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A NOTE ON THE REVERSE ORDER LAWS FOR $\{1,2,3\}$- AND $\{1,2,4\}$-INVERSES OF MULTIPLE MATRIX PRODUCTS*

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Abstract. Motivated by the equivalent conditions for the inclusions

$$A_n\{1,2,i\}\cdots A_2\{1,2,i\}A_1\{1,2,i\} \subseteq (A_1A_2\cdots A_n)\{1,2,i\} \quad (i=3,4)$$

presented in [B. Zheng and Z. Xiong. The reverse order laws for {1,2,3}- and {1,2,4}-inverses of multiple matrix products. *Linear Multilinear Algebra*, 58:765–782, 2010.], we show that for $i \in \{3,4\}$,

$$A_n\{1,2,i\}\cdots A_2\{1,2,i\}A_1\{1,2,i\} = (A_1A_2\cdots A_n)\{1,2,i\}$$

is equivalent to

$$A_n\{1,2,i\}\cdots A_2\{1,2,i\}A_1\{1,2,i\} \subseteq (A_1A_2\cdots A_n)\{1,2,i\}.$$

Key words. Reverse order law, Maximal and minimal ranks, Generalized inverse, Generalized Schur complement.

AMS subject classifications. 15A09.

1. Introduction. Throughout this paper, we let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$. For $A \in \mathbb{C}^{m\times n}$, its rank and conjugate transpose are denoted by $r(A)$ and A^*, respectively. The identity matrix of order n is denoted by I_n.

For convenience, the following notations will be used in this paper. For $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$ and $X_i \in \mathbb{C}^{l_{i+1}\times l_i}$ $(i = 1, 2, \ldots, n)$, we set

$$\mathscr{A}_i^j = A_iA_{i+1}\cdots A_j, \quad \mathscr{X}_i^j = X_i^*X_{i+1}^*\cdots X_j^*, \quad 1 \le i \le j \le n, \tag{1.1}$$

and

$$\mathscr{X}_{n+1}^n = I_{l_{n+1}} \text{ and } \mathscr{X}_1^0 = I_{l_1}.$$

Recall that the *Moore-Penrose inverse* $A^\dagger$ of a matrix $A \in \mathbb{C}^{m\times n}$ is defined to be the unique solution of the four *Penrose equations* (see, for example, [1])

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA. \tag{1.2}$$

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Let ($\emptyset \neq$) $\eta \subseteq \{1,2,3,4\}$. Then $A\eta$ denotes the set of all matrices X which satisfy property (i) in (1.2) for each $i \in \eta$. Any matrix $X \in A\eta$ is called an *η-inverse* of A. One usually denotes any $\{1\}$-inverse of A as A^-. Any $\{1,2,3\}$-inverse of A is denoted by $A^{(1,2,3)}$, and any $\{1,2,4\}$-inverse of A is denoted by $A^{(1,2,4)}$. Set $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

As we know, for nonsingular matrices $A_i \in \mathbb{C}^{n\times n}$, $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}$ A_1^{-1}. However, this so-called *reverse order law* is not necessarily true for the generalized inverses. The reverse order law for the generalized inverses of the multiple-matrix products yields a class of interesting problems that are fundamental in the theory of generalized inverses of matrices, and statistics. They have attracted considerable attention since the mid 1960s, and many interesting results have been studied. There are three important relations with which we are concerned in the study of the reverse order law for any η-inverse of the multiple-matrix products $A_1A_2\cdots A_n$. These are

$$\begin{aligned} A_n\eta\cdots A_2\eta A_1\eta &\subseteq (A_1A_2\cdots A_n)\eta, \\ A_n\eta\cdots A_2\eta A_1\eta &\supseteq (A_1A_2\cdots A_n)\eta, \\ A_n\eta\cdots A_2\eta A_1\eta &= (A_1A_2\cdots A_n)\eta. \end{aligned}$$

Since the Moore-Penrose inverse of a matrix is unique, for Moore-Penrose inverse $(A_1A_2\cdots A_n)^\dagger$, the above three relations reduce to one, namely $(A_1A_2\cdots A_n)^\dagger = A_n^\dagger\cdots A_2^\dagger A_1^\dagger$. In [5] Tian derived equivalent conditions for the equality to hold. For $\{1\}$-inverse and $\{1,2\}$-inverse, using Product Singular Value Decomposition (P-SVD), in [6] Wei presented the equivalent conditions for $A_n\{1\}\cdots A_2\{1\}A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$ and $A_n\{1,2\}\cdots A_2\{1,2\}A_1\{1,2\} = (A_1A_2\cdots A_n)\{1,2\}$, and in [8] Zheng and Xiong obtained conditions on the ranks of the known matrices satisfying $A_n\{1\}\cdots A_2\{1\}\ A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$ by using the maximal rank of the generalized Schur complement, but the cases $A_n\{1\}\cdots A_2\{1\}A_1\{1\} \supseteq (A_1A_2\cdots A_n)\{1\}$ and $A_n\{1\}\cdots A_2\{1\}\ A_1\{1\} = (A_1A_2\cdots A_n)\{1\}$ are still open. For the $\{1,3\}$-inverse, by using P-SVD, in [7] Liu and Wei gave the necessary and sufficient conditions for the reverse order law

$$A_n\{1,3\}\cdots A_2\{1,3\}A_1\{1,3\} \subseteq (\supseteq)(A_1A_2\cdots A_n)\{1,3\},$$

but these results require information about the sub-block produced by P-SVD.

For {1,2,3}- and {1,2,4}-inverses, in [9] Xiong and Zheng presented necessary and sufficient conditions for $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\}$ $(i = 3,4)$ to hold. Recently, by applying the maximal and minimal ranks of generalized Schur complements, in [2] Zheng and Xiong derived equivalent conditions for one side inclusion relations of η-inverses

$$A_n\{1,2,i\}\cdots A_2\{1,2,i\}A_1\{1,2,i\} \subseteq (A_1A_2\cdots A_n)\{1,2,i\} \quad (i = 3,4),$$

and the other two relations remain open. We restate these results below.

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THEOREM 1.1. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$, and $\mathscr{A}_i^j$, $1 \le i \le j \le n$ be as in (1.1). Then the following statements are equivalent:*

1. $A_n\{1,2,3\} \cdots A_2\{1,2,3\} A_1\{1,2,3\} \subseteq (A_1 A_2 \cdots A_n)\{1,2,3\}$;
2. $r\left(\begin{bmatrix} (\mathscr{A}_1^n)^* \mathscr{A}_1^{n-1} E_{A_n} & (\mathscr{A}_1^n)^* \mathscr{A}_1^{n-2} E_{A_{n-1}} & \cdots & (\mathscr{A}_1^n)^* \mathscr{A}_1^{1} E_{A_2} \end{bmatrix}\right) = 0$ *and*

$$r(A_1 A_2 \cdots A_n) = \min\{r(A_1), r(A_2), \ldots, r(A_n)\}$$
$$= \sum_{i=1}^n r(A_i) - r\left(\begin{bmatrix} A_2^* & 0 & \cdots & 0 \\ 0 & A_3^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^* \\ \mathscr{A}_1^1 & \mathscr{A}_1^2 & \cdots & \mathscr{A}_1^{n-1} \end{bmatrix}\right).$$

THEOREM 1.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$, and $\mathscr{A}_i^j$, $1 \le i \le j \le n$ be as in (1.1). Then the following statements are equivalent:*

1. $A_n\{1,2,4\} \cdots A_2\{1,2,4\} A_1\{1,2,4\} \subseteq (A_1 A_2 \cdots A_n)\{1,2,4\}$;
2. $r\left(\begin{bmatrix} F_{A_1} \mathscr{A}_2^n (\mathscr{A}_1^n)^* \\ F_{A_2} \mathscr{A}_3^n (\mathscr{A}_1^n)^* \\ \vdots \\ F_{A_{n-1}} \mathscr{A}_n^n (\mathscr{A}_1^n)^* \end{bmatrix}\right) = 0$ *and*

$$r(A_1 A_2 \cdots A_n) = \min\{r(A_1), r(A_2), \ldots, r(A_n)\}$$
$$= \sum_{i=1}^n r(A_i) - r\left(\begin{bmatrix} A_{n-1}^* & 0 & \cdots & 0 & \mathscr{A}_n^n \\ 0 & A_{n-2}^* & \cdots & 0 & \mathscr{A}_{n-1}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_1^* & \mathscr{A}_2^n \end{bmatrix}\right).$$

We remark that since $E_A = F_{A^*}$ and $E_{A^*} = F_A$, we use the E_A and F_A instead of $E_{A^*} = F_A$ and $E_A = F_{A^*}$ as in [2].

It is not easy to establish equivalent conditions of the inclusions

$$A_n\{1,2,i\} \cdots A_2\{1,2,i\} A_1\{1,2,i\} \supseteq (A_1 A_2 \cdots A_n)\{1,2,i\}, \quad (i = 3, 4),$$

since these involve complicated extreme ranks operations. In [2] the authors present equivalent conditions of the inclusion

$$A_n\{1,2,i\} \cdots A_2\{1,2,i\} A_1\{1,2,i\} \subseteq (A_1 A_2 \cdots A_n)\{1,2,i\}.$$

In this paper, by applying the extremal ranks of generalized Schur complements, we prove that for $i \in \{3, 4\}$,

$$A_n\{1,2,i\} \cdots A_2\{1,2,i\} A_1\{1,2,i\} = (A_1 A_2 \cdots A_n)\{1,2,i\}$$

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is actually equivalent to

$$A_n\{1,2,i\}\cdots A_2\{1,2,i\}A_1\{1,2,i\} \subseteq (A_1A_2\cdots A_n)\{1,2,i\}.$$

For the theory of generalized inverses, we refer the reader to [1]. In the following proposition, we have compiled some basic facts about the $\{1,2,3\}$- and $\{1,2,4\}$-inverses, which will be used throughout the paper.

PROPOSITION 1.3. *Let $A \in \mathbb{C}^{m\times n}$. Then*

1. $r(A^{(1,2,3)}) = r(A^{(1,2,4)}) = r(A)$;
2. $A^{(1,2,3)}AA^{\dagger} = A^{(1,2,3)}$ *and* $A^{\dagger}AA^{(1,2,4)} = A^{(1,2,4)}$.

In our development we will need the following lemmas.

LEMMA 1.4. [4] *Let X_1 and X_2 be a pair of outer inverses of a matrix A, that is, $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Then*

$$r(X_1 - X_2) = r\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) + r\left(\begin{bmatrix} X_1 & X_2 \end{bmatrix}\right) - r(X_1) - r(X_2).$$

LEMMA 1.5. [3] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then*

$$r(D - CA^{\dagger}B) = r\left(\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}\right) - r(A),$$

$$\max_{A^{(1,2,3)}} r(D - CA^{(1,2,3)}B) = \min\left\{r\left(\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}\right) - r(A), r\left(\begin{bmatrix} A^*B \\ D \end{bmatrix}\right)\right\},$$

$$\begin{aligned} &\min_{A^{(1,2,3)}} r(D - CA^{(1,2,3)}B) \\ &= r\left(\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}\right) + r\left(\begin{bmatrix} A^*B \\ D \end{bmatrix}\right) - r\left(\begin{bmatrix} A & 0 \\ 0 & A^*B \\ C & D \end{bmatrix}\right), \end{aligned} \tag{1.3}$$

$$\max_{A^{(1,2,4)}} r(D - CA^{(1,2,4)}B) = \min\left\{r\left(\begin{bmatrix} CA^* & D \end{bmatrix}\right),\ r\left(\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix}\right) - r(A)\right\},$$

$$\begin{aligned} \min_{A^{(1,2,4)}} r(D - CA^{(1,2,4)}B) &= r\left(\begin{bmatrix} CA^* & D \end{bmatrix}\right) + r\left(\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix}\right) \\ &\quad - r\left(\begin{bmatrix} A & 0 & B \\ 0 & CA^* & D \end{bmatrix}\right). \end{aligned}$$

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Lemma 1.6. [3] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then*

$$r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = r(A) + r[(I - AA^\dagger)B], \text{ and}$$
$$r\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) = r(A) + r[C(I - A^\dagger A)].$$

2. Main results. In this section, we will prove that for $i = 3, 4$,

$$A_n\{1,2,i\} \cdots A_2\{1,2,i\}A_1\{1,2,i\} = (A_1A_2\cdots A_n)\{1,2,i\}$$

is equivalent to

$$A_n\{1,2,i\} \cdots A_2\{1,2,i\}A_1\{1,2,i\} \subseteq (A_1A_2\cdots A_n)\{1,2,i\}.$$

Before giving the main results, we first prove some auxiliary ones.

Lemma 2.1. *Let $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,\ldots,n$, and $\mathscr{A}_i^j$, $\mathscr{X}_i^j$, $1 \le i \le j \le n$ be as in (1.1). Then*

$$\begin{aligned} &\min_{X_1,X_2,\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \\ &= r(A_1) + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_1^1E_{A_2} & \cdots & \mathscr{A}_1^{n-1}E_{A_n} & \mathscr{A}_1^n \end{bmatrix}\right). \end{aligned} \tag{2.1}$$

Proof. Formula (1.3) gives

$$\begin{aligned} &\min_{X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \\ &= \min_{X_n} r\left(\begin{bmatrix} -(\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \\ &= \min_{X_n} r\left(\begin{bmatrix} 0 & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix} - X_n\begin{bmatrix} (\mathscr{X}_1^{n-1})^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A_n^*A_n & A_n^*(\mathscr{X}_1^{n-1})^* & 0 \\ I_{l_{n+1}} & 0 & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r\left(\begin{bmatrix} A_n^*(\mathscr{X}_1^{n-1})^* & 0 \\ 0 & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \\ &\quad - r\left(\begin{bmatrix} A_n & 0 & 0 \\ 0 & A_n^*(\mathscr{X}_1^{n-1})^* & 0 \\ I_{l_{n+1}} & 0 & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A_n^*(\mathscr{X}_1^{n-1})^* & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r[(\mathscr{A}_1^n)^{(1,2,3)}] - r[A_n(\mathscr{A}_1^n)^{(1,2,3)}] \\ &= r\left(\begin{bmatrix} A_n^*(\mathscr{X}_1^{n-1})^* & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \end{aligned} \tag{2.2}$$

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Applying (1.3) to (2.2), we have

$$\begin{aligned}
&\min_{X_{n-1},X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= \min_{X_{n-1}} r\left(\begin{bmatrix} A_n^*(\mathscr{X}_1^{n-1})^* & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-1}^*A_{n-1} & A_{n-1}^*(\mathscr{X}_1^{n-2})^* & 0 \\ A_n^* & 0 & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r\left(\begin{bmatrix} A_{n-1}^*(\mathscr{X}_1^{n-2})^* & 0 \\ 0 & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad -r\left(\begin{bmatrix} A_{n-1} & 0 & 0 \\ 0 & A_{n-1}^*(\mathscr{X}_1^{n-2})^* & 0 \\ A_n^* & 0 & A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-1}^*(\mathscr{X}_1^{n-2})^* & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r[A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)}]\\
&\quad -r\left(\begin{bmatrix} A_{n-1}E_{A_n} & \mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-1}^*(\mathscr{X}_1^{n-2})^* & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r(\mathscr{A}_1^n)\\
&\quad -r\left(\begin{bmatrix} A_{n-1}E_{A_n} & \mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\min_{X_{n-2},X_{n-1},X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= \min_{X_{n-2}} r\left(\begin{bmatrix} A_{n-1}^*(\mathscr{X}_1^{n-2})^* & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) + r[A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)}]\\
&\quad -r\left(\begin{bmatrix} A_{n-1}E_{A_n} & \mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-2}^*A_{n-2} & A_{n-2}^*(\mathscr{X}_1^{n-3})^* & 0 & 0 \\ A_{n-1}^* & 0 & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad +r\left(\begin{bmatrix} A_{n-2}^*(\mathscr{X}_1^{n-3})^* & 0 & 0 \\ 0 & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad -r\left(\begin{bmatrix} A_{n-2} & 0 & 0 & 0 \\ 0 & A_{n-2}^*(\mathscr{X}_1^{n-3})^* & 0 & 0 \\ A_{n-1}^* & 0 & A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad +r[A_n^*A_n(\mathscr{A}_1^n)^{(1,2,3)}] - r\left(\begin{bmatrix} A_{n-1}E_{A_n} & \mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-2}^*(\mathscr{X}_1^{n-3})^* & A_{n-2}^*\mathscr{A}_{n-2}^{n-2}E_{A_{n-1}} & A_{n-2}^*\mathscr{A}_{n-2}^{n-1}E_{A_n} & A_{n-2}^*\mathscr{A}_{n-2}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad +r\left(\begin{bmatrix} A_{n-1}^*A_{n-1}E_{A_n} & A_{n-1}^*\mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad -r\left(\begin{bmatrix} \mathscr{A}_{n-2}^{n-2}E_{A_{n-1}} & \mathscr{A}_{n-2}^{n-1}E_{A_n} & \mathscr{A}_{n-2}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad +r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} A_{n-1}E_{A_n} & \mathscr{A}_{n-1}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-2}^*(\mathscr{X}_1^{n-3})^* & A_{n-2}^*\mathscr{A}_{n-2}^{n-2}E_{A_{n-1}} & A_{n-2}^*\mathscr{A}_{n-2}^{n-1}E_{A_n} & A_{n-2}^*\mathscr{A}_{n-2}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad +r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_{n-2}^{n-2}E_{A_{n-1}} & \mathscr{A}_{n-2}^{n-1}E_{A_n} & \mathscr{A}_{n-2}^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right).
\end{aligned}$$

In general, for $1 \le i \le n-1$, we have

$$\min_{X_{n-i},X_{n-i+1},\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right) \tag{2.3}$$

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$$
\begin{aligned}
&= r\left(\begin{bmatrix} A_{n-i}^*(\mathscr{X}_1^{n-i-1})^* & A_{n-i}^*\mathscr{A}_{n-i}^{n-i}E_{A_{n-i+1}} & \cdots & A_{n-i}^*\mathscr{A}_{n-i}^{n-1}E_{A_n} & A_{n-i}^*\mathscr{A}_{n-i}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_{n-i}^{n-i}E_{A_{n-i+1}} & \cdots & \mathscr{A}_{n-i}^{n-1}E_{A_n} & \mathscr{A}_{n-i}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right).
\end{aligned}
$$

Notice that, for $i = 1$ or 2, the rank identity (2.3) has been proved. Assume that the statement (2.3) is true for $i-1$ $(i \geq 2)$, i.e.,

$$
\begin{aligned}
&\min_{X_{n-i+1},\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-i+1}^*(\mathscr{X}_1^{n-i})^* & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n-i+1}E_{A_{n-i+1}} & \cdots & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_{n-i+1}^{n-i+1}E_{A_{n-i+1}} & \cdots & \mathscr{A}_{n-i+1}^{n-1}E_{A_n} & \mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right).
\end{aligned}
$$

Substituting (1.3) into above equation yields

$$
\begin{aligned}
&\min_{X_{n-i},X_{n-i+1},\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-i}^*A_{n-i} & A_{n-i}^*(\mathscr{X}_1^{n-i-1})^* & 0 & \cdots & 0\\ A_{n-i+1}^* & 0 & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n-i+1}E_{A_{n-i+1}} & \cdots & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r\left(\begin{bmatrix} A_{n-i}^*(\mathscr{X}_1^{n-i-1})^* & 0 & \cdots & 0\\ 0 & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n-i+1}E_{A_{n-i+1}} & \cdots & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad - r\left(\begin{bmatrix} A_{n-i}^*A_{n-i} & 0 & 0 & \cdots & 0\\ 0 & A_{n-i}^*(\mathscr{X}_1^{n-i-1})^* & 0 & \cdots & 0\\ A_{n-i+1}^* & 0 & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n-1}E_{A_n} & \cdots & A_{n-i+1}^*\mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_{n-i+1}^{n-i+1}E_{A_{n-i+1}} & \cdots & \mathscr{A}_{n-i+1}^{n-1}E_{A_n} & \mathscr{A}_{n-i+1}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_{n-i}^*(\mathscr{X}_1^{n-i-1})^* & A_{n-i}^*\mathscr{A}_{n-i}^{n-i}E_{A_{n-i+1}} & \cdots & A_{n-i}^*\mathscr{A}_{n-i}^{n-1}E_{A_n} & A_{n-i}^*\mathscr{A}_{n-i}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_{n-i}^{n-i}E_{A_{n-i+1}} & \cdots & \mathscr{A}_{n-i}^{n-1}E_{A_n} & \mathscr{A}_{n-i}^{n}(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right).
\end{aligned}
$$

That is to say the statement (2.3) is also true for i.
In particular, we take $i = n-1$, then

$$
\begin{aligned}
&\min_{X_1,X_2,\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{X}_1^n)^* & (\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r\left(\begin{bmatrix} A_1^*(\mathscr{X}_1^0)^* & A_1^*\mathscr{A}_1^1E_{A_2} & \cdots & A_1^*\mathscr{A}_1^{n-1}E_{A_n} & A_1^*\mathscr{A}_1^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&\quad + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_1^1E_{A_2} & \cdots & \mathscr{A}_1^{n-1}E_{A_n} & \mathscr{A}_1^n(\mathscr{A}_1^n)^{(1,2,3)} \end{bmatrix}\right)\\
&= r(A_1) + r(\mathscr{A}_1^n) - r\left(\begin{bmatrix} \mathscr{A}_1^1E_{A_2} & \cdots & \mathscr{A}_1^{n-1}E_{A_n} & \mathscr{A}_1^n \end{bmatrix}\right)
\end{aligned}
$$

and the proof is complete. □

Lemma 2.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,\ldots,n$, and $\mathscr{A}_i^j$, $\mathscr{X}_i^j$, $1 \leq i \leq j \leq n$ be as in (1.1). If $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} \subseteq (A_1A_2\cdots A_n)\{1,2,3\}$, then, for each $(\mathscr{A}_1^n)^{(1,2,3)}$ and $(\mathscr{X}_1^n)^*$, we have*

$$
r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{X}_1^n)^* \end{bmatrix}\right) = r(\mathscr{A}_1^n). \tag{2.4}
$$

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Proof. Because $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} \subseteq (A_1A_2\cdots A_n)\{1,2,3\}$, for each $(\mathscr{X}_1^n)^*$, there exists a $(\mathscr{A}_1^n)^{(1,2,3)'}$ such that $(\mathscr{A}_1^n)^{(1,2,3)'} = (\mathscr{X}_1^n)^*$. Thus

$$r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{X}_1^n)^* \end{bmatrix}\right) = r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{A}_1^n)^{(1,2,3)'} \end{bmatrix}\right).$$

The formula $r(AB) \le \min\{r(A), r(B)\}$ together with the fact that

$$\begin{bmatrix} (\mathscr{A}_1^n)^*\mathscr{A}_1^n & 0 \\ 0 & (\mathscr{A}_1^n)^*\mathscr{A}_1^n \end{bmatrix}\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{A}_1^n)^{(1,2,3)'} \end{bmatrix} = \begin{bmatrix} (\mathscr{A}_1^n)^* \\ (\mathscr{A}_1^n)^* \end{bmatrix},$$

and

$$\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)}((\mathscr{A}_1^n)^\dagger)^* & 0 \\ 0 & (\mathscr{A}_1^n)^{(1,2,3)'}((\mathscr{A}_1^n)^\dagger)^* \end{bmatrix}\begin{bmatrix} (\mathscr{A}_1^n)^* \\ (\mathscr{A}_1^n)^* \end{bmatrix} = \begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{A}_1^n)^{(1,2,3)'} \end{bmatrix},$$

means that

$$r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{A}_1^n)^{(1,2,3)'} \end{bmatrix}\right) = r\left(\begin{bmatrix} (\mathscr{A}_1^n)^* \\ (\mathscr{A}_1^n)^* \end{bmatrix}\right) = r(\mathscr{A}_1^n).$$

Now the statement (2.4) readily follows. $\square$

Based on the above auxiliary conclusions, in the following, we present the equivalent conditions for $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} = (A_1A_2\cdots A_n)\{1,2,3\}$.

THEOREM 2.3. *Let $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$ for $i = 1,2,\ldots,n$, and $\mathscr{A}_i^j$ $(1 \le i \le j \le n)$ be as in (1.1). Then the following statements are equivalent:*

1. $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} \subseteq (A_1A_2\cdots A_n)\{1,2,3\}$;
2. $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} = (A_1A_2\cdots A_n)\{1,2,3\}$;
3. $r\left(\begin{bmatrix} (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}E_{A_n} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2}E_{A_{n-1}} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1E_{A_2} \end{bmatrix}\right) = 0$ *and*

$$\begin{aligned} r(A_1A_2\cdots A_n) &= \min\{r(A_1), r(A_2), \ldots, r(A_n)\} \\ &= \sum_{i=1}^n r(A_i) - r\left(\begin{bmatrix} A_2^* & 0 & \cdots & 0 \\ 0 & A_3^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^* \\ \mathscr{A}_1^1 & \mathscr{A}_1^2 & \cdots & \mathscr{A}_1^{n-1} \end{bmatrix}\right). \end{aligned}$$

Proof. Since (1) and (3) are equivalent, (2)⇒(1) and (2)⇒(3) are obvious, therefore, we only need to prove that (1)⇒(2).

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We remark that $A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} = (A_1A_2\cdots A_n)\{1,2,3\}$ is equivalent to

$$A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} \subseteq (A_1A_2\cdots A_n)\{1,2,3\},$$

and

$$A_n\{1,2,3\}\cdots A_2\{1,2,3\}A_1\{1,2,3\} \supseteq (A_1A_2\cdots A_n)\{1,2,3\}.$$

Here, we only need to prove that the latter inclusion holds under the conditions of part (1) or (3). The second side inclusion relation is equivelent to

$$\max_{(\mathscr{A}_1^n)^{(1,2,3)}} \min_{X_1,X_2,\ldots,X_n} r[(\mathscr{A}_1^n)^{(1,2,3)} - (\mathscr{X}_1^n)^*] = 0. \tag{2.5}$$

If part (1) holds, then $(\mathscr{A}_1^n)^{(1,2,3)}$ and $(\mathscr{X}_1^n)^*$ are a pair of outer inverses of matrix $\mathscr{A}_1^n$. Using Lemma 1.4 and Lemma 2.2, we have

$$\begin{aligned}
&(2.6)\quad r[(\mathscr{A}_1^n)^{(1,2,3)} - (\mathscr{X}_1^n)^*]\\
&= r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} \\ (\mathscr{X}_1^n)^* \end{bmatrix}\right) + r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} & (\mathscr{X}_1^n)^* \end{bmatrix}\right) - r((\mathscr{A}_1^n)^{(1,2,3)}) - r((\mathscr{X}_1^n)^*)\\
&= r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} & (\mathscr{X}_1^n)^* \end{bmatrix}\right) - r((\mathscr{X}_1^n)^*)\\
&= r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} & (\mathscr{X}_1^n)^* \end{bmatrix}\right) - r(\mathscr{A}_1^n).
\end{aligned}$$

Combining (2.6) with (2.1), we have

$$\begin{aligned}
&\max_{(\mathscr{A}_1^n)^{(1,2,3)}} \min_{X_1,X_2,\ldots,X_n} r[(\mathscr{A}_1^n)^{(1,2,3)} - (\mathscr{X}_1^n)^*]\\
&= \max_{(\mathscr{A}_1^n)^{(1,2,3)}} \min_{X_1,X_2,\ldots,X_n} r\left(\begin{bmatrix} (\mathscr{A}_1^n)^{(1,2,3)} & (\mathscr{X}_1^n)^* \end{bmatrix}\right) - r(\mathscr{A}_1^n)\\
&= r(A_1) - r\left(\begin{bmatrix} \mathscr{A}_1^1 E_{A_2} & \cdots & \mathscr{A}_1^{n-1}E_{A_n} & \mathscr{A}_1^n \end{bmatrix}\right).
\end{aligned} \tag{2.7}$$

Note that if $P^*Q = 0$, then $r\left(\begin{bmatrix} P & Q \end{bmatrix}\right) = r(P)+r(Q)$. From part (3), (2.7) reduces to

$$\begin{aligned}
&\max_{(\mathscr{A}_1^n)^{(1,2,3)}} \min_{X_1,X_2,\ldots,X_n} r[(\mathscr{A}_1^n)^{(1,2,3)} - (\mathscr{X}_1^n)^*]\\
&= r(A_1) - r\left(\begin{bmatrix} \mathscr{A}_1^1 E_{A_2} & \cdots & \mathscr{A}_1^{n-1}E_{A_n} \end{bmatrix}\right) - r(\mathscr{A}_1^n)\\
&= \sum_{i=1}^n r(A_i) - r\left(\begin{bmatrix} A_2^* & 0 & \cdots & 0 \\ 0 & A_3^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^* \\ \mathscr{A}_1^1 & \mathscr{A}_1^2 & \cdots & \mathscr{A}_1^{n-1} \end{bmatrix}\right) - r(\mathscr{A}_1^n).
\end{aligned} \tag{2.8}$$

According to part (3), we know that the right-hand side of (2.8) is equal to zero. □

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For the reverse order law for {1, 2, 4}-inverse of multiple matrix products we have a similar result. Here, we present it below without proof.

THEOREM 2.4. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$, and $\mathscr{A}_i^j$, $1 \le i \le j \le n$ be as in (1.1). Then the following statements are equivalent:*

1. $A_n\{1,2,4\}\cdots A_2\{1,2,4\}A_1\{1,2,4\} \subseteq (A_1A_2\cdots A_n)\{1,2,4\}$;
2. $A_n\{1,2,4\}\cdots A_2\{1,2,4\}A_1\{1,2,4\} = (A_1A_2\cdots A_n)\{1,2,4\}$;
3. $r\left(\begin{bmatrix} F_{A_1}\mathscr{A}_2^n(\mathscr{A}_1^n)^* \\ F_{A_2}\mathscr{A}_3^n(\mathscr{A}_1^n)^* \\ \vdots \\ F_{A_{n-1}}\mathscr{A}_n^n(\mathscr{A}_1^n)^* \end{bmatrix}\right) = 0$ *and*

$$\begin{aligned} r(A_1A_2\cdots A_n) &= \min\{r(A_1), r(A_2), \ldots, r(A_n)\} \\ &= \sum_{i=1}^n r(A_i) - r\left(\begin{bmatrix} A_{n-1}^* & 0 & \cdots & 0 & \mathscr{A}_n^n \\ 0 & A_{n-2}^* & \cdots & 0 & \mathscr{A}_{n-1}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_1^* & \mathscr{A}_2^n \end{bmatrix}\right). \end{aligned}$$

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