(iii), we conclude that $\mathbf{P}-\mathbf{Q}$ is nonsingular if and only if $\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D}) = r-n$. Thus, in order to verify whether $\mathbf{P}-\mathbf{Q}$ is invertible or not it is necessary to know only the ranks of $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$, without need to derive an expression for the difference. Another related fact is that eigenvalues of $\mathbf{P}-\mathbf{Q}$ belonging to the set $(0,1) \cup (-1,0)$ are also linked with the principal angles between subspaces, the fact asserted in [18, Theorem 2.17] or [13, Theorem 26].

Among further functions of the projectors $\mathbf{P}$ and $\mathbf{Q}$ considered in [5] are $\mathbf{PQ}$, $(\mathbf{P}-\mathbf{Q})^2$, and $\mathbf{PQ}-\mathbf{QP}$ covered by Theorems 3, 4, and 9, respectively. Note that the last of these functions is a skew-Hermitian matrix, which means that its eigenvalues are purely imaginary. This, of course, does not impose any constraints on possibilities of expressing numbers of different eigenvalues in terms of ranks of the matrices involved in the joint decomposition specified in (1.1). Subsequently, we consider the function $\mathbf{PQ}+\mathbf{QP}$, which was not treated in [5].

THEOREM 2.6. *Let the orthogonal projectors* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be partitioned as in* (1.1). *Then:*

(i) *eigenvalues of* $\mathbf{PQ}+\mathbf{QP}$ *belong to the set* $[-\frac{1}{4}, 2]$,

(ii) $\mathbf{PQ}+\mathbf{QP}$ *has* $\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})$ *eigenvalues equal to* 2,

(iii) $\mathbf{PQ}+\mathbf{QP}$ *has* $n-\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})$ *eigenvalues equal to* 0,

(iv) $\mathbf{PQ}+\mathbf{QP}$ *has* $\mathrm{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $(0,2)$,

(v) $\mathbf{PQ}+\mathbf{QP}$ *has* $\mathrm{rk}(\mathbf{B})$ *eigenvalues belonging to the set* $[-\frac{1}{4}, 0)$.

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Proof. Let us denote by $\lambda \in \mathbb{C}$ the eigenvalues of $\mathbf{PQ}+\mathbf{QP}$, and assume first that $\lambda \neq 0$. Applying formula (9.3.1) in [22] to the determinant of $\mathbf{PQ}+\mathbf{QP}-\lambda\mathbf{I}_n$ leads to

$$\det(\mathbf{PQ}+\mathbf{QP}-\lambda\mathbf{I}_n) = \det(-\lambda\mathbf{I}_{n-r})\det(2\mathbf{A}-\lambda\mathbf{I}_r+\frac{1}{\lambda}\mathbf{BB}^*),$$

from where, on account of $\mathbf{A}=\mathbf{A}^2+\mathbf{BB}^*$, the identity being a direct consequence of $\mathbf{Q}^2=\mathbf{Q}$, we obtain

$$\det(\mathbf{PQ}+\mathbf{QP}-\lambda\mathbf{I}_n) = (-1)^{n-r}\lambda^{n-2r}\det[-\mathbf{A}^2+(2\lambda+1)\mathbf{A}-\lambda^2\mathbf{I}_r].$$

Hence, it is seen that nonzero eigenvalues of $\mathbf{PQ}+\mathbf{QP}$ are such that

$$\det[-\mathbf{A}^2+(2\lambda+1)\mathbf{A}-\lambda^2\mathbf{I}_r]=0. \tag{2.8}$$

As was already mentioned, $\mathbf{A}$ has $\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})$ unit eigenvalues and $\mathrm{rk}(\mathbf{B})$ eigenvalues belonging to the set $(0,1)$. To shorten the notation, let us denote $k \equiv \mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})$, $l \equiv \mathrm{rk}(\mathbf{B})$, and $m \equiv r-k-l$, i.e., m stands for the number of zero eigenvalues of $\mathbf{A}$. Moreover, let us use the symbols α_j, $j=1,2,\ldots,l$, to represent the eigenvalues of $\mathbf{A}$ belonging to the set $(0,1)$. As $\mathbf{A}$ is Hermitian, there exists unitary $\mathbf{V}\in\mathbb{C}_{r,r}$, such that

$$\mathbf{A}=\mathbf{V}\mathrm{diag}(\underbrace{1,...,1}_{k\,\text{times}},\alpha_1,...,\alpha_l,\underbrace{0,...,0}_{m\,\text{times}})\mathbf{V}^*.$$

In consequence, (2.8) can be rewritten as

$$\lambda^{2m}[\lambda(2-\lambda)]^k\prod_{j=1}^{l}[-\alpha_j^2+(2\lambda+1)\alpha_j-\lambda^2]=0. \tag{2.9}$$

From (2.9) we conclude that besides k eigenvalues equal to 2, which establishes point (ii) of the theorem, the remaining nonzero eigenvalues of $\mathbf{PQ}+\mathbf{QP}$ constitute solutions to the equation

$$\prod_{j=1}^{l}[-\alpha_j^2+(2\lambda+1)\alpha_j-\lambda^2]=0. \tag{2.10}$$

Straightforward calculations show that λs satisfying (2.10) are

$$\lambda=\alpha_j-\sqrt{\alpha_j} \quad \text{and} \quad \lambda=\alpha_j+\sqrt{\alpha_j}, \tag{2.11}$$

where $j=1,2,\ldots,l$. Since $\alpha_j\in(0,1)$ for every j, it is seen that $\alpha_j-\sqrt{\alpha_j}\in[-\frac{1}{4},0)$, whereas $\alpha_j+\sqrt{\alpha_j}\in(0,2)$. Thus, the nonzero eigenvalues of $\mathbf{PQ}+\mathbf{QP}$ belong to the set $[-\frac{1}{4},0)\cup(0,2]$, with $\mathrm{rk}(\mathbf{B})$ eigenvalues belonging to each of the subsets $[-\frac{1}{4},0)$ and $(0,2)$. Thus, also points (i), (iv), and (v) are confirmed. The fact that the number of zero eigenvalues of $\mathbf{PQ}+\mathbf{QP}$ is equal to $n-\mathrm{rk}(\mathbf{A})-\mathrm{rk}(\mathbf{B})$ follows directly from the remaining points of the theorem or from the expression for rank of $\mathbf{PQ}+\mathbf{QP}$ recalled in (1.4). ∎

Theorem 2.6(iii) asserts that $\mathrm{rk}(\mathbf{PQ}+\mathbf{QP})=n$ if and only if $\mathrm{rk}(\mathbf{A})+\mathrm{rk}(\mathbf{B})=n$. Thus, sole knowledge of ranks of $\mathbf{A}$ and $\mathbf{B}$ enables to verify whether $\mathbf{PQ}+\mathbf{QP}$ is singular or not.

Observe that Theorem 2.6(i) was also established in [19] and [29]. Another fact is that from (2.11) it follows that $\lambda=1$ whenever $\alpha=\frac{1}{2}(3-\sqrt{5})$ or $\alpha=\frac{1}{2}(3+\sqrt{5})$. As the latter value is not in the set $(0,1)$, we conclude that $\mathbf{PQ}+\mathbf{QP}$ has a unit eigenvalue whenever $\mathbf{A}$ has an eigenvalue $\alpha=\frac{1}{2}(3-\sqrt{5})$. The number of such eigenvalues of $\mathbf{A}$ cannot exceed l, and when this upper bound is reached, then $\mathbf{PQ}+\mathbf{QP}$ has $\mathrm{rk}(\mathbf{B})$ unit eigenvalues and no eigenvalues belonging to the set $(0,1)\cup(1,2)$.

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3. Examples. The subsequent three examples demonstrate applicability of the results provided in the preceding section.

EXAMPLE 3.1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be orthogonal projectors of the forms*

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \textit{and} \quad \mathbf{Q} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Parenthetically note that $\mathbf{P}$ *is a unique irreducible (idempotent) doubly stochastic* 3×3 *matrix, and that doubly stochastic matrices play an important role in majorization theory; see e.g.,* [20, Chapter 2]*. The joint decomposition* (1.1) *enables to factorize the projectors as*

$$\mathbf{P} = \mathbf{U} \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \mathbf{U}' \quad \textit{and} \quad \mathbf{Q} = \mathbf{U} \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{array}\right) \mathbf{U}',$$

with

$$\mathbf{U} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Visibly, $r = 1$, $\text{rk}(\mathbf{A}) = 1$, $\text{rk}(\mathbf{B}) = 0$, *and* $\text{rk}(\mathbf{D}) = 1$, *whence, on account of Theorems* 2.1, 2.5, *and* 2.6, *we straightforwardly conclude that eigenvalues of* $\mathbf{P} + \mathbf{Q}$, $\mathbf{P} - \mathbf{Q}$, *and* $\mathbf{PQ} + \mathbf{QP}$ *are as provided in Table* 1.

TABLE 1
Eigenvalues of the three functions of orthogonal projectors determined on account of Theorems 2.1, 2.5, *and* 2.6.

Function	Eigenvalues
$\mathbf{P} + \mathbf{Q}$	$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$
$\mathbf{P} - \mathbf{Q}$	$\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 0$
$\mathbf{PQ} + \mathbf{QP}$	$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$

It should be emphasized that the eigenvalues listed in Table 1 *were determined directly from knowledge of the joint decomposition of* $\mathbf{P}$ *and* $\mathbf{Q}$, *and that there was no need to derive explicit representations of the three functions* $\mathbf{P}+\mathbf{Q}$, $\mathbf{P}-\mathbf{Q}$, *and* $\mathbf{PQ}+\mathbf{QP}$. *Furthermore, the very knowledge suffices to formulate corresponding conclusions about eigenvalues of several other functions of* $\mathbf{P}$ *and* $\mathbf{Q}$, *such as* $\mathbf{PQ}$, $\mathbf{PQP}$, $\mathbf{I}_n - \mathbf{PQ}$, $\mathbf{PQ} - \mathbf{QP}$, $\mathbf{I}_n - \mathbf{P} - \mathbf{Q}$, $\mathbf{P} + \mathbf{Q} - \mathbf{PQ}$, *or* $(\mathbf{PQ})^\dagger$, $(\mathbf{P} - \mathbf{PQ})^\dagger$, *each of which was for different reasons already considered in the literature.*

EXAMPLE 3.2. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be orthogonal projectors of the forms*

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{Q} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

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On account of (1.1)*, the projectors can be expressed as*

$$\mathbf{P} = \mathbf{U}\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)\mathbf{U} \quad \textit{and} \quad \mathbf{Q} = \mathbf{U}\left(\begin{array}{cc|c} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \hline -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)\mathbf{U},$$

with symmetric "flip matrix"

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since clearly $r = 2$, $\mathrm{rk}(\mathbf{A}) = 1$, $\mathrm{rk}(\mathbf{B}) = 1$, *and* $\mathrm{rk}(\mathbf{D}) = 1$, *from Theorems* 2.1, 2.3, 2.5, *and* 2.6 *it straightforwardly follows that eigenvalues of* $\mathbf{P}+\mathbf{Q}$, $\mathbf{P}-\mathbf{Q}$, *and* $\mathbf{PQ}+\mathbf{QP}$ *are as provided in Table* 2.

Table 2
Eigenvalues of the three functions of orthogonal projectors determined on account of Theorems 2.1, 2.3, 2.5, *and* 2.6.

Function	Eigenvalues
$\mathbf{P}+\mathbf{Q}$	$\lambda_1 = 1$, $\lambda_2 \in (0,1)$, $\lambda_3 \in (1,2)$
$\mathbf{P}-\mathbf{Q}$	$\lambda_1 = 1$, $\lambda_2 \in (-1,0)$, $\lambda_3 \in (0,1)$
$\mathbf{PQ}+\mathbf{QP}$	$\lambda_1 = 0$, $\lambda_2 \in [-\frac{1}{4},0)$, $\lambda_3 \in (0,2)$

Needless to say, the values of eigenvalues and their assignments to the subsets indicated in Table 2 *were determined without prior specification of actual representations of the three functions of orthogonal projectors. Further analysis shows that spectrum: of* $\mathbf{P}+\mathbf{Q}$ *contains* $\lambda_2 = 1-\frac{\sqrt{2}}{2}$, $\lambda_3 = 1+\frac{\sqrt{2}}{2}$, *of* $\mathbf{P}-\mathbf{Q}$ *contains* $\lambda_2 = -\frac{\sqrt{2}}{2}$, $\lambda_3 = \frac{\sqrt{2}}{2}$, *whereas of* $\mathbf{PQ}+\mathbf{QP}$ *contains* $\lambda_2 = \frac{1}{2}(1-\sqrt{2})$, $\lambda_3 = \frac{1}{2}(1+\sqrt{2})$.

Example 3.3. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be orthogonal projectors of the forms*

$$\mathbf{P} = \frac{1}{14}\begin{pmatrix} 1 & 2 & -3i \\ 2 & 4 & -6i \\ 3i & 6i & 9 \end{pmatrix} \quad \textit{and} \quad \mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

On account of (1.1)*, the projectors can be expressed as*

$$\mathbf{P} = \mathbf{U}\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)\mathbf{U}^* \quad \textit{and} \quad \mathbf{Q} = \mathbf{U}\left(\begin{array}{c|cc} \frac{4}{7} & \frac{2}{7\sqrt{13}}(1+2i) & \frac{1}{\sqrt{182}}(1-5i) \\ \hline \frac{2}{7\sqrt{13}}(1-2i) & \frac{16}{21} & \frac{1}{39\sqrt{14}}(63-31i) \\ \frac{1}{\sqrt{182}}(1+5i) & \frac{1}{39\sqrt{14}}(63+31i) & \frac{2}{3} \end{array}\right)\mathbf{U}^*,$$

with

$$\mathbf{U} = \begin{pmatrix} -\frac{1}{\sqrt{14}} & -\sqrt{\frac{13}{14}} & 0 \\ -\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{182}} & \frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{14}}i & -\frac{3}{\sqrt{182}}i & -\frac{2}{\sqrt{13}}i \end{pmatrix}.$$

Since clearly $r = 1$, $\mathrm{rk}(\mathbf{A}) = 1$, $\mathrm{rk}(\mathbf{B}) = 1$, *and* $\mathrm{rk}(\mathbf{D}) = 2$, *from Theorems* 2.1, 2.3, 2.5, *and* 2.6 *it directly follows that eigenvalues of* $\mathbf{P}+\mathbf{Q}$, $\mathbf{P}-\mathbf{Q}$, *and* $\mathbf{PQ}+\mathbf{QP}$ *are as provided in Table* 3.

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Table 3
Eigenvalues of the three functions of orthogonal projectors determined on account of Theorems 2.1, 2.3, 2.5, and 2.6.

Function	Eigenvalues
$\mathbf{P}+\mathbf{Q}$	$\lambda_1 = 1$, $\lambda_2 \in (0,1)$, $\lambda_3 \in (1,2)$
$\mathbf{P}-\mathbf{Q}$	$\lambda_1 = -1$, $\lambda_2 \in (-1,0)$, $\lambda_3 \in (0,1)$
$\mathbf{PQ}+\mathbf{QP}$	$\lambda_1 = 0$, $\lambda_2 \in [-\frac{1}{4},0)$, $\lambda_3 \in (0,2)$

Also in this case, the values of eigenvalues and their assignments to the subsets indicated in Table 3 were determined without prior specification of actual representations of the three functions of orthogonal projectors. Further analysis shows that spectrum: of $\mathbf{P}+\mathbf{Q}$ contains $\lambda_2 = 1-\frac{2}{\sqrt{7}}$, $\lambda_3 = 1+\frac{2}{\sqrt{7}}$, of $\mathbf{P}-\mathbf{Q}$ contains $\lambda_2 = -\sqrt{\frac{3}{7}}$, $\lambda_3 = \sqrt{\frac{3}{7}}$, whereas of $\mathbf{PQ}+\mathbf{QP}$ contains $\lambda_2 = \frac{2}{7}(2-\sqrt{7})$, $\lambda_3 = \frac{2}{7}(2+\sqrt{7})$.

It is worthwhile to consider the three examples provided above from the perspective of the so-called Schur–Horn theorem, which—briefly speaking—asserts that the diagonal entries of a Hermitian matrix are majorized by the eigenvalues of the matrix; see e.g., [10] or [20, p. 4]. Confronting the theorem with the approach based on the joint decomposition of a pair of orthogonal projectors leads to the conclusion that knowledge of ranks of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ provides information regarding eigenvalues of a Hermitian matrix determined by $\mathbf{P}$ and $\mathbf{Q}$, which cannot be withdrawn by examining its diagonal entries. For example, the greatest diagonal entry of $\mathbf{P}+\mathbf{Q}$ in Example 3.1 is $d_{max} = \frac{4}{3}$, in Example 3.2 $d_{max} = \frac{3}{2}$, and in Example 3.3 $d_{max} = \frac{55}{42}$. By utilizing the Schur–Horn theorem, one can conclude that the largest eigenvalue of $\mathbf{P}+\mathbf{Q}$ in each example belongs to the set $[d_{max}, 2]$, but no more precise information is attainable. This reflects the fact that by means of the Schur–Horn theorem, one can conclude that $\mathbf{P}+\mathbf{Q}$ has two in its spectrum merely when there is two among the diagonal entries of the matrix. No such constraints are imbedded in the analysis based on the decomposition (1.1), as here one knows the number of eigenvalues of $\mathbf{P}+\mathbf{Q}$ equal to two as soon as the value of $\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$ is determined.

4. Square roots of functions of a pair of orthogonal projectors. A sum of two orthogonal projectors is Hermitian nonnegative definite, which means that its square root exists. Clearly, if λ is an eigenvalue of $\mathbf{P}+\mathbf{Q}$, then $\sqrt{\lambda}$ is an eigenvalue of $(\mathbf{P}+\mathbf{Q})^{1/2}$. Thus, from Theorems 2.1 and 2.3 we obtain the following result.

Theorem 4.1. *Let the orthogonal projectors $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ be partitioned as in (1.1). Then:*

(i) *eigenvalues of $(\mathbf{P}+\mathbf{Q})^{1/2}$ belong to the set $[0, \sqrt{2}]$,*

(ii) *$(\mathbf{P}+\mathbf{Q})^{1/2}$ has $\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$ eigenvalues equal to $\sqrt{2}$,*

(iii) *$(\mathbf{P}+\mathbf{Q})^{1/2}$ has $r - \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D})$ eigenvalues equal to 1,*

(iv) *$(\mathbf{P}+\mathbf{Q})^{1/2}$ has $n - r - \mathrm{rk}(\mathbf{D})$ eigenvalues equal to 0,*

(v) *$(\mathbf{P}+\mathbf{Q})^{1/2}$ has $\mathrm{rk}(\mathbf{B})$ eigenvalues belonging to the set $(0,1)$,*

(vi) *$(\mathbf{P}+\mathbf{Q})^{1/2}$ has $\mathrm{rk}(\mathbf{B})$ eigenvalues belonging to the set $(1,2)$.*

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Clearly, analogous results to Theorem 4.1 could be obtained from Theorem 2.5 concerned with $(\mathbf{P}-\mathbf{Q})^{1/2}$ and from Theorem 2.6 concerned with $(\mathbf{PQ}+\mathbf{QP})^{1/2}$. In fact, this kind of characterizations could be straightforwardly formulated with respect to cubic, quartic, and further roots, which likely were so far never considered in the literature. This observation constitutes yet another evidence that the joint decomposition specified in (1.1) can be advantageously exploited to deal with a variety of problems involving a pair of orthogonal projectors.

In what follows we provide an additional demonstration of the usefulness of the decomposition (1.1), now by establishing an alternative solution to the problem posed in [28], which aims at showing that orthogonal projectors $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ satisfy

$$\begin{aligned}\operatorname{tr}(\mathbf{P}+\mathbf{Q}) - (2-\sqrt{2})\operatorname{rk}(\mathbf{PQ}) &\leqslant \operatorname{tr}[(\mathbf{P}+\mathbf{Q})^{1/2}] \\ &\leqslant (\sqrt{2}-1)\operatorname{tr}(\mathbf{P}+\mathbf{Q}) + (2-\sqrt{2})\operatorname{rk}(\mathbf{P}+\mathbf{Q}),\end{aligned} \tag{4.12}$$

and, moreover, that the equalities hold in (4.12) if and only if $\mathbf{PQ} = \mathbf{QP}$. The solution to the problem given in [14] is over five pages long and proves to be quite involved. In what follows it is demonstrated that the proof based on the decomposition (1.1) is not only shorter (given the knowledge available), but also distinctly clearer.

First observe that by summing up the eigenvalues of $(\mathbf{P}+\mathbf{Q})^{1/2}$, from Theorem 4.1 one concludes that

$$\operatorname{tr}[(\mathbf{P}+\mathbf{Q})^{1/2}] = r + (\sqrt{2}-1)\operatorname{rk}(\mathbf{A}) - (\sqrt{2}+1)\operatorname{rk}(\mathbf{B}) + \operatorname{rk}(\mathbf{D}) + s, \tag{4.13}$$

where

$$s = \sum_{i=1}^{\operatorname{rk}(\mathbf{B})} \sqrt{\alpha_i} + \sum_{i=1}^{\operatorname{rk}(\mathbf{B})} \sqrt{2-\alpha_i},$$

with α_i, $i = 1, 2, ..., \operatorname{rk}(\mathbf{B})$, denoting eigenvalues of $\mathbf{P}+\mathbf{Q}$ belonging to the set $(0,1)$.

In the light of (1.3), (2.6), and (4.13), the former inequality in (4.12) can be equivalently expressed as

$$\sqrt{2}\operatorname{rk}(\mathbf{B}) \leqslant s. \tag{4.14}$$

The first observation is that when $\mathbf{PQ} = \mathbf{QP}$, which happens if and only if $\mathbf{B} = \mathbf{0}$, then both sides of the inequality (4.14) are equal to zero.

Let us now assume that $\mathbf{B} \neq \mathbf{0}$, and consider the function $f\colon [0,2] \to \mathbb{R}$ of the form $f(x) = \sqrt{x} + \sqrt{2-x}$. With basic calculus it can be shown that the minimal value of $f(x)$ is $\sqrt{2}$, which is attained for $x = 0$ and $x = 2$, whereas the maximal value of $f(x)$ is 2, which is achieved for $x = 1$. In consequence, it follows that in the set $\mathcal{S}$, the function $f(x)$ satisfies $\sqrt{2} < f(x) < 2$. This leads to the conclusion that

$$\sqrt{2}\operatorname{rk}(\mathbf{B}) < s < 2\operatorname{rk}(\mathbf{B}). \tag{4.15}$$

Thus, the inequality (4.15) necessarily holds.

In fact, the above arguments can be utilized to confirm validity also of the latter inequality in (4.12), which, on account of (1.3), (2.6), and (4.13), can be equivalently expressed as

$$s \leqslant 2\operatorname{rk}(\mathbf{B}). \tag{4.16}$$

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When $\mathbf{PQ} = \mathbf{QP}$, then both sides of the inequality (4.16) are equal to zero, and when $\mathbf{P}$ and $\mathbf{Q}$ do not commute, then the inequality in (4.16) becomes sharp and holds by (4.15). These observations accomplish the solution to the problem posed in [28].

The paper is concluded with a result concerned with the square root of a product of two orthogonal projectors, which is represented by a not necessarily Hermitian nonnegative definite matrix. Two alternative representations of the root were derived in [2] and [12]. Yet another expression for $(\mathbf{PQ})^{\frac{1}{2}}$ is established in the next theorem, the result which demonstrates further applicability of the joint decomposition specified in (1.1).

Theorem 4.2. *Let the orthogonal projectors* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be partitioned as in* (1.1). *Then:*

$$[(\mathbf{PQP})^{\frac{1}{2}}]^{\dagger}\mathbf{PQ} = \mathbf{U}\begin{pmatrix} (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A} & (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{4.17}$$

is a square root of $\mathbf{PQ}$.

Proof. The fact that $[(\mathbf{PQP})^{\frac{1}{2}}]^{\dagger}\mathbf{PQ}$ has the representation provided in (4.17) follows directly by substituting $\mathbf{P}$ and $\mathbf{Q}$ of the forms given in (1.1). Hence, it remains to show that the second power of the matrix on the right-hand side of (4.17) coincides with the representation of $\mathbf{PQ}$, or, in other words, we need to verify that the identity

$$\begin{pmatrix} (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A} & (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A} & (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

is satisfied. Since $\mathbf{A}$ is necessarily Hermitian, we have $(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}^{\frac{1}{2}} = \mathbf{A}^{\frac{1}{2}}(\mathbf{A}^{\frac{1}{2}})^{\dagger}$. This property yields $(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A} = (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}^{\frac{1}{2}}(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}^{\frac{1}{2}}\mathbf{A} = (\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}^{\frac{1}{2}}\mathbf{A} = \mathbf{A}^{\frac{1}{2}}(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}} = \mathbf{A}$. Similar arguments lead to the identities $(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{A}(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{B} = \mathbf{A}(\mathbf{A}^{\frac{1}{2}})^{\dagger}(\mathbf{A}^{\frac{1}{2}})^{\dagger}\mathbf{B} = \mathbf{A}\mathbf{A}^{\dagger}\mathbf{B} = \mathbf{B}$, with the last equality valid on account of [5, Lemma 2(vii)]. The proof is complete. □

In the context of Theorem 4.2, it is worth pointing out that $\mathbf{A}$ is a Hermitian nonnegative definite matrix, which in general should simplify the task of determining an actual representation of the Moore–Penrose inverse of $\mathbf{A}^{\frac{1}{2}}$, involved in (4.17). In fact, for the orthogonal projectors given in the three examples provided in Section 3, this task is actually trivial, as in Example 3.1 the inverse is given by $(\mathbf{A}^{\frac{1}{2}})^{\dagger} = \begin{pmatrix} 1 \end{pmatrix}$, in Example 3.2 by $(\mathbf{A}^{\frac{1}{2}})^{\dagger} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$, and in Example 3.3 by $(\mathbf{A}^{\frac{1}{2}})^{\dagger} = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{7}{3}} \end{pmatrix}$.

For computational aspects concerned with determining the square root of a matrix the reader is advised to have a look into [15, Chapter 7] or [25] and the references therein.

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