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GRAPH PRODUCTS THAT ALLOW TWO DISTINCT EIGENVALUES*

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Abstract. The parameter $q(G)$ of a graph G is the minimum number of distinct eigenvalues of a symmetric matrix whose pattern is given by G. We introduce a novel graph product by which we construct new infinite families of graphs that achieve $q(G) = 2$. Several graph families for which it is already known that $q(G) = 2$ can also be thought of as arising from this new product.

Key words. Inverse eigenvalue problem for graphs, Graph products, q-parameter.

AMS subject classifications. 05C50, 15A29, 15A18.

1. Introduction.

Graph inverse eigenvalue problems have been the subject of extensive research for many years now, and most recently, considerable attention has been given to the parameter $q(G)$ of a graph G, which is the minimum number of distinct eigenvalues that can be achieved by a symmetric matrix whose pattern is given by G. More specifically, for a graph $G = (V(G), E(G))$, we define

$$\mathcal{S}(G) = \{\text{symmetric matrices } A = [a_{ij}] \mid a_{ij} = 0 \text{ for } i \neq j \iff ij \notin E(G)\}.$$

Note that no restriction is put on the diagonal entries of A. The *Inverse Eigenvalue Problem for Graphs (IEPG)* asks what spectra can be achieved by matrices $A \in \mathcal{S}(G)$ given the graph G (see [4, 13]). This is a difficult problem in general, and most work in the area addresses subquestions such as the minimum rank, maximum nullity, and maximum multiplicity problems [2, 11, 13].

One important subproblem of the IEPG is to understand the possible multiplicities of eigenvalues of matrices in $\mathcal{S}(G)$. This entails understanding the possible numbers of distinct eigenvalues of matrices in $\mathcal{S}(G)$. To this end, for a symmetric matrix A, we first define $q(A)$ to be the number of distinct eigenvalues of A, and then

$$q(G) = \min\{q(A) \mid A \in \mathcal{S}(G)\}.$$

The $q(G)$ parameter was introduced in [16] and has been extensively studied. In general, determining $q(G)$ can still be quite challenging, but many results have been found [3, 8, 16, 13, 17, 18]. Note that it is easy to characterize graphs for which $q(G) = 1$: since the matrices in $\mathcal{S}(G)$ are symmetric, then $q(A) = 1$ if and only if A is a multiple of the identity, which implies that $q(G) = 1$ if and only if G has no edges. The problem becomes much more subtle when investigating graphs that achieve exactly two distinct eigenvalues. Graphs for which $q(G) = 2$ have turned out to be interesting and difficult to characterize [3, 5]. Families of graphs with $q(G) = 2$ have been constructed and studied in [6, 7]. Of note, results in [3] imply that any connected graph can appear as an induced subgraph of a graph with $q(G) = 2$, making a complete characterization of these graphs challenging (see also [1]).

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In this note, we attempt to unify some of the families of graphs with $q(G) = 2$ by viewing them as resulting from graph products. Some of our constructions will involve the well-known strong product of graphs $G \boxtimes H$. We will also introduce a new graph product, a variant of the strong product, which we call the *modified strong product*, which we will denote using $G \mathbin{\overline{\underline{\times}}} H$. See Definition 2.1 below. Matrices whose pattern corresponds to strong products and modified strong products can be described easily using tensor products of matrices for the factors. Using this we construct new families of graphs with $q(G) = 2$, as well as view some known examples with $q(G) = 2$ as arising from this new product. A surprising connection with the chromatic index of a graph will be explored. Specifically, for our main results, we will prove that all of the following families of graphs can achieve 2 distinct eigenvalues (see Theorems 3.2, 3.3, and 3.6 below):

- $G \mathbin{\overline{\underline{\times}}} K_k$ where G is k regular with chromatic index k.
- $G \boxtimes K_{k+1}$ where G has maximum degree k.
- $G \boxtimes K_c$ where G arises from a uniform linear hypergraph with maximum degree k and chromatic index c with $c > k$.
- $G \boxtimes K_{c+1}$ where G is as above, except $k = c$.
- $G \mathbin{\overline{\underline{\times}}} K_c$ in the same context, but when the hypergraph is k-regular, and again, $c = k$.

Each of these theorems is proven by decomposing the edge sets of the graphs into partitions where each piece has 2 eigenvalues and then examining the tensor product of the corresponding 0-1 matrices with a special set of orthogonal idempotents.

We remark that in [12, Table (b) of page 42], a brief description of the modified strong product does appear in an exhaustive description of all possible graph products. This product is observed there to be a nonassociative graph product and is thus dismissed, not considering it "worthy of attention" [12], and no name for this product is given there. We thus feel justified in calling the modified strong product a "new" graph product, and one of the contributions of this paper is to demonstrate its usefulness.

We also note that other work has already been done in studying the q parameter in the context of graph products. In particular, work in [9] gives bounds, and in some cases explicit formulas, for q of cartesian products, strong products, and tensor products of some families of graphs.

The rest of the paper will be organized as follows. In Section 2, we will introduce the modified strong product and investigate the relevant spectral properties. In Section 3, we prove the main results and describe examples achieving $q(G) = 2$ from constructions using the modified strong product and the strong product. We will end with some examples, as well as some modifications of the main results.

2. The modified strong product. We will begin with the definition of the modified strong product. We first remind the reader of the definition of the strong product of two graphs. Recall that the strong product $G \boxtimes H$ is the graph whose vertex set is $V(G) \times V(H)$ containing all edges of the form

- $(g, h)(g, h')$ where $hh' \in E(H)$,
- $(g, h)(g', h)$ where $gg' \in E(G)$,
- $(g, h)(g', h')$ where $gg' \in E(G)$ and $hh' \in E(H)$.

The adjacency matrix of $G \boxtimes H$ is $(A_G + I) \otimes (A_H + I) - I$ [12], where $\otimes$ denotes the Kronecker product of matrices, i.e., for an $m \times n$ matrix M and $p \times q$ matrix N, $M \otimes N$ is the $mp \times nq$ matrix whose entries can be thought of in block form, where the entries of the i, j block are $m_{i,j}N$. Since the pattern allows the diagonal entries to vary freely, the matrix $(A_G + I) \otimes (A_H + I)$ has the same pattern as the adjacency matrix of $G \boxtimes H$. This is a fact that we will use later.

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(a) $K_2 \boxtimes P_3$

(b) $P_3 \boxtimes K_2$

Figure 1: Two examples of modified strong products. Note in particular that these are not isomorphic.

Definition 2.1. *The* modified strong product *of G and H, which we will denote by $G \boxtimes H$, is defined as the graph on vertex set $V(G) \times V(H)$ and whose edges are*

- $(g, h)(g', h)$ *where* $gg' \in E(G)$,
- $(g, h)(g', h')$ *where* $gg' \in E(G)$ *and* $hh' \in E(H)$.

Note that this is very similar to the strong product of G and H, but we are not including the edges $(g, h)(g, h')$ for $hh' \in E(H)$. In particular, this means that $G \boxtimes H$ is not necessarily isomorphic to $H \boxtimes G$. See Fig. 1. Note that the adjacency matrix of $G \boxtimes H$ is $A_G \otimes (A_H + I)$.

Definition 2.2. *For some index set $K \subseteq [k]$, let D_K be the diagonal matrix with $d_{ii} = 1$ if and only if $i \in K$. For ease of notation, let $D_i = D_{\{i\}}$.*

Lemma 2.3. *For any k, there exists an orthogonal k by k matrix Q such that QD_KQ^T is a matrix with all nonzero entries for all $K \subseteq [k]$ except for $K = \emptyset, [k]$.*

Proof. We will obtain a Q with the desired properties from a *Householder matrix*

$$Q = H(u) := I - 2uu^T,$$

where u is a unit vector. It is easy to see that for any unit vector u, $H(u)H(u)^T = H(u)^2 = I$, so this always yields an orthogonal matrix.

Let K be a nonempty proper subset of $[k]$ and given a unit vector u, let u_K be the vector indexed by the entries of K, and $u_{\overline{K}}$ the remaining entries of the vector. By reordering the entries if necessary, we may assume without loss of generality the elements of K are indexed first. Then

$$D_K = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix},$$

and a straightforward computation yields

$$H(u)D_KH(u)^T = \begin{bmatrix} I_K - 4(1 - u_K^T u_K)u_K u_K^T & -2(1 - 2u_K^T u_K)u_K u_{\overline{K}}^T \\ -2(1 - 2u_K^T u_K)u_{\overline{K}} u_K^T & 4(u_K^T u_K)u_{\overline{K}} u_{\overline{K}}^T \end{bmatrix}.$$

□

We need to choose u so that this is nowhere zero for every nonempty, proper subset K. For any k, there are many possible choices for such a u. We leave the details to the reader.

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This lemma immediately leads to the following corollary.

COROLLARY 2.4. *Let Q be defined by the lemma above, and then let $J_i = QD_iQ^T$ for all $i \in [k]$. Then any nonempty sum of the J_is will have all nonzero entries except for the sum of all of them which is equal to the identity.*

We can also see that these J_is give a set of orthogonal $k \times k$ idempotents, as we see in the following lemma. The proof is straightforward and we omit the details.

LEMMA 2.5. *For all $i \neq j \in [k]$:*

$$J_i^2 = J_i,$$
$$J_iJ_j = 0.$$

Let the ith column of Q be q_i. Then q_i, q_j are eigenvectors of J_i with eigenvalues $1, 0$, respectively.

Using this we can prove the main lemma which we will use for our results.

LEMMA 2.6. *Suppose we have $n \times n$ matrices $A_1, \dots, A_k$, and suppose A_i has eigenvalues $\lambda_{i,1}, \dots \lambda_{i,n}$ with corresponding eigenvectors $v_{i,1}, \dots, v_{i,n}$. Let $q_1, ..., q_k$ be the columns of Q, and let*

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k).$$

Then

$$M(v_{i,l} \otimes q_i) = \lambda_{i,l}(v_{i,l} \otimes q_i),$$

for $i = 1, ..., k$, $l = 1, ...n$. Moreover, this gives us all the nk eigenvalues of M, counting multiplicities.

Proof. A straightforward computation shows that $M(v_{i,l} \otimes q_i) = \lambda_{i,l}v_{i,l} \otimes q_i$. To see that the vectors given are all linearly independent (and therefore we have all the eigenvalues, including multiplicities), we note that

$$M = \sum_{i\in[k]}(A_i \otimes J_i) = (I \otimes Q)\left(\sum_{i\in[k]}(A_i \otimes D_i)\right)(I \otimes Q)^T,$$

which shows M is similar to $\sum_{i\in[k]}(A_i \otimes D_i)$ whose eigenvectors $v_{j,l} \otimes e_l$ ($j = 1, ..., k$, and $l = 1, ..., n$) are clearly linearly independent and lead to the eigenvectors $v_{j,l} \otimes q_l$ of M after applying the similarity. □

The next lemma addresses the pattern of zeros and nonzeros of our matrices. We will find it helpful to talk about two matrices sharing the same pattern. We will say that a matrix $M = [m_{ij}]$ *has the pattern of* the matrix $N = [n_{ij}]$ if $m_{ij} = 0$ if and only if $n_{ij} = 0$. Thus, a matrix has the pattern of the all ones matrix J if all its entries are nonzero. Likewise, it has the pattern of the identity matrix I if nonzero entries occur if and only if they are on the diagonal.

LEMMA 2.7. *If $A_1, \dots, A_k$ are $n \times n$ matrices with 0,1 entries, then the pattern of*

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k),$$

is determined by the entries of:

$$A = A_1 + \cdots + A_k.$$

Specifically, M is an $nk \times nk$ matrix with blocks of size k by k whose pattern is given as follows: for A given by $A = [a_{i,j}]_{i,j=1}^n$.

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- *If* $a_{l,m} = 0$*, then the* l, m *block will be all zeros.*
- *If* $0 < a_{l,m} < k$*, then the* l, m *block will have all nonzero entries (it has the pattern of the all ones matrix* J*).*
- *If* $a_{l,m} = k$*, then the* l, m *block will have nonzeros only on the diagonal (it has the pattern of the identity matrix* I*).*

Proof. Since each A_i is a 0,1 matrix, the matrices $A_i \otimes J_i$ will be block matrices with blocks equal to J_i or a block of all 0. Therefore, the matrix M will consist of blocks that are sums of the J_i. Then the result follows from Corollary 2.4. □

3. Chromatic index and products with cliques. These constructions will use the adjacency matrix of a complete graph, whose eigenvalues are well known in spectral graph theory.

LEMMA 3.1. *The adjacency matrix of* K_n *has eigenvalue* $n-1$ *with multiplicity* 1 *and* -1 *with multiplicity* $n-1$*.*

Recall that the chromatic index of a graph G, denoted $\chi'(G)$, is the minimum number of colors needed to give a proper edge coloring of the graph; i.e. no two edges of the same color share a vertex. By Vizing's Theorem, a graph G with maximum degree k has $k \leq \chi'(G) \leq k+1$. It is NP-Complete to determine whether an arbitrary graph G has $\chi'(G) = k$ [14]. Note that subgraph of G given by all the edges of a fixed color in a proper edge coloring gives a matching in the graph. If G is a k-regular graph with chromatic index k, then each color set will be a perfect matching, meaning every vertex of the graph will be incident to an edge of the matching.

THEOREM 3.2. *If* G *is* k*-regular with chromatic index* k*, then* $q(G \boxtimes K_k) = 2$*.*

Proof. Let G_i be the subgraph of G on the same vertex set but only containing the edges of color i and let A_i be its adjacency matrix. Since each A_i consists of a disjoint union of $n/2$ copies of K_2, its eigenvalues will be -1 and 1 (by Lemma 3.1), each with multiplicity $n/2$ where n is the total number of vertices. Let M be the matrix:

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k).$$ □

By Lemma 2.6, the eigenvalues of M will be $-1, 1$ each with multiplicity $nk/2$.

We can also see that $M \in \mathcal{S}(G \boxtimes K_k)$. Using Lemma 2.7, the pattern of M is determined by $A_1 + \cdots + A_k = A$, which is equal to the adjacency matrix of G. By that lemma, M will have a block of all nonzero elements corresponding to a one in the adjacency matrix of G and a block of zeros corresponding to a zero in the adjacency matrix of G. This is equivalent to saying that the pattern of M is the same as that of the matrix $A \otimes J = A \otimes (I + A_{K_k}) \in \mathcal{S}(G \boxtimes K_k)$ as desired.

König showed that every k-regular bipartite graph has chromatic index k [15]. Also, Chetwynd and Hilton showed that any k-regular, $2n$-vertex graph G with $k \geq \frac{12}{7}n$ has chromatic index k [10]. Therefore, there are an infinite family of examples which Theorem 3.2 can apply to.

THEOREM 3.3. *If* G *is connected and has maximum degree* k*, then* $q(G \boxtimes K_{k+1}) = 2$*.*

Proof. Since G has maximum degree k, by Vizing's theorem, it has a $k+1$-coloring of the edges. This coloring will be such that for any vertex v, at least one color will miss that vertex. Let G_i be the subgraph of G on the same vertex set but only containing the edges of color i. Construct matrices $A_1, \ldots, A_{k+1}$ such that A_i has entries:

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$$A_{i,l,m} = \begin{cases} 1 & \text{if } l = m \text{ and } l \text{ is not incident to any edges of color } i \\ 1 & \text{if } l \text{ adjacent to } m \text{ by an edge of color } i \\ 0 & \text{otherwise} \end{cases}.$$

Define $M = (A_1 \otimes J_1) + \cdots + (A_{k+1} \otimes J_{k+1})$. Notice that each A_i is the adjacency matrix of the disjoint union of some number of copies of K_2 with some ones added on the diagonal. However, the ones on the diagonal are specifically added where there are not K_2s. This means that up to a permutation of the vertices, A_i consists of blocks of the two forms:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \begin{bmatrix} 1 \end{bmatrix}.$$

□

This matrix will have a pair of eigenvalues $-1, 1$ for each K_2 (by Lemma 3.1) and an eigenvalue 1 for each vertex which is missed by the color i. The exact multiplicities of these eigenvalues will depend on the edge coloring. Therefore, by Lemma 2.6, the eigenvalues of M will be $-1, 1$ with the multiplicities being the sum of the multiplicities of those eigenvalues for each A_i.

We can also see that $M \in \mathcal{S}(G \boxtimes K_{k+1})$. Using Lemma 2.7, the pattern of M is determined by $A_1 + \cdots + A_k = B$. The off-diagonal entries of B correspond exactly to the adjacency matrix of G, since every edge of G is exactly one color, and so is present in exactly one matrix A_i. Therefore, M will have a block of all nonzero elements corresponding to a one in the adjacency matrix of G, and a block of zeros corresponding to a zero in the adjacency matrix of G. Since every vertex of G fails to be incident to some color, the diagonal entries of B are nonzero. Since G is connected, every vertex has at least one edge incident to it, and so has at least one color incident to it, therefore, no diagonal entry of B is equal to $k+1$. And so the diagonal entries of B are greater than 0 and less than $k+1$. And so, by Lemma 2.7, the diagonal blocks of M will have all nonzero entries. Let A be the adjacency matrix of G. We have shown that the pattern of M is the same as the pattern of $(A+I) \otimes J = (A+I) \otimes (I + A_{K_{k+1}}) \in \mathcal{S}(G \boxtimes K_{k+1})$. Therefore, a symmetric matrix in $\mathcal{S}(G \boxtimes K_{k+1})$ has only two eigenvalues, $-1, 1$, meaning that $q(G \boxtimes K_{k+1}) = 2$.

Definition 3.4. *Given a hypergraph $\mathcal{H}$, the* representing graph *G has the same vertex set as $\mathcal{H}$ and an edge connecting any two vertices which are in a hyperedge together.*

Definition 3.5. *A hypergraph $\mathcal{H}$ is l-*uniform *if every edge is of size l. The hypergraph is* linear *if every pair of edges intersects in at most one vertex.*

Note that a linear, l-uniform hypergraph $\mathcal{H}$ is a partition of the edges of the underlying graph G into cliques of size l.

We want to color the hyperedges of $\mathcal{H}$. Each color class of the edges will correspond to a subgraph of the representing graph G, which consists of disjoint copies of K_l. We can then use the fact that K_l has only two eigenvalues to do a similar trick to Theorem 3.3 and show that $q(G \boxtimes K_k)$ is 2 (where k is the chromatic index of $\mathcal{H}$).

Theorem 3.6. *Let G be the representing graph of a l-uniform linear hypergraph $\mathcal{H}$ with the maximum degree of the hypergraph being k. Let $\chi'(\mathcal{H}) = c$. Then:*

(a) If $c > k$, then $q(G \boxtimes K_c) = 2$.
(b) If $c = k$, then $q(G \boxtimes K_{c+1}) = 2$.
(c) If $\mathcal{H}$ is k-regular and $c = k$, then $q(G \boxtimes K_c) = 2$.

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Proof. In cases (a), let A_i be:

$$A_{i,a,b} = \begin{cases} c-1 & \text{if } a = b \text{ and } a \text{ is not incident to a hyperedge of color } i \\ 1 & \text{if } a \text{ and } b \text{ share a hyperedge of color } i \\ 0 & \text{otherwise} \end{cases}.$$

We have chosen the diagonal element so that the eigenvalues of A_i will be -1 and $c-1$ (with multiplicities determined by the exact coloring). Then defining:

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k).$$

We get from Lemma 2.6 that the eigenvalues of M will be -1 and $c-1$.

Now, we need only show that the pattern of M matches the pattern of $G \boxtimes K_c$. The off-diagonal blocks of M will consist of a single J_i, where i depends on the color of the corresponding hyperedge. The diagonal blocks of M will consist of $c-1$ times a sum of the J_is, which correspond to colors which do not occur on hyperedges incident to this vertex. Since, by Corollary 2.4, the nonempty sum of any number of J_is except all of them will have all nonzero entries, and we are multiplying this by a nonzero number, the pattern for this block is all nonzero. We know that the sum of J_is involved cannot be all of them since at least one hyperedge (colored some color) will be incident to this vertex.

Case (b) is shown similarly, except that we first construct a coloring on $c+1$ colors to ensure that at least one color misses each vertex, and when constructing the A_i the diagonal values are c instead of $c-1$.

In case (c), let A_i be:

$$A_{i,a,b} = \begin{cases} 1 \text{ if } a \text{ and } b \text{ share a hyperedge of color } i \\ 0 \text{ otherwise} \end{cases}.$$

Then defining:

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k).$$

We can see the eigenvalues of M will be -1 and $c-1$ from Lemma 2.6 and 3.1. We can also see that the pattern of M will match the pattern of $G \boxtimes K_c$ by Lemma 2.7.

Therefore, we see in all cases that the appropriate graphs have q equal to 2. ∎

See Fig. 2 for some examples of graphs which this theorem applies to.

In Fig. 2(a), the edges of the graph can be partitioned into two cliques of size 3. Since the maximum degree of the resulting hypergraph is two, and we can color the edges of this hypergraph with two colors, we see that this falls into case (b).

In Fig. 2(b), the edges of the graph can be partitioned into four cliques of size 3. Each clique is incident to every other, and so we need four colors to color them. Since the maximum degree of the resulting hypergraph is two, we see this falls into case (a).

Figure 2(c) is particularly illustrative of the power of this theorem. Any graph $K_l \square K_l$ can have its edges partitioned into two spanning subgraphs (the "rows" and the "columns") whose components are each K_l.

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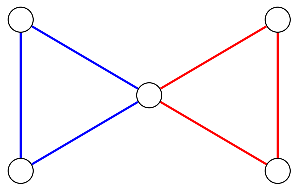
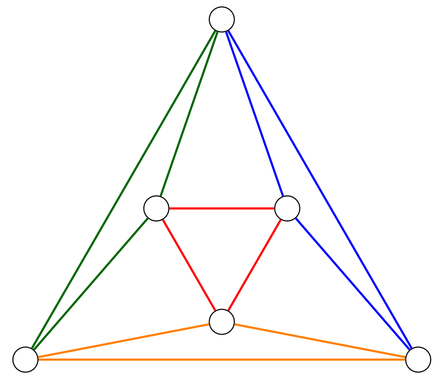
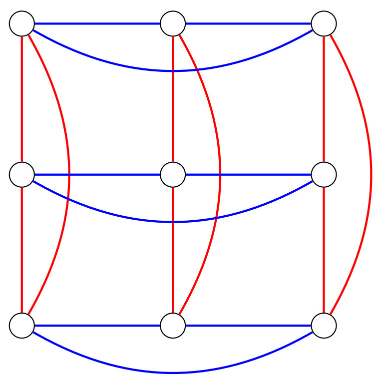


(a) Bowtie graph,
$q(\text{Bowtie} \boxtimes K_3) = 2$



(b) Octahedron,
$q(\text{Octahedron} \boxtimes K_4) = 2$



(c) $K_3 \square K_3$,
$q((K_3 \square K_3) \mathbin{\overline{\times}} K_2) = 2$
$q((K_3 \square K_3) \boxtimes K_3) = 2$

Figure 2: Some examples of graphs where Theorem 3.6 applies.

The maximum degree of the resulting hypergraph is two, since each vertex is in one K_l from a "row" and one K_l from a "column". Therefore, this class of graphs fall into case (c), and so $q((K_l \square K_l) \mathbin{\overline{\times}} K_2) = 2$ for all l. We can also use case (b) to conclude that $q((K_l \square K_l) \boxtimes K_3) = 2$.

How does one apply Theorem 3.6 to, for example, the Bowtie graph (Fig. 2(a))? As stated above, the edges of this graph can be partitioned into two cliques of size 3. If we number the vertices of graph so that the vertex shared between the two cliques is in the middle, then we can write the adjacency matrix of this graph as the following sum of two matrices to represent this partition.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

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The matrices A_1, A_2 in the theorem proof will be almost the same as the addends in this equation, except we will modify the diagonal elements to represent those vertices which are missed by that clique:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

At this point, we still have slight problem, since the middle vertex is present in both cliques, that diagonal element is zero in the sum $A_1 + A_2$. Therefore, if we were to construct the matrix M using just these two, the pattern by Lemma 2.7 will have a block of zeros on the diagonal, which will not match up with the pattern of the strong product Bowtie $\boxtimes K_2$, and so we add a third color, a third matrix, which misses every vertex because it is not really there[1]

$$A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then the sum:

$$A_1 + A_2 + A_3 = \begin{bmatrix} 4 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}.$$

Has the same pattern of nonzero entries as the adjacency matrix plus the identity. By Lemma 2.7, the pattern of the matrix M built from this will be the same of the pattern of the adjacency matrix plus the identity Kronecker product with a 3×3 all ones matrix, which has the pattern of Bowtie $\boxtimes K_3$.

[1] In larger graphs we could also modify the coloring of the cliques so that the additional color touches some vertices, but that is not possible in this case since there are only two cliques to color.

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Using the construction from Lemma 2.3 with $u = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, we get the blocks:

$$J_1 = \frac{1}{9}\begin{bmatrix}1 & -2 & -2\\ -2 & 4 & 4\\ -2 & 4 & 4\end{bmatrix},$$

$$J_2 = \frac{1}{9}\begin{bmatrix}4 & -2 & 4\\ -2 & 1 & -2\\ 4 & -2 & 4\end{bmatrix},$$

$$J_3 = \frac{1}{9}\begin{bmatrix}4 & 4 & -2\\ 4 & 4 & -2\\ -2 & -2 & 1\end{bmatrix}.$$

And so the matrix M is

$$\begin{aligned} M &= A_1 \otimes J_1 + A_2 \otimes J_2 + A_3 \otimes J_3 \\ &= \frac{1}{9}\begin{bmatrix}
16 & 4 & 4 & 1 & -2 & -2 & 1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & 10 & -8 & -2 & 4 & 4 & -2 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & -8 & 10 & -2 & 4 & 4 & -2 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & -2 & 16 & 4 & 4 & 1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 4 & 4 & 4 & 10 & -8 & -2 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 4 & 4 & 4 & -8 & 10 & -2 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & -2 & 1 & -2 & -2 & 8 & 8 & -4 & 4 & -2 & 4 & 4 & -2 & 4\\
-2 & 4 & 4 & -2 & 4 & 4 & 8 & 8 & -4 & -2 & 1 & -2 & -2 & 1 & -2\\
-2 & 4 & 4 & -2 & 4 & 4 & -4 & -4 & 2 & 4 & -2 & 4 & 4 & -2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 4 & 10 & 4 & -8 & 4 & -2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -2 & 4 & 16 & 4 & -2 & 1 & -2\\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 4 & -8 & 4 & 10 & 4 & -2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 4 & 4 & -2 & 4 & 10 & 4 & -8\\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -2 & -2 & 1 & -2 & 4 & 16 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 4 & 4 & -2 & 4 & -8 & 4 & 10
\end{bmatrix}. \end{aligned}$$

This matrix is in $\mathcal{S}(\text{Bowtie} \boxtimes K_3)$ and has only two distinct eigenvalues, specifically, the eigenvalues are -1 with multiplicity 4 and 2 with multiplicity 11.

The paper [6] describes the *closed candles* pictured in Fig. 3a and prove that they satisfy $q = 2$. We note that half of these arise simply as $C_{2k} \boxtimes K_2$, and so Theorem 3.2 gives another proof that these satisfy $q = 2$. The paper also shows that $q(C_{2k+1} \boxtimes K_2) = 2$, which our results fail to cover. However, Theorem 3.3 does show that $q(C_{2k+1} \boxtimes K_3) = 2$, which as far as we are aware is new information.

Finally, we remark that work in [5] describes families of graphs which they call *single-ended candles* and *double-ended candles* (example in Fig. 3b and 3c), which they show satisfies $q = 2$. These graphs are very similar to $P_k \boxtimes K_2$. The missing vertices correspond exactly to the vertices of P_k, which are not of degree 2. If it were not for those vertices, we would be able to apply Theorem 3.2 to this case. This suggests that there may be a generalization of our theorem for graphs that have chromatic index k and are "almost" k-regular.

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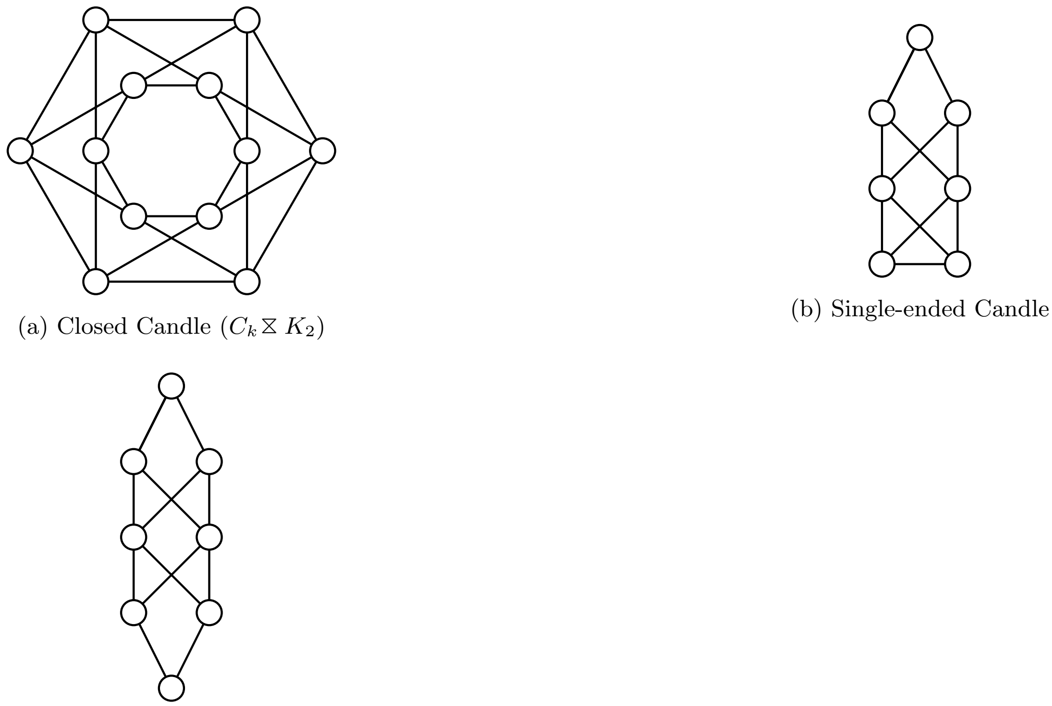

(a) Closed Candle ($C_k \text{ ⧖ } K_2$)

(b) Single-ended Candle

(c) Double-ended Candle

Figure 3: Graphs from [5] and [6] satisfying $q = 2$.

In all our results, we used conditions on G that would guarantee that the construction from Lemma 2.6 would guarantee the pattern of the matrix corresponds to a strong product or a modified strong product. Of course, Lemma 2.6 can apply more generally, but it may not always be as straightforward to describe the graph obtained. As an instance of this idea, the following result can be viewed as a generalization of Theorem 3.2.

Theorem 3.7. *Let G be any graph with chromatic index k. Then we may construct some graph G' which is a subgraph of $G \boxtimes K_k$ that contains $G \text{ ⧖ } K_k$ which has $q(G') = 2$.*

Proof. We proceed as in the proof of Theorem 3.2. Let G_i be the subgraph of G with the same vertex set and containing only the edges of color i, and let A_i be its adjacency matrix, and add a 1 to the diagonal corresponding to any isolated vertex in G_i. Then the eigenvalues of A_i are -1 and 1 with some multiplicities. Then let

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k).$$

Letting $A = A_1 + \cdots + A_k$, we see that A is the adjacency matrix of G, possibly with extra nonzero entries on the diagonal corresponding to vertices not reached by a color. Then by Lemma 2.7, the pattern of M will include every edge in $G \text{ ⧖ } K_k$, and whenever there is a nonzero diagonal entry less than k, the corresponding $k \times k$ block will also have all nonzero entries, so the corresponding graph will contain a clique on those vertices, just as the strong product does. ∎

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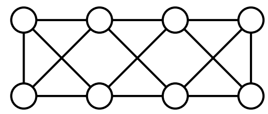

Figure 4: The graph G_4.

As an example of the preceding theorem, let us consider how this applies to paths. Let P_n denote the path on n vertices, with the vertices labeled $1, ..., n$. Let G_n be the graph obtained from $P_n \boxtimes K_2$ by adding the edges $(1,1)(1,2)$ and $(n,1)(n,2)$. See Fig. 4.

Corollary 3.8. *The graphs G_n constructed above satisfy $q(G_n) = 2$.*

Proof. Note that $\chi'(P_n) = 2$. If n is even, the subgraph given by one color contains every vertex of P_n, and the other color will either includes every vertex except the two end vertices. If n is odd, then both colors include every vertex except one end vertex. In either case, the matrix A from the proof of Theorem 3.7 is the adjacency matrix of the path together with a 1 on both ending diagonals. Then the matrix M from the same proof belongs to $\mathcal{S}(G_n)$. ◻

We note the similarity between these graphs and the various types of "candles" studied in [5]. It is an interesting open question if other known families of graphs satisfying $q = 2$ might arise from other "product-like" constructions.

As a final note, an alternative generalization of Theorem 3.2 can be obtained which retains better control over the pattern of the graph if we sacrifice insisting on only 2 eigenvalues.

Theorem 3.9. *If G is any graph with chromatic index k, then $q(G \boxtimes K_k) \leq 3$.*

Proof. We proceed exactly as in the proof of Theorem 3.7, but we will not add an extra 1 to the diagonal. Let G_i be the subgraph of G containing the edges of color i, and let A_i be the adjacency matrix of G_i. Each G_i consists of disjoint edges or disjoint vertices, so the eigenvalues of A_i are -1, 1, and possible 0 if G_i is not a perfect matching. Then

$$M = (A_1 \otimes J_1) + \cdots + (A_k \otimes J_k),$$

is a matrix in $\mathcal{S}(G \boxtimes K_k)$ with at most 3 eigenvalues, -1, 1, and possibly 0. ◻

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