



## ON THE RANK OF $M \times 2 \times 2$ AND $M \times 3 \times 2$ TENSORS OVER ARBITRARY FIELDS\*

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**Abstract.** In this paper, we provide exact rank computations for  $m \times 2 \times 2$  and  $m \times 3 \times 2$  tensors over arbitrary fields. By analyzing the structural properties of slice matrices, we reduce the tensor rank problem to computations involving matrix ranks and diagonalizations. This yields a complete and explicit rank classification for these families of tensors and provides a clearer structural understanding on rank of small tensors.

**Key words.** Tensor decomposition, Tensor rank, Arbitrary field.

**AMS subject classification.** 15A03,15A69.

**1. Introduction.** A classical problem in algebraic computational complexity is to determine the minimal number of nonscalar multiplications required to compute a family of bilinear forms  $\sum_{i,j} \alpha_{ijk} x_i y_j$ ,  $k = 1, 2, \dots, p$  in noncommuting variables  $x_1, \dots, x_m, y_1, \dots, y_n$ . This number can be variously regarded as the rank of the 3-tensor  $(\alpha_{ijk})$  or as the minimum number of rank-1 matrices whose linear span contains the  $m \times n$  matrices  $A_k = (\alpha_{ijk})$ ,  $k = 1, 2, \dots, p$  [2, 7]. There is no doubt that this is a difficult subject. Actually, Hillar and Lim [10] have proved that determining the rank even for a 3-tensor is NP-hard in numerical linear algebra. Nonetheless, this problem has attracted much attention and has found applications in various fields, including numerical analysis [23], image analysis [12], engineering [13], and so on [9].

For the rank of 3-tensors, only a few specific cases have been solved. As Martin points out that even the simplest case (the rank of a  $2 \times 2 \times 2$  tensor) is highly nontrivial and may provide insight into the larger problem [15]. Recall that the rank of  $2 \times 2 \times 2$  tensors has been studied by some scholars in the literature, and they have accumulated meaningful results. For example, we show some recent work: Ten Berge [21] and Martin [15] consider the rank of  $2 \times 2 \times 2$  tensors over the real number field. Later, Bremner et al. [6] apply Ten Berge's proof to the complex number field with minor changes. Stavrou used computer algebra to argue the tensor rank and canonical forms of small tensors (low order or low dimension) over certain prime fields [5, 14, 17]. For more details and source about  $2 \times 2 \times 2$  tensors, we may refer to [16].

For  $m \times n \times 2$  tensors, Ja'Ja' explored the tensor rank in [11] by Kronecker's canonical form. However, his result is subject to infinite fields with sufficiently large cardinality. In contrast, our results are applicable to arbitrary fields. In this paper, we provide a method for computing the rank of  $m \times 2 \times 2$  and  $m \times 3 \times 2$  tensors over arbitrary fields by the intrinsic properties of two slice matrices.

For a general 3-tensor, most scholars focus on giving a good bound. For example, for  $m \times n \times 2$  tensors, Ja'Ja' and Kruskal proposed an upper bound when the base field satisfies certain conditions [11, 13], later Song et al. supplemented their results and proved their upper bounds are sharp [22]. In this paper, we

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compute the rank of a given  $m \times n \times 2$  tensor over an arbitrary field. So far, we have solved  $m \times 2 \times 2$  and  $m \times 3 \times 2$  tensors in this paper. If the readers are interested in the bounds on the rank of tensors, we recommend consulting [1, 3, 4, 8, 20, 18].

The outline of this paper is organized as follows. The notions and known results are listed in Section 2, the rank of an  $m \times 2 \times 2$  tensor is shown in Section 3, and the rank of an  $m \times 3 \times 2$  tensor is discussed in Section 4.

**2. Preliminaries.** We first recall some basic facts and set terminology. Let  $\mathbb{F}$  be a field.  $|\mathbb{F}|$  denotes the cardinality of  $\mathbb{F}$ . We denote by

$$\mathbb{F}^n = \{(a_1, \dots, a_n)^T \mid a_i \in \mathbb{F}, i = 1, 2, \dots, n\},$$

the  $n$ -dimensional vector space over  $\mathbb{F}$ , and denote by  $\mathbb{F}^{m \times n}$  the vector space of  $m \times n$  matrices. We consider an  $m \times n \times p$  tensor  $\mathcal{T}$  with entries in an arbitrary field  $\mathbb{F}$  of scalars:

$$\mathcal{T} = (a_{ijk}), a_{ijk} \in \mathbb{F}, 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p.$$

For a 3-order tensor, slices are two-dimensional submatrices obtained by fixing one index. All slices occurring in this paper are frontal slices, which are given by fixing the third index of a 3-tensor. For an  $m \times n \times p$  tensor  $\mathcal{T} = [X_1|X_2|\dots|X_p]$ , we define its unfolding as the  $m \times np$  matrix  $A = (X_1, X_2, \dots, X_p)$ , and the rank of this matrix is denoted by  $\text{rank}(A)$  or  $\text{rank}(X_1, X_2, \dots, X_p)$ . The rank of  $\mathcal{T} = [X_1|X_2|\dots|X_p]$  is the smallest  $r$  such that every slice  $X_i$  can be written as a linear combination of the same  $r$  rank-1 matrices, denoted by  $r(\mathcal{T})$ . We define  $R_{\mathbb{F}}(m, n, 2)$  to be the maximum rank attainable by  $m \times n \times 2$  tensors over a field  $\mathbb{F}$ .  $\lfloor x \rfloor$  is the largest integer equal to or below  $x$ , and  $\lceil x \rceil$  is the smallest integer equal to or above  $x$ . We denote by  $I_s$  the identity matrix with the size  $s \times s$ . Next we introduce some rank-preserving transformations based on slices.

LEMMA 1 ([22, Lemma 3.2]). *Suppose  $\mathcal{T} = [X_1|X_2|\dots|X_p]$  is an  $m \times n \times p$  tensor over  $\mathbb{F}$ , then*

- (1)  $r [X_1|X_2|\dots|X_p] = r [X_1^T|X_2^T|\dots|X_p^T]$ ;
- (2)  $r [X_1|\dots|X_k|\dots|X_p] = r [X_1|\dots|\alpha X_k|\dots|X_p], \alpha \in \mathbb{F} \setminus \{0\}$ ;
- (3)  $r [X_1|\dots|X_k|\dots|X_l|\dots|X_p] = r [X_1|\dots|X_l|\dots|X_k|\dots|X_p]$ ;
- (4)  $r [X_1|\dots|X_k|\dots|X_l|\dots|X_p] = r [X_1|\dots|X_k|\dots|X_l + \alpha X_k|\dots|X_p], \alpha \in \mathbb{F}$  and  $k \neq l$ ;
- (5)  $r [O|X_2|\dots|X_p] = r [X_2|\dots|X_p]$ ;
- (6)  $r [X_1|X_2|\dots|X_p] = r [PX_1Q|PX_2Q|\dots|PX_pQ]$ , where  $P, Q$  are invertible matrices of order- $m$  and order- $n$ , respectively.

The following lemma offers the maximal rank of  $m \times n \times 2$  tensors over an arbitrary field.

LEMMA 2 ([22, Theorem 1.1 and 1.2]).

- (1) *Suppose  $\mathbb{F}$  is an arbitrary field with  $|\mathbb{F}| > 2$ , and  $m \geq n \geq 2$ , then*

$$R_{\mathbb{F}}(m, n, 2) = \begin{cases} n + \lfloor m/2 \rfloor, & \text{if } n \leq m \leq 2n - 1. \\ 2n, & \text{if } m \geq 2n. \end{cases}$$

- (2) *Let  $\mathbb{F}_2$  be the field with two elements, and  $m \geq n \geq 2$ , then*

$$R_{\mathbb{F}_2}(m, n, 2) = \begin{cases} n + \lfloor m/2 \rfloor, & \text{if } n \leq m \leq 2n - 2. \\ 2n - 1, & \text{if } m = 2n - 1. \\ 2n, & \text{if } m \geq 2n. \end{cases}$$

The next lemma shows the relationship between the rank of an  $m \times n \times 2$  tensor and the rank of its unfolding.

LEMMA 3 ([22, Lemma 3.7 and Theorem 3.8]). *Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times n \times 2$  tensor over  $\mathbb{F}$ , and  $m \geq n \geq 2$ , then*

- (1)  $\text{rank}(X_1, X_2) \leq r(\mathcal{T}) \leq \text{rank}(X_1) + \text{rank}(X_2)$ ;
- (2) if  $\text{rank}(X_1, X_2) = \text{rank}(X_1) + \text{rank}(X_2)$ , then  $r(\mathcal{T}) = \text{rank}(X_1) + \text{rank}(X_2)$ ;
- (3) if  $\text{rank}(X_1, X_2) = 2n - 1$ , then  $r(\mathcal{T}) = 2n - 1$ ;
- (4) if  $\text{rank}(X_1, X_2) = 2n$ , then  $r(\mathcal{T}) = 2n$ .

The following result is well known and will be used frequently throughout this article.

LEMMA 4 ([11, Lemma 3.1]). *Suppose  $\mathcal{T} = [I_n|X]$  is an  $n \times n \times 2$  tensor over  $\mathbb{F}$ , then  $r(\mathcal{T}) = n$  if and only if  $X$  is diagonalizable.*

**3.  $m \times 2 \times 2$  tensors.** Since an  $m \times n \times 2$  tensor just has two frontal slices, we can assume without loss of generality that  $\text{rank}(X_1) \geq \text{rank}(X_2)$ . If  $\text{rank}(X_1) < \text{rank}(X_2)$ , then swap and redefine  $X_1$  and  $X_2$ ; this will not change the rank by Lemma 1.

**3.1.  $2 \times 2 \times 2$  tensors.** It is well known that the rank of  $2 \times 2 \times 2$  tensor in  $\mathbb{R}$  and  $\mathbb{C}$  has been solved; see [6, Section 3], [15, Section 3], and [16, 19]. In what follows we will fully characterize the rank of  $2 \times 2 \times 2$  tensors and prove this result over any field.

THEOREM 5. *Suppose  $\mathcal{T} = [X_1|X_2]$  is a nonzero  $2 \times 2 \times 2$  tensor over  $\mathbb{F}$ , and  $\text{rank}(X_1) \geq \text{rank}(X_2)$ , then*

$$r(\mathcal{T}) = \begin{cases} 1, & \text{if } \text{rank}(X_1) = 1, X_2 = kX_1, k \in \mathbb{F}. \\ 2, & \text{if } \text{rank}(X_1) = 1, X_2 \neq kX_1, \forall k \in \mathbb{F}. \\ 2, & \text{if } \text{rank}(X_1) = 2, X_1^{-1}X_2 \text{ is diagonalizable.} \\ 3, & \text{if } \text{rank}(X_1) = 2, X_1^{-1}X_2 \text{ is not diagonalizable.} \end{cases}$$

*Proof.* When  $\text{rank}(X_1) = 1$ , it follows from Lemma 3 and  $\text{rank}(X_1) \geq \text{rank}(X_2)$  that

$$1 \leq r(\mathcal{T}) \leq \text{rank}(X_1) + \text{rank}(X_2) \leq 2.$$

If  $X_2 = kX_1, k \in \mathbb{F}$ , then it is easy to see that  $r(\mathcal{T}) = 1$ . If  $X_2 \neq kX_1, \forall k \in \mathbb{F}$ , then  $r(\mathcal{T}) \neq 1$ . Otherwise there is a  $k \in \mathbb{F}$  such that  $X_2 = kX_1$ , a contradiction. So  $r(\mathcal{T}) = 2$ .

When  $\text{rank}(X_1) = 2$ , it follows from Lemma 2 that

$$2 \leq r(\mathcal{T}) \leq R_{\mathbb{F}}(2, 2, 2) = 3,$$

then from Lemma 4 and (6) of Lemma 1 we obtain the conclusion. □

**3.2.  $3 \times 2 \times 2$  tensors.** Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 2 \times 2$  tensor over  $\mathbb{F}$ , then it follows from Lemma 2 and Lemma 3 that

$$(1) \quad \text{rank}(X_1, X_2) \leq R_{\mathbb{F}}(3, 2, 2) = 3.$$

If  $\text{rank}(X_1, X_2) \leq 2$ , then there exists an invertible matrix  $P$  such that

$$P \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix},$$

where  $Y_1, Y_2$  are  $2 \times 2$  matrices, which means the  $3 \times 2 \times 2$  tensor  $\mathcal{T}$  degenerates to a  $2 \times 2 \times 2$  tensor (refer to Theorem 5). So it suffices to prove the case of  $\text{rank}(X_1, X_2) = 3$ , which follows immediately from (1).

**THEOREM 6.** *Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 2 \times 2$  tensor over  $\mathbb{F}$  and*

$$\text{rank}(X_1, X_2) = 3,$$

*then  $r(\mathcal{T}) = 3$ .*

**3.3.  $m \times 2 \times 2$  tensors.** From the same construction in Section 3.2, it suffices to consider the case of  $\text{rank}(X_1, X_2) = 4$ .

**THEOREM 7.** *Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times 2 \times 2$  tensor over  $\mathbb{F}$ , and  $\text{rank}(X_1, X_2) = 4$ , then  $r(\mathcal{T}) = 4$ .*

*Proof.* According to Lemma 2 and Lemma 3,

$$4 = \text{rank}(X_1, X_2) \leq r(\mathcal{T}) \leq R_{\mathbb{F}}(m, 2, 2) = 4,$$

so  $r(\mathcal{T}) = 4$ . □

#### 4. $m \times 3 \times 2$ tensors.

**4.1.  $3 \times 3 \times 2$  tensors.** Recall that the rank of  $m \times 3 \times 2$  tensors may be different over different fields [11, 13]. So we need to discuss the rank in two cases: one is in  $\mathbb{F}_2$ , the other is in any field except  $\mathbb{F}_2$ . First, we consider the case of  $\mathbb{F}_2$ . Before giving the theorem, let us introduce two lemmas over  $\mathbb{F}_2$ , which will be useful in the following proofs.

**LEMMA 8.** *Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times n \times 2$  tensor over  $\mathbb{F}_2$ , then*

$$r(\mathcal{T}) \geq \frac{\text{rank}(X_1) + \text{rank}(X_2) + \text{rank}(X_1 + X_2)}{2}.$$

*Proof.* It follows from (3) and (4) of Lemma 1 that

$$(2) \quad r(\mathcal{T}) = r[X_1|X_2] = r[X_2|X_1] = r[X_1|X_1 + X_2].$$

Assume that  $r(\mathcal{T}) = r$ ,  $\text{rank}(X_1) = r_1$ ,  $\text{rank}(X_2) = r_2$ ,  $\text{rank}(X_1 + X_2) = r_3$ , and according to (2), we can assume  $r_3 \leq r_2 \leq r_1$ ; otherwise, we can adjust the positions of  $X_1, X_2, X_1 + X_2$  and rename them to ensure the inequality holds. Moreover, we can assume  $r_1 \leq r$  by the definition of rank. It follows from  $r(\mathcal{T}) = r$  that there exists an  $m \times r$  matrix  $G$ , an  $r \times n$  matrix  $F$ , and  $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r \in \mathbb{F}_2$  such that

$$X_1 = G \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{pmatrix} F \quad \text{and} \quad X_2 = G \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_r \end{pmatrix} F.$$

Since  $r_1 = \text{rank}(X_1) \leq r(\mathcal{T}) = r$ , so there are at least  $r_1$  1's in  $\lambda_1, \dots, \lambda_r$ . Similarly, there are at least  $r_2$  1's in  $\mu_1, \dots, \mu_r$ . Due to  $r \leq r_1 + r_2$ , there exist at least  $r_1 + r_2 - r$  0's in  $\lambda_1 + \mu_1, \dots, \lambda_r + \mu_r$  over  $\mathbb{F}_2$ , that is, at most  $2r - (r_1 + r_2)$  1's. It follows from

$$X_1 + X_2 = G \begin{pmatrix} \lambda_1 + \mu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 + \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r + \mu_r \end{pmatrix} F$$

that  $r_3 \leq 2r - (r_1 + r_2)$ . Thus  $2r \geq r_1 + r_2 + r_3$ , concluding the proof. □

LEMMA 9. Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $n \times n \times 2$  tensor over  $\mathbb{F}_2$ , and  $n \geq 2$ . If

$$\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = n,$$

then

$$r(\mathcal{T}) = n + \left\lceil \frac{n}{2} \right\rceil.$$

*Proof.* From Lemmas 2 and 8 it follows that

$$\frac{3n}{2} \leq r(\mathcal{T}) \leq R_{\mathbb{F}_2}(n, n, 2) = n + \left\lceil \frac{n}{2} \right\rceil,$$

concluding the proof. □

Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 3 \times 2$  tensor over  $\mathbb{F}$ , then  $\text{rank}(X_1, X_2) \leq 3$ . If  $\text{rank}(X_1, X_2) \leq 2$ , then  $\mathcal{T}$  can degenerate to a  $3 \times 2 \times 2$  tensor as in the case of Subsections 3.2 and 3.3. So it suffices to consider the case of  $\text{rank}(X_1, X_2) = 3$ . According to Lemma 1, we can assume

$$X_1 = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix};$$

and assume  $s = \text{rank}(X_1) \geq \text{rank}(X_2) \geq \text{rank}(X_1 + X_2)$ . (If  $\text{rank}(X_2) < \text{rank}(X_1 + X_2)$ , then swap and redefine  $X_2$  and  $X_1 + X_2$ , since we are working over  $\mathbb{F}_2$ , the rank remains unchanged.) It follows that

$$(3) \quad 3 = \text{rank}(X_1, X_2) \leq r(\mathcal{T}) \leq \text{rank}(X_2) + \text{rank}(X_1 + X_2) \leq 6.$$

In the following theorem, we discuss the rank of  $3 \times 3 \times 2$  tensors over  $\mathbb{F}_2$  by the above discussion of  $\text{rank}(X_1), \text{rank}(X_2)$  and  $\text{rank}(X_1 + X_2)$ .

THEOREM 10. Let

$$X_1 = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

be  $3 \times 3$  matrices over  $\mathbb{F}_2$ , where  $s = \text{rank}(X_1) \geq \text{rank}(X_2) \geq \text{rank}(X_1 + X_2)$ ,  $\text{rank}(X_1, X_2) = 3$ . Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 3 \times 2$  tensor over  $\mathbb{F}_2$ , then

$$r(\mathcal{T}) = \begin{cases} 3, & \text{if } \text{rank}(X_2) + \text{rank}(X_1 + X_2) = 3. \\ 3, & \text{if } \text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, x_{33} \neq 0. \\ 3, & \text{if } \text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, (x_{13}, x_{23}, x_{33}) = 0. \\ 4, & \text{if } \text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, x_{33} = 0, (x_{13}, x_{23}) \neq 0. \\ 4, & \text{if } \text{rank}(X_1) = 3, 4 \leq \text{rank}(X_2) + \text{rank}(X_1 + X_2) \leq 5. \\ 5, & \text{if } \text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 3. \end{cases}$$

*Proof.* (1) When  $\text{rank}(X_2) + \text{rank}(X_1 + X_2) = 3$ , it follows from (3) that  $r(\mathcal{T}) = 3$ .

(2) When  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, x_{33} \neq 0$ , without loss of generality we assume

$$X_1 = \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), X_2 = \left( \begin{array}{ccc|c} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), X_1 + X_2 = \left( \begin{array}{ccc|c} x_{11} + 1 & x_{12} & 0 & 0 \\ x_{21} & x_{22} + 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Since  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2$ , then

$$\text{rank} \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) = \text{rank} \left( \begin{array}{cc} x_{11} + 1 & x_{12} \\ x_{21} & x_{22} + 1 \end{array} \right) = 1.$$

It follows that

$$\det \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) = \det \left( \begin{array}{cc} x_{11} + 1 & x_{12} \\ x_{21} & x_{22} + 1 \end{array} \right) = 0.$$

As a result, both 0 and 1 are eigenvalues of  $\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)$ , implying that  $\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)$  is diagonalizable. According to Lemma 4, we have

$$r \left[ \begin{array}{cc|cc} 1 & 0 & x_{11} & x_{12} \\ 0 & 1 & x_{21} & x_{22} \end{array} \right] = 2.$$

So

$$r(\mathcal{T}) = r \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & x_{11} & x_{12} & 0 \\ 0 & 1 & 0 & x_{21} & x_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \leq r \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & x_{11} & x_{12} & 0 \\ 0 & 1 & 0 & x_{21} & x_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] + r \left[ \begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] = 3.$$

Since  $r(\mathcal{T}) \geq \text{rank}(X_1, X_2) = 3$ , thus  $r(\mathcal{T}) = 3$ .

(3) When  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, (x_{13}, x_{23}, x_{33}) = 0$ , without loss of generality we may assume

$$X_1 = \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ and } X_2 = \left( \begin{array}{ccc|c} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & x_{32} & 0 & 0 \end{array} \right).$$

Then  $\mathcal{T}$  degenerates into a  $3 \times 2 \times 2$  tensor, since  $r(\mathcal{T}) \leq R_{\mathbb{F}}(3, 2, 2) = 3$ , then  $r(\mathcal{T}) = 3$ .

(4) When  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 2, x_{33} = 0, (x_{13}, x_{23}) \neq 0$ , it turns out that  $r(\mathcal{T}) = 4$ . It is easy to see  $r(\mathcal{T}) \leq \text{rank}(X_1) + \text{rank}(X_2) = 4$ . If  $r(\mathcal{T}) \leq 3$ , then there are  $3 \times 3$  matrices

$$P = (p_{ij}) = (\alpha_1 \quad \alpha_2 \quad \alpha_3), \quad Q = (q_{ij}) = (\beta_1 \quad \beta_2 \quad \beta_3)^T$$

over  $\mathbb{F}_2$ , where  $\alpha_i, \beta_i \in \mathbb{F}_2^3, i = 1, 2, 3$ , and  $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 \in \mathbb{F}_2$ , such that

$$X_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) = (\alpha_1 \quad \alpha_2 \quad \alpha_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \beta_1^T \\ \beta_2^T \\ \beta_3^T \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Q,$$

$$X_2 = \left( \begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{array} \right) = (\alpha_1 \quad \alpha_2 \quad \alpha_3) \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \begin{pmatrix} \beta_1^T \\ \beta_2^T \\ \beta_3^T \end{pmatrix} = P \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} Q.$$

Since  $\text{rank}(X_1, X_2) = 3$ , and the column vectors of  $(X_1, X_2)$  can be represented linearly by  $\alpha_1, \alpha_2, \alpha_3$ , then  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent, that is, matrix  $P$  is invertible. Similarly,  $\beta_1, \beta_2, \beta_3$  are linearly independent, which means matrix  $Q$  is invertible.

Since  $\text{rank}(X_1) = 2$ , without loss of generality, we assume  $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$ , then  $\mu_3 \neq 0$ . (Otherwise  $r(\mathcal{T}) = 2$ , a contradiction) So  $\mu_3 = 1$ .

Since the third row and the third column of  $X_1$  are all 0, it follows that

$$\begin{pmatrix} p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} p_{31} & p_{32} & 0 \end{pmatrix} Q = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$

$$P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_{13} \\ q_{23} \\ q_{33} \end{pmatrix} = P \begin{pmatrix} q_{13} \\ q_{23} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since  $P, Q$  are invertible, then  $p_{31} = p_{32} = 0, q_{13} = q_{23} = 0$ . Moreover, it is observed from the element in the third row and the third column of  $X_2$  that

$$0 = \begin{pmatrix} 0 & 0 & p_{33} \end{pmatrix} \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q_{33} \end{pmatrix} = p_{33}q_{33}.$$

So  $p_{33} = 0$  or  $q_{33} = 0$ , which contradicts the assumption that  $P, Q$  are invertible.

(5) When  $\text{rank}(X_1) = 3, \text{rank}(X_2) + \text{rank}(X_1 + X_2) = 4$ , it follows from Lemma 8 that  $r(\mathcal{T}) \geq \frac{7}{2}$ , and since

$$r(\mathcal{T}) \leq \text{rank}(X_2) + \text{rank}(X_1 + X_2) = 4,$$

so  $r(\mathcal{T}) = 4$ .

When  $\text{rank}(X_1) = 3, \text{rank}(X_2) + \text{rank}(X_1 + X_2) = 5$ , it turns out that

$$\text{rank}(X_1) = \text{rank}(X_2) = 3, \text{rank}(X_1 + X_2) = 2.$$

It follows from Lemma 8 that

$$4 \leq r(\mathcal{T}) \leq R_{\mathbb{F}_2}(3, 3, 2) = 5.$$

Since  $r(\mathcal{T}) \neq 3$ , it follows from Lemma 4 that  $X_2$  is not diagonalizable. According to the rational canonical form theory [11, section 3, p. 453], we may assume that

$$X_1 = \begin{pmatrix} 1 & | & 0 & 0 \\ 0 & | & 1 & 0 \\ 0 & | & 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} x_{11} & | & 0 & 0 \\ 0 & | & x_{22} & x_{23} \\ 0 & | & x_{32} & x_{33} \end{pmatrix},$$

or

$$X_1 = \begin{pmatrix} 1 & | & 0 & 0 \\ 0 & | & 1 & 0 \\ 0 & | & 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & | & 0 & x_{13} \\ 1 & | & 0 & x_{23} \\ 0 & | & 1 & x_{33} \end{pmatrix}.$$

Moreover, according to  $\text{rank}(X_2) = 3, X_2 \in \mathbb{F}_2^{3 \times 3}$ , we assume  $X_2$  is one of the following two forms:

$$(a) : X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}; \quad (b) : X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x_{23} \\ 0 & 1 & x_{33} \end{pmatrix}.$$

In case (a),

$$r(\mathcal{T}) = r[X_1|X_2] \leq r \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] + r \left[ \begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{22} & x_{23} \\ 0 & 0 & 1 & 0 & x_{32} & x_{33} \end{array} \right] \leq 1 + R_{\mathbb{F}_2}(2, 2, 2) = 4.$$

Thus,  $r(\mathcal{T}) = 4$ .

In case (b),

$$X_1 + X_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & x_{23} \\ 0 & 1 & x_{33} + 1 \end{pmatrix}.$$

Since  $\text{rank}(X_1 + X_2) = 2$ , then

$$\begin{pmatrix} 0 & 1 & x_{33} + 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & x_{23} \end{pmatrix},$$

and so  $x_{23} = x_{33}$ .

Let

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & x_{23} + 1 \\ 0 & 1 & x_{23} + 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 3},$$

be a rank-1 matrix, then

$$C + X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since there exists an invertible matrix

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

over  $\mathbb{F}_2$ , such that

$$P^{-1}(C + X_2)P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so  $C + X_2$  is diagonalizable. It follows from Lemma 4 that  $r[X_1|C + X_2] = 3$ . So

$$r(\mathcal{T}) = r[X_1|X_2] \leq r[X_1|X_2 + C] + r[O|C] = 3 + 1 = 4.$$

Thus,  $r(\mathcal{T}) = 4$ .

(6) When  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 3$ , then it follows from Lemma 9 that  $r(\mathcal{T}) = 5$ . □

According to Theorem 10, the following result is trivial.

**COROLLARY 11.** *Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 3 \times 2$  tensor over  $\mathbb{F}_2$ , then  $r(\mathcal{T}) = 5$  if and only if*

$$\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(X_1 + X_2) = 3.$$

In what follows we still assume that  $\text{rank}(X_1) \geq \text{rank}(X_2)$  and  $\text{rank}(X_1, X_2) = 3$ , so that

$$3 = \text{rank}(X_1, X_2) \leq r(\mathcal{T}).$$

Next we will discuss the rank of  $3 \times 3 \times 2$  tensors over  $\mathbb{F}$  with  $|\mathbb{F}| > 2$  by the relation of  $\text{rank}(X_1)$  and  $\text{rank}(X_2)$ .

THEOREM 12. *Let*

$$X_1 = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

be  $3 \times 3$  matrices over  $\mathbb{F}$ , where  $|\mathbb{F}| > 2$ , and  $s = \text{rank}(X_1) \geq \text{rank}(X_2)$ ,  $\text{rank}(X_1, X_2) = 3$ . Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $3 \times 3 \times 2$  tensor over  $\mathbb{F}$ , then

- (1) If  $\text{rank}(X_1) + \text{rank}(X_2) = 3$ , then  $r(\mathcal{T}) = 3$ .
- (2) If  $\text{rank}(X_1) = \text{rank}(X_2) = 2$ , and  $x_{33} = 0$ , then

$$r(\mathcal{T}) = \begin{cases} 3, & \text{if } (x_{13}, x_{23}) = 0. \\ 4, & \text{if } (x_{13}, x_{23}) \neq 0. \end{cases}$$

- (3) If  $\text{rank}(X_1) = \text{rank}(X_2) = 2$ , and  $x_{33} \neq 0$ , then there must exist a  $k \in \mathbb{F}$  such that  $(kX_1 + X_2)$  is invertible, and then

$$r(\mathcal{T}) = \begin{cases} 3, & \text{if } (kX_1 + X_2)^{-1}X_1 \text{ is diagonalizable.} \\ 4, & \text{otherwise.} \end{cases}$$

- (4) If  $\text{rank}(X_1) = 3$ , then

$$r(\mathcal{T}) = \begin{cases} 3, & \text{if } X_1^{-1}X_2 \text{ is diagonalizable.} \\ 4, & \text{otherwise.} \end{cases}$$

*Proof.* Since

$$3 = \text{rank}(X_1, X_2) \leq r(\mathcal{T}) \leq R_{\mathbb{F}}(3, 3, 2) = 4,$$

so it suffices to discuss whether the rank is 3 or 4.

- (1) If  $\text{rank}(X_1) + \text{rank}(X_2) = 3$ , then  $r(\mathcal{T}) \leq \text{rank}(X_1) + \text{rank}(X_2) \leq 3$ , and thus  $r(\mathcal{T}) = 3$ .

(2) If  $\text{rank}(X_1) = \text{rank}(X_2) = 2$ , and  $(x_{13}, x_{23}, x_{33}) = 0$ , then  $\mathcal{T}$  can degenerate to a  $3 \times 2 \times 2$  tensor. Since  $R_{\mathbb{F}}(3, 2, 2) = 3$ , so  $r(\mathcal{T}) \leq 3$ , and thus  $r(\mathcal{T}) = 3$ .

If  $\text{rank}(X_1) = \text{rank}(X_2) = 2$ ,  $x_{33} = 0$ , and  $(x_{13}, x_{23}) \neq 0$ , the argument is analogous to that in (4) of Theorem 10, so the proof is omitted here.

- (3) If  $\text{rank}(X_1) = \text{rank}(X_2) = 2$ , and  $x_{33} \neq 0$ , without the loss of generality we assume

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since  $|\mathbb{F}| > 2$ , then there is a  $k \in \mathbb{F}$ , such that  $\det(kX_1 + X_2) \neq 0$ , that is,  $(kX_1 + X_2)$  is invertible. Due to Lemma 1,  $r[X_1|kX_1 + X_2] = r[X_1|X_2]$ . Finally, the conclusion follows according to Lemma 4.

- (4) This part is straightforward by Lemma 4. □

**4.2.  $4 \times 3 \times 2$  tensors.** Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $4 \times 3 \times 2$  tensor over  $\mathbb{F}$ , then  $\text{rank}(X_1, X_2) \leq 4$ . If  $\text{rank}(X_1, X_2) \leq 3$ , then the  $4 \times 3 \times 2$  tensor  $\mathcal{T}$  can degenerate to a  $3 \times 3 \times 2$  tensor in the same way as shown before, and then we can turn to Section 4.1. So it suffices to consider the case of  $\text{rank}(X_1, X_2) = 4$ . Since  $R_{\mathbb{F}}(4, 3, 2) = 5$ , so  $4 \leq r(\mathcal{T}) \leq 5$ , our task now is to discuss whether the rank is 4 or 5. In the following

theorem, we propose a necessary and sufficient condition for the rank of  $m \times n \times 2$  tensor to be  $m$  under the condition of  $\text{rank}(X_1, X_2) = m$ .

THEOREM 13. *Let*

$$X_1 = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} B \\ C \end{pmatrix},$$

be  $m \times n$  matrices over  $\mathbb{F}$ , where  $B \in \mathbb{F}^{n \times n}$ ,  $C \in \mathbb{F}^{(m-n) \times n}$ ,  $m > n \geq 2$ . Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times n \times 2$  tensor over  $\mathbb{F}$ . If  $\text{rank}(X_1, X_2) = m$ , then  $r(\mathcal{T}) = m$  if and only if there is an  $n \times (m-n)$  matrix  $K$  such that  $B + KC$  is diagonalizable.

*Proof.* Sufficiency. Since  $\text{rank}(X_1, X_2) = m$ , then  $\text{rank}(C) = m - n$ . Since  $B + KC$  is diagonalizable, then we have  $r \left[ \begin{array}{c|c} I_n & B + KC \end{array} \right] = n$  by Lemma 4. So

$$r [X_1|X_2] = r \left[ \begin{array}{c|c} I_n & B \\ \hline 0 & C \end{array} \right] = r \left[ \begin{array}{c|c} I_n & B + KC \\ \hline 0 & C \end{array} \right] \leq r \left[ \begin{array}{c|c} I_n & B + KC \end{array} \right] + r [0|C] = n + m - n = m.$$

Since  $r [X_1|X_2] \geq \text{rank}(X_1, X_2) = m$ , we conclude that  $r(\mathcal{T}) = m$ .

Necessity. Suppose  $r [X_1|X_2] = m$ , then there are  $\gamma_i \in \mathbb{F}^m$ ,  $\beta_i \in \mathbb{F}^n$  and  $\lambda_i, \mu_i \in \mathbb{F}$ ,  $i = 1, 2, \dots, m$  such that

$$X_1 = \sum_{i=1}^m \lambda_i \gamma_i \beta_i^T \quad \text{and} \quad X_2 = \sum_{i=1}^m \mu_i \gamma_i \beta_i^T.$$

Since  $\text{rank}(X_1) = n$ , then

$$\text{rank} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \geq n,$$

which means that there are at least  $n$  of  $\lambda_1, \dots, \lambda_m$  that are nonzero. Assume that  $\lambda_1, \dots, \lambda_s$  are nonzero,  $\lambda_{s+1} = \cdots = \lambda_m = 0$ , where  $n \leq s \leq m$ .

Let

$$\begin{aligned} \alpha_i &= \lambda_i \gamma_i, \quad i = 1, \dots, s; \alpha_i = \gamma_i, \quad i = s + 1, \dots, m. \\ \kappa_i &= \lambda_i^{-1} \mu_i, \quad i = 1, \dots, s; \kappa_i = \mu_i, \quad i = s + 1, \dots, m. \end{aligned}$$

Then

$$(4) \quad \begin{aligned} X_1 &= \sum_{i=1}^s \alpha_i \beta_i^T = (\alpha_1 \quad \cdots \quad \alpha_m) \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_{s-n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_m^T \end{pmatrix}, \\ X_2 &= \sum_{i=1}^m \kappa_i \alpha_i \beta_i^T = (\alpha_1 \quad \cdots \quad \alpha_m) \begin{pmatrix} \kappa_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_m \end{pmatrix} \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_m^T \end{pmatrix}. \end{aligned}$$

Since  $\text{rank}(X_1, X_2) = m$ , and the column vectors of  $(X_1, X_2)$  can be represented linearly by  $\alpha_1, \dots, \alpha_m$ , so  $\alpha_1, \dots, \alpha_m$  are linearly independent, that is,  $(\alpha_1 \quad \cdots \quad \alpha_m)$  is invertible.

Let

$$P = (\alpha_1 \quad \cdots \quad \alpha_m) = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix},$$

where  $P_{11} \in \mathbb{F}^{n \times n}$ ,  $P_{22} \in \mathbb{F}^{(s-n) \times (s-n)}$ ,  $P_{33} \in \mathbb{F}^{(m-s) \times (m-s)}$ . Let also

$$\Lambda_1 = \text{diag}(\kappa_1, \dots, \kappa_n), \quad \Lambda_2 = \text{diag}(\kappa_{n+1}, \dots, \kappa_s), \quad \Lambda_3 = \text{diag}(\kappa_{s+1}, \dots, \kappa_m),$$

and let

$$Q = \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_m^T \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} \beta_1^T \\ \vdots \\ \beta_n^T \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \beta_{n+1}^T \\ \vdots \\ \beta_s^T \end{pmatrix}, \quad Q_3 = \begin{pmatrix} \beta_{s+1}^T \\ \vdots \\ \beta_m^T \end{pmatrix}.$$

If some submatrices of  $P$  have size 0, then it should be considered absent, along with the corresponding submatrices of  $Q$ . According to (4), we have

$$\begin{pmatrix} I_n \\ 0 \end{pmatrix} = X_1 = P \begin{pmatrix} I_n & & \\ & I_{s-n} & \\ & & 0 \end{pmatrix} Q = \begin{pmatrix} (P_{11} \quad P_{12}) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \\ (P_{21} \quad P_{22}) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \\ (P_{31} \quad P_{32}) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \end{pmatrix}.$$

So  $I_n = (P_{11} \quad P_{12}) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ , thus  $\text{rank} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = n$ . It follows from

$$0 = \begin{pmatrix} P_{21} & P_{22} \\ P_{31} & P_{32} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},$$

that

$$\text{rank} \begin{pmatrix} P_{21} & P_{22} \\ P_{31} & P_{32} \end{pmatrix} \leq s - n.$$

If this inequality is strict, then

$$\begin{aligned} \text{rank} \begin{pmatrix} P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} &\leq \text{rank} \begin{pmatrix} P_{21} & P_{22} \\ P_{31} & P_{32} \end{pmatrix} + \text{rank} \begin{pmatrix} P_{23} \\ P_{33} \end{pmatrix} \\ &< s - n + m - s = m - n. \end{aligned}$$

This contradicts the condition that  $P$  is invertible. So

$$\text{rank} \begin{pmatrix} P_{21} & P_{22} \\ P_{31} & P_{32} \end{pmatrix} = s - n,$$

then by adjusting the order of  $\alpha_1, \dots, \alpha_s$  if necessary, we may assume

$$\text{rank} \begin{pmatrix} P_{22} \\ P_{32} \end{pmatrix} = \text{rank} \begin{pmatrix} P_{21} & P_{22} \\ P_{31} & P_{32} \end{pmatrix} = s - n.$$

So

$$m - n = \text{rank} \begin{pmatrix} P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} = \text{rank} \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix},$$

and so  $\begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}$  is invertible.

Let

$$K = - \begin{pmatrix} P_{12} & P_{13} \end{pmatrix} \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}^{-1}, \quad R = \begin{pmatrix} I_n & K \\ 0 & I_{m-n} \end{pmatrix},$$

then

$$RP = \begin{pmatrix} \tilde{P}_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix},$$

where

$$\tilde{P}_{11} = P_{11} - \begin{pmatrix} P_{12} & P_{13} \end{pmatrix} \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}^{-1} \begin{pmatrix} P_{21} \\ P_{31} \end{pmatrix}.$$

Since

$$X_1 = \begin{pmatrix} I_n \\ 0 \end{pmatrix} = RX_1 = RP \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_{s-n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} \tilde{P}_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} \tilde{P}_{11}Q_1 \\ * \\ * \end{pmatrix},$$

so  $I_n = \tilde{P}_{11}Q_1$ , and so  $Q_1 = \tilde{P}_{11}^{-1}$ . Then

$$\begin{aligned} \begin{pmatrix} B + KC \\ C \end{pmatrix} &= R \begin{pmatrix} B \\ C \end{pmatrix} = RX_2 = RP \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} Q \\ &= \begin{pmatrix} \tilde{P}_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{P}_{11}\Lambda_1Q_1 \\ * \\ * \end{pmatrix}. \end{aligned}$$

So  $B + KC = \tilde{P}_{11}\Lambda_1Q_1 = \tilde{P}_{11}\Lambda_1\tilde{P}_{11}^{-1}$ . □

Consider the case that  $m = 4$  and  $n = 3$  in Theorem 13. Then, we can determine when the rank is 4 for  $4 \times 3 \times 2$  tensors. Recall the assumption that  $\text{rank}(X_1) \geq \text{rank}(X_2)$ ,  $\text{rank}(X_1, X_2) = 4$ . When  $\text{rank}(X_1, X_2) = 4$ ,  $\text{rank}(X_1)$  is at least 2. (Otherwise,  $\text{rank}(X_2) \leq \text{rank}(X_1) < 2$ , a contradiction to  $\text{rank}(X_1, X_2) = 4$ .) In the following theorem, we discuss the rank of  $4 \times 3 \times 2$  tensors mainly by  $\text{rank}(X_1)$ .

COROLLARY 14. *Let*

$$X_1 = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} B \\ \beta^T \end{pmatrix},$$

be  $4 \times 3$  matrices over  $\mathbb{F}$ , where  $B \in \mathbb{F}^{3 \times 3}$ ,  $\beta \in \mathbb{F}^3$ ,  $s = \text{rank}(X_1) \geq \text{rank}(X_2)$ , and  $\text{rank}(X_1, X_2) = 4$ . Suppose  $\mathcal{T} = [X_1|X_2]$  is a  $4 \times 3 \times 2$  tensor over  $\mathbb{F}$ . Then

(1) If  $\text{rank}(X_1) = 2$ , then  $r(\mathcal{T}) = 4$ .

(2) If  $\text{rank}(X_1) = 3$ , then

$$r(\mathcal{T}) = \begin{cases} 4, & \text{if there is a vector } k \text{ such that } B + k\beta^T \text{ is diagonalizable.} \\ 5, & \text{otherwise.} \end{cases}$$

*Proof.* The conclusion is obvious by Theorem 13. □

**4.3.  $m \times 3 \times 2$  tensors.** Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times 3 \times 2$  tensor over  $\mathbb{F}$ , then  $\text{rank}(X_1, X_2) \leq 6$ . If  $\text{rank}(X_1, X_2) \leq 4$ , then  $\mathcal{T}$  can degenerate to a  $4 \times 3 \times 2$  tensor as stated in the previous section and then we turn to Section 4.2. So we only need to consider the case of  $\text{rank}(X_1, X_2) = 5$  and  $\text{rank}(X_1, X_2) = 6$ .

**THEOREM 15.** *Suppose  $\mathcal{T} = [X_1|X_2]$  is an  $m \times 3 \times 2$  tensor over  $\mathbb{F}$ , and  $\text{rank}(X_1, X_2) \geq 5$ . Then*

$$r(\mathcal{T}) = \text{rank}(X_1, X_2).$$

*Proof.* (1) If  $\text{rank}(X_1, X_2) = 5$ , then  $\mathcal{T}$  can degenerate to a  $5 \times 3 \times 2$  tensor as mentioned before. It follows from Lemma 3 that  $r(\mathcal{T}) = 5$ .

(2) If  $\text{rank}(X_1, X_2) = 6$ , it follows from

$$6 = \text{rank}(X_1, X_2) \leq r(\mathcal{T}) \leq \text{rank}(X_1) + \text{rank}(X_2) \leq 6,$$

that  $r(\mathcal{T}) = 6$ . □

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