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SOME NEW LOWER BOUNDS FOR THE MINIMUM EIGENVALUE OF THE HADAMARD PRODUCT OF AN M-MATRIX AND ITS INVERSE*

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Abstract. For the Hadamard product $A \circ A^{-1}$ of an M-matrix A and its inverse A^{-1}, some new lower bounds for the minimum eigenvalue of $A \circ A^{-1}$ are given. These bounds improve the results of [H.B. Li, T.Z. Huang, S.Q. Shen, and H. Li. Lower bounds for the minimum eigenvalue of Hadamard product of an M-matrix and its inverse. *Linear Algebra Appl.*, 420:235-247, 2007] and [Y.T. Li, F.B. Chen, and D.F. Wang. New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse. *Linear Algebra Appl.*, 430:1423-1431, 2009].

Key words. Hadamard product, M-matrix, Inverse, Minimum eigenvalue, Lower bounds.

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1. Introduction. For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. For $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we write $A \geq 0$ ($A > 0$) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all $i, j \in N$. If $A \geq 0$, we say A is a *nonnegative matrix*, and if $A > 0$, we say A is a *positive matrix*. The *Perron eigenvalue* of an $n \times n$ nonnegative matrix P is denoted by $\rho(P)$.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if there exists a nonnegative matrix B and a nonnegative real number λ, such that $A = \lambda I - B$ with $\lambda \geq \rho(B)$, where I is the identity matrix. If $\lambda > \rho(B)$ (resp., $\lambda = \rho(B)$), then the M-matrix A is nonsingular (resp., singular); see [1].

For $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, define $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A.

The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{R}^{n\times n}$ is the matrix $A \circ B = [a_{ij}b_{ij}] \in \mathbb{R}^{n\times n}$. If A and B are M-matrices, then it was proved in [5] that $A \circ B^{-1}$ is also an M-matrix. For an M-matrix A, Fiedler et al. showed in [4] that $0 < \tau(A \circ A^{-1}) \leq 1$. In [5], Fiedler and Markham gave a lower bound on

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$\tau(A \circ A^{-1})$,

$$\tau(A \circ A^{-1}) \geq \frac{1}{n}, \tag{1.1}$$

and proposed the following conjecture:

$$\tau(A \circ A^{-1}) \geq \frac{2}{n}. \tag{1.2}$$

This conjecture has been proved by Yong ([13, 14]), Song ([10]) and Chen ([3]) independently.

In [12], Xiang used the spectral radius of the Jacobi iterative matrix of an $n \times n$ M-matrix A, and proved that

$$\tau(A \circ A^{-1}) \geq 1 - \rho(J_A)^2, \tag{1.3}$$

and

$$\tau(A \circ A^{-1}) \geq \frac{1 + \rho(J_A)^{\frac{1}{n+2}}}{1 + (n-1)\rho(J_A)^{\frac{1}{n+2}}}, \tag{1.4}$$

where $\rho(J_A)$ denotes the spectral radius of the Jacobi iterative matrix of A.

Obviously, the lower bounds (1.1) and (1.2) are simple, but they are not accurate enough. For the lower bounds (1.3) and (1.4), it is difficult to calculate the lower bound of $\tau(A \circ A^{-1})$ by using these formulas, since it is difficult to calculate $\rho(J_A)$ when the order of A is large.

In [7], Li obtained the following result:

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum\limits_{j \neq i} s_{ji}} \right\}, \tag{1.5}$$

which only depends on the entries of $A = [a_{ij}]$, where $R_i = \sum\limits_{k \neq i} |a_{ik}|$, $d_i = \frac{R_i}{|a_{ii}|}$, $i \in N$; $s_{ji} = \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| d_k}{|a_{jj}|}$, $j \neq i$, $j \in N$; $s_i = \max\limits_{j \neq i}\{s_{ij}\}$, $i \in N$. In [8], Li improved the bound (1.5) in some cases, and obtained the following result:

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum\limits_{j \neq i} m_{ji}} \right\}, \tag{1.6}$$

where $r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum\limits_{k \neq l,i} |a_{lk}|}$, $l \neq i$; $r_i = \max\limits_{l \neq i}\{r_{li}\}$, $i \in N$; $m_{ji} = \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| r_i}{|a_{jj}|}$, $j \neq i$; $m_i = \max\limits_{j \neq i}\{m_{ij}\}$, $i \in N$.

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Recently, in [9], Li has proved the following bound:

$$\tau(B \circ A^{-1}) \geq \min_i \left\{ \frac{b_{ii} - n_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\},$$

where $r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{lk}|}$, $l \neq i$; $r_i = \max_{l \neq i}\{r_{li}\}$, $i \in N$; $n_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|}$, $j \neq i$; $n_i = \max_{j \neq i}\{n_{ij}\}$, $i \in N$. When $B = A$, the bound gives a lower bound of $\tau(A \circ A^{-1})$:

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - n_i R_i}{1 + \sum_{j \neq i} n_{ji}} \right\}. \tag{1.7}$$

In this paper, we present some new lower bounds on $\tau(A \circ A^{-1})$. The bounds improve the results in [7, 8].

2. Preliminaries and notation. In this section, we give some lemmas which give bounds on the entries of the inverse matrix A^{-1} of a nonsingular matrix A. The following is the list of notations that we use throughout: For $i, j, k, l \in N$,

$$R_i = \sum_{k \neq i} |a_{ik}|, \ C_i = \sum_{k \neq i} |a_{ki}|, \ d_i = \frac{R_i}{|a_{ii}|}, \ \hat{c}_i = \frac{C_i}{|a_{ii}|};$$

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{lk}|}, \ l \neq i; \ r_i = \max_{l \neq i}\{r_{li}\}, \ i \in N;$$

$$c_{il} = \frac{|a_{il}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{kl}|}, \ l \neq i; \ c_i = \max_{l \neq i}\{r_{il}\}, \ i \in N;$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \ j \neq i; \ m_i = \max_{j \neq i}\{m_{ij}\}, \ i \in N;$$

$$n_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|}, \ j \neq i; \ n_i = \max_{j \neq i}\{n_{ij}\}, \ i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{|a_{jj}|}, \ j \neq i, \ j \in N; \ s_i = \max_{j \neq i}\{s_{ij}\}, \ i \in N;$$

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$$T_{ji} = \min\{m_{ji}, n_{ji}\},\ j \neq i;\ T_i = \max_{j\neq i}\{T_{ij}\},\ i \in N.$$

LEMMA 2.1. [8, Lemma 2.2] *Let A be an $n \times n$ real matrix.*

(a) *If $A = [a_{ij}]$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{a_{jj}} b_{ii},\ i, j \in N,\ i \neq j.$$

(b) *If $A = [a_{ij}]$ is a strictly column diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ij} \leq \frac{|a_{ij}| + \sum_{k\neq j,i} |a_{kj}| c_i}{a_{jj}} b_{ii},\ i, j \in N,\ i \neq j.$$

LEMMA 2.2. [9, Lemma 2.2] *Let A be an $n \times n$ real matrix.*

(a) *If $A = [a_{ij}]$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_k}{a_{jj}} b_{ii},\ i, j \in N,\ i \neq j.$$

(b) *If $A = [a_{ij}]$ is a strictly column diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ij} \leq \frac{|a_{ij}| + \sum_{k\neq j,i} |a_{kj}| c_k}{a_{jj}} b_{ii},\ i, j \in N,\ i \neq j.$$

LEMMA 2.3. *If $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ji} \leq T_{ji} b_{ii},\ i, j \in N,\ i \neq j.$$

Proof. By Lemma 2.1 (a) and Lemma 2.2 (a), we have

$$b_{ji} \leq n_{ji} b_{ii},\ b_{ji} \leq m_{ji} b_{ii},\ i, j \in N,\ i \neq j.$$

From $T_{ji} = \min\{m_{ji}, n_{ji}\}$, we get

$$b_{ji} \leq T_{ji} b_{ii},\ i, j \in N,\ i \neq j. \quad \square$$

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LEMMA 2.4. [8, Theorem 3.1] *If $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is an M-matrix and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then*

$$b_{ii} \geq \frac{1}{1 + \sum\limits_{j\neq i} m_{ji}},\ i \in N.$$

LEMMA 2.5. [7, Theorem 2.1] *Let A be an $n \times n$ real matrix.*

(a) *If $A = [a_{ij}]$ is a strictly row diagonally dominant matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$|b_{ji}| \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}|d_k}{|a_{jj}|}|b_{ii}|,\ i,j \in N,\ i \neq j.$$

(b) *If $A = [a_{ij}]$ is a strictly column diagonally dominant matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$|b_{ij}| \leq \frac{|a_{ij}| + \sum\limits_{k\neq j,i} |a_{kj}|\hat{c}_k}{|a_{jj}|}|b_{ii}|,\ i,j \in N,\ i \neq j.$$

LEMMA 2.6. [7, Theorem 2.3] *If $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies*

$$b_{ii} \geq \frac{1}{a_{ii}},\ i \in N.$$

LEMMA 2.7. [14, Lemma 2.3] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = [1, 1, \ldots, 1]^T$.*

LEMMA 2.8. [11, P. 719] *Let $A = [a_{ij}]$ be an $n \times n$ complex matrix and x_1, x_2, $\ldots$, x_n be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_i \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq x_i \sum_{j\neq i} \frac{1}{x_j}|a_{ji}|,\ i \in N \right\}.$$

LEMMA 2.9. [14, Lemma 2.1] *If P is an irreducible M-matrix, and $Pz \geq kz$ for a nonnegative nonzero vector z, then $\tau(P) \geq k$.*

The following result can be found in [2].

LEMMA 2.10. *If $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is an M-matrix, then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix.*

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LEMMA 2.11. [6, Lemma 5.1.2] *Let $A,\ B \in \mathbb{R}^{n\times n}$, and suppose that $D \in \mathbb{R}^{n\times n}$ and $E \in \mathbb{R}^{n\times n}$ are diagonal matrices. Then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

3. Main results. In this section, we present some new lower bounds for $\tau(A \circ A^{-1})$.

THEOREM 3.1. *If $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is an M-matrix, and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then*

$$b_{ii} \geq \frac{1}{1 + \sum\limits_{j\neq i} n_{ji}}, i \in N;\ \text{and}\ b_{ii} \geq \frac{1}{1 + \sum\limits_{j\neq i} T_{ji}},\ i \in N.$$

Proof. We first prove $b_{ii} \geq \frac{1}{1+\sum\limits_{j\neq i} n_{ji}}, i \in N$. Since A^{-1} is doubly stochastic, by Lemma 2.7, we know that $Ae = e$, so A is a strictly diagonally dominant matrix by row. By Lemma 2.2 (a), for $i \in N$,

$$\begin{aligned}
1 &= b_{ii} + \sum_{j\neq i} |b_{ji}| \\
&\leq b_{ii} + \sum_{j\neq i} \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_k}{|a_{jj}|} b_{ii} \\
&= \left(1 + \sum_{j\neq i} \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_k}{|a_{jj}|}\right) b_{ii} \\
&= (1 + \sum_{j\neq i} n_{ji}) b_{ii},
\end{aligned}$$

i.e.,

$$b_{ii} \geq \frac{1}{1 + \sum\limits_{j\neq i} n_{ji}},\ i \in N.$$

Similarly, we can prove $b_{ii} \geq \frac{1}{1+\sum\limits_{j\neq i} T_{ji}},\ i \in N.$ ∎

THEOREM 3.2. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then*

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum\limits_{j\neq i} T_{ji}} \right\}.$$

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Proof. Since A is irreducible, from Lemma 2.7, we know that $Ae = e$, so A is a strictly diagonally dominant matrix by row. Therefore, $0 < T_i < 1$, $i = 1, 2, \ldots, n$.

Let $\tau(A \circ A^{-1}) = \lambda$. By Lemma 2.8, there exists $i_0 \in N$, such that

$$|\lambda - a_{i_0 i_0} b_{i_0 i_0}| \leq T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j i_0} b_{j i_0}|.$$

Hence,

$$\begin{aligned}
|\lambda| &\geq a_{i_0 i_0} b_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j i_0} b_{j i_0}| \\
&\geq a_{i_0 i_0} b_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j i_0}| T_{j i_0} b_{i_0 i_0} \text{ (by Lemma 2.3)} \\
&\geq (a_{i_0 i_0} - T_{i_0} R_{i_0}) b_{i_0 i_0} \\
&\geq \frac{a_{i_0 i_0} - T_{i_0} R_{i_0}}{1 + \sum\limits_{j \neq i_0} T_{j i_0}} \text{ (by Theorem 3.1)} \\
&\geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum\limits_{j \neq i} T_{ji}} \right\}. \quad \square
\end{aligned}$$

Remark 3.3. If A is reducible, without loss of generality, we can assume that A is a block upper triangular matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \cdots \\ & & & A_{kk} \end{bmatrix}$$

with irreducible diagonal blocks A_{ii}, $i \in K = \{1, 2, \ldots, k\}$. Then $\tau(A \circ A^{-1}) = \min_{i \in K} \tau(A_{ii} \circ A_{ii}^{-1})$. Thus, the problem of the reducible matrix A is reduced to those of irreducible diagonal blocks A_{ii}, $i \in K$. The result of Theorem 3.2 also holds.

Theorem 3.4. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible strictly row diagonally dominant M-matrix. Then*

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \right\}.$$

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Proof. Since A is irreducible, then $A^{-1} = [b_{ij}] > 0$, and $A \circ A^{-1}$ is again irreducible. Note that

$$\tau(A \circ A^{-1}) = \tau((A \circ A^{-1})^T) = \tau(A^T \circ (A^T)^{-1})).$$

Let

$$(A^T \circ (A^T)^{-1})e = [g_1, g_2, \ldots, g_n]^T,$$

where $e = [1, 1, \ldots, 1]^T$. Without loss of generality, we may assume that $g_1 = \min_i \{g_i\}$, by Lemma 2.3 , we have

$$\begin{aligned} g_1 &= \sum_{j=1}^{n} |a_{j1} b_{j1}| \\ &= a_{11} b_{11} - \sum_{j \neq 1} |a_{j1} b_{j1}| \\ &\geq a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| T_{j1} b_{11} \text{ (by Lemma 2.3)} \\ &= (a_{11} - \sum_{j \neq 1} |a_{j1}| T_{j1}) b_{11} \\ &\geq \frac{a_{11} - \sum_{j \neq 1} |a_{j1}| T_{j1}}{a_{11}} \text{ (by Lemma 2.6)} \\ &\geq 1 - \frac{1}{a_{11}} \sum_{j \neq 1} |a_{j1}| T_{j1}. \end{aligned}$$

Therefore, $(A^T \circ (A^T)^{-1})e \geq (1 - \frac{1}{a_{11}} \sum_{j \neq 1} |a_{j1}| T_{j1})e$. From Lemma 2.9, we have

$$\tau(A \circ A^{-1}) = \tau(A^T \circ (A^T)^{-1}) \geq \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \right\}. \quad \square$$

REMARK 3.5. If A is an M-matrix, then by Lemma 2.10, we know that there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix. So the result of Theorem 3.4 also holds for a general M-matrix.

THEOREM 3.6. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an M-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then*

$$\min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\} \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\}.$$

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Proof. Since $T_{ji} = \min\{m_{ji}, n_{ji}\}$,

$$T_{ji} \le m_{ji},\ j \ne i,\ j \in N;\ T_i \le m_i,\ i \in N.$$

Hence,

$$a_{ii} - T_i R_i \ge a_{ii} - m_i R_i,\ \frac{1}{1 + \sum\limits_{j \ne i} T_{ji}} \ge \frac{1}{1 + \sum\limits_{j \ne i} m_{ji}}.$$

Therefore,

$$\min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum\limits_{j \ne i} T_{ji}} \right\} \ge \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum\limits_{j \ne i} m_{ji}} \right\}. \qquad \square$$

Remark 3.7. Theorem 3.6 shows that the result of Theorem 3.2 is better than that of Theorem 3.2 in [10].

Theorem 3.8. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an M-matrix. Then*

$$\min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| T_{ji} \right\} \ge \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| m_{ji} \right\}.$$

Proof. By the proof of Theorem 3.6, we have

$$T_{ji} \le m_{ji},\ j \ne i.$$

So

$$1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| T_{ji} \ge 1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| m_{ji}.$$

Thus,

$$\min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| T_{ji} \right\} \ge \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| m_{ji} \right\}. \qquad \square$$

Remark 3.9. Theorem 3.8 shows that the result of Theorem 3.4 is better than that of Theorem 3.4 in [10].

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THEOREM 3.10. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then*

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i \sum\limits_{j\neq i} \frac{|a_{ji}|n_{ji}}{s_j}}{1 + \sum\limits_{j\neq i} m_{ji}} \right\}.$$

Proof. Since A^{-1} is doubly stochastic, by Lemma 2.7, we have $Ae = e$, $A^T e = e$, so A is a strictly diagonally dominant M-matrix, and

$$a_{ii} = \sum_{k\neq i} |a_{ik}| + 1 = \sum_{k\neq i} |a_{ki}| + 1, \; a_{ii} > 1$$

and

$$d_i = \frac{\sum\limits_{k\neq i} |a_{ik}|}{|a_{ii}|} < 1, \; i \in N.$$

For convenience, we denote

$$\tilde{R}_j = \sum_{k\neq j} |a_{jk}| d_k, \; j \in N.$$

Then, for any $j \in N$ with $j \neq i$, we have

$$\tilde{R}_j \leq |a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k \leq R_j = \sum_{k\neq j} |a_{jk}| \leq a_{jj}.$$

Therefore, there exists a real number α_{ji} ($0 \leq \alpha_{ji} \leq 1$), such that

$$|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k| = \alpha_{ji} R_j + (1 - \alpha_{ji})\tilde{R}_j.$$

Let $\alpha_j = \max\limits_{i\neq j}\{\alpha_{ji}\}$. Then $0 < \alpha_j \leq 1$, (if $\alpha_j = 0$, then A is reducible, which is a contradiction). So, from the definition of s_{ij} , we have

$$s_j = \max_{i\neq j}\{s_{ji}\} = \frac{\alpha_j R_j + (1 - \alpha_j)\tilde{R}_j}{a_{jj}}, \; j \in N.$$

Since $0 < \alpha_j \leq 1$, we get $0 < s_j \leq 1$.

Let $\tau(A \circ A^{-1}) = \lambda$. By Lemma 2.8, there exists $i_0 \in N$, such that

$$|\lambda - a_{i_0 i_0} b_{i_0 i_0}| \leq s_{i_0} \sum_{j\neq i_0} \frac{1}{s_j} |a_{j i_0} b_{j i_0}|.$$

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Hence,

$$
\begin{aligned}
|\lambda| &\geq a_{i_0i_0}b_{i_0i_0} - s_{i_0}\sum_{j\neq i_0}\frac{1}{s_j}|a_{ji_0}b_{ji_0}| \\
&\geq a_{i_0i_0}b_{i_0i_0} - s_{i_0}\sum_{j\neq i_0}\frac{1}{s_j}|a_{ji_0}|\frac{|a_{ji_0}| + \sum\limits_{k\neq j,i_0}|a_{jk}|r_k}{a_{jj}}b_{i_0i_0} \text{ (by Lemma 2.2 (a))} \\
&= (a_{i_0i_0} - s_{i_0}\sum_{j\neq i_0}\frac{1}{s_j}|a_{ji_0}|n_{ji_0})b_{i_0i_0} \\
&\geq \frac{a_{i_0i_0} - s_{i_0}\sum\limits_{j\neq i_0}\frac{1}{s_j}|a_{ji_0}|n_{ji_0}}{1+\sum\limits_{j\neq i_0}m_{ji_0}} \text{ (by Lemma 2.4)} \\
&\geq \min_i\{\frac{a_{ii} - s_i\sum\limits_{j\neq i}\frac{|a_{ji}|n_{ji}}{s_j}}{1+\sum\limits_{j\neq i}m_{ji}}\}. \quad \square
\end{aligned}
$$

REMARK 3.11. When A is reducible, without loss of generality, we can assume that A is a block upper triangular matrix of the form

$$
A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \cdots \\ & & & A_{kk} \end{bmatrix}
$$

with irreducible diagonal blocks A_{ii}, $i \in K$. Then $\tau(A \circ A^{-1}) = \min\limits_{i\in K}\tau(A_{ii} \circ A_{ii}^{-1})$. Thus, the problem of the reducible matrix A is reduced to those of irreducible diagonal blocks A_{ii}, $i \in K$. The result of Theorem 3.10 also holds.

By using Lemma 2.6, Lemma 2.10 and Theorem 3.10, we can get the following corollary.

COROLLARY 3.12. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be an M-matrix. Then*

$$
\tau(A \circ A^{-1}) \geq \min_i\left\{\frac{a_{ii} - s_i\sum\limits_{j\neq i}\frac{|a_{ji}|n_{ji}}{s_j}}{a_{ii}}\right\}.
$$

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4. Examples.

Example 4.1. (See also Example 3.1 in [9]) Let

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

By $Ae = e$ and $A^T e = e$, we know that A^{-1} is a doubly stochastic matrix. By calculating with *Matlab 7.0*, we have

$$A^{-1} = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2333 & 0.3667 & 0.2 & 0.2 \\ 0.1667 & 0.2333 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 4 \end{bmatrix}.$$

If we apply the conjecture of Fiedler and Markham, we have

$$\tau(A \circ A^{-1}) \geq \frac{2}{n} = 0.5;$$

if we apply Theorem 3.1 of [9], we have

$$\tau(A \circ A^{-1}) \geq 0.6624;$$

if we apply Theorem 3.2 of [10], we have

$$\tau(A \circ A^{-1}) \geq 0.7999.$$

But, if we apply Theorem 3.2, we have

$$\tau(A \circ A^{-1}) \geq 0.85;$$

if we apply Theorem 3.10, we have

$$\tau(A \circ A^{-1}) \geq 0.8602.$$

In fact, $\tau(A \circ A^{-1}) = 0.9755$.

Example 4.2. Let

$$A = \begin{bmatrix} 5 & -1 & -2 & -1 \\ -1 & 12 & -7 & -2 \\ -1 & -1 & 15 & -4 \\ -2 & -3 & 0 & 10 \end{bmatrix}.$$

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By calculating with *Matlab 7.0*, we have

$$A^{-1} = \begin{bmatrix} 0.2372 & 0.0364 & 0.0486 & 0.0505 \\ 0.0512 & 0.1043 & 0.0555 & 0.0482 \\ 0.0360 & 0.0197 & 0.0806 & 0.0398 \\ 0.0628 & 0.0386 & 0.0264 & 0.1245 \end{bmatrix}.$$

Therefore, A is a nonsingular M-matrix.

If we apply the conjecture of Fiedler and Markham, we have

$$\tau(A \circ A^{-1}) \geq \frac{2}{n} = 0.5;$$

if we apply Theorem 3.5 of [9], we have

$$\tau(A \circ A^{-1}) \geq 0.5689;$$

if we apply Theorem 3.4 of [10], we have

$$\tau(A \circ A^{-1}) \geq 0.5422.$$

But, if we apply Theorem 3.4, we have

$$\tau(A \circ A^{-1}) \geq 0.5959;$$

if we apply Corollary 3.12, we have

$$\tau(A \circ A^{-1}) \geq 0.6021.$$

In fact, $\tau(A \circ A^{-1}) = 0.9548$.

REMARK 4.3. The numerical examples show that in these cases the bounds of Theorem 3.2 and Theorem 3.10 are sharper than Theorem 3.1 in [9] and Theorem 3.2 in [10]; the bounds in Theorem 3.4 and Corollary 3.12 are sharper than Theorem 3.5 in [9] and Theorem 3.4 in [10].

REFERENCES

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York, 1979.

[2] J.L. Chen. *Special Matrix.* Tsinghua University Press, Beijing, 2000.

[3] S.C. Chen. A lower bound for the minimum eigenvalue of the Hadamard product of matrices. *Linear Algebra Appl.*, 378:159–166, 2004.

[4] M. Fiedler, C.R. Johnson, T. Markham, and M. Neumann. A trace inequality for M-matrices and the symmetrizability of a real matrix by a positive diagonal matrix. *Linear Algebra Appl.*, 71:81–94, 1985.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 630-643, June 2011

[5] M. Fiedler and T. Markham. An inequality for the Hadamard product of an M-matrix and inverse M-matrix. *Linear Algebra Appl.*, 101:1–8, 1988.

[6] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, 1991.

[7] H.B. Li, T.Z. Huang, S.Q. Shen, and H. Li. Lower bounds for the minimum eigenvalue of Hadamard product of an M-matrix and its inverse. *Linear Algebra Appl.*, 420:235–247, 2007.

[8] Y.T. Li, F.B. Chen, and D.F. Wang. New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse. *Linear Algebra Appl.*, 430:1423–1431, 2009.

[9] Y.T. Li, Y.Y. Li, R.W. Wang, and Y.Q. Wang. Some new bounds on eigenvalues of the Hadamard product and the Fan product of matrices. *Linear Algebra Appl.*, 432:536–545, 2010.

[10] Y.Z. Song. On an inequality for the Hadamard product of an M-matrix and its inverse. *Linear Algebra Appl.*, 305:99–105, 2000.

[11] R.S. Varga. Minimal Gerschgorin sets. *Pacific J. Math.*, 15(2):719–729, 1965.

[12] S.H. Xiang. On an inequality for the Hadamard product of an M-matrix or an H-matrix and its inverse. *Linear Algebra Appl.*, 367:17–27, 2003.

[13] X.R. Yong. Proof of a conjecture of Fiedler and Markham. *Linear Algebra Appl.*, 320:167–171, 2000.

[14] X.R. Yong and Z. Wang. On a conjecture of Fiedler and Markham. *Linear Algebra Appl.*, 288:259–267, 1999.