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PAIRS OF MATRICES, ONE OF WHICH COMMUTES WITH THEIR COMMUTATOR*

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Abstract. Let A, B be $n \times n$ complex matrices such that $C = AB - BA$ and A commute. For $n = 2$, we prove that A, B are simultaneously triangularizable. For $n \geq 3$, we give an example of matrices A, B such that the pair (A, B) does not have property L of Motzkin-Taussky, and such that B and C are not simultaneously triangularizable. Finally, we estimate the complexity of the Alp'in-Koreshkov's algorithm that checks whether two matrices are simultaneously triangularizable. Practically, one cannot test a pair of numerical matrices of dimension greater than five.

Key words. Nilpotent matrix, Property L, Commutator, Quasi-commute.

AMS subject classifications. 15A27, 15A22.

1. Introduction.

Definition 1.1. i) We say that the $n \times n$ complex matrices A, B *quasi-commute* if both A and B commute with $AB - BA$.
ii) The $n \times n$ complex matrices A, B are said to be *simultaneously triangularizable* (ST) if there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular.

Consider the following standard result.

Theorem 1.2. (Little McCoy's Theorem [6]) *If A and B quasi-commute, then they are ST .*

In this article, we deal with pairs of $n \times n$ complex matrices (A, B) such that only A commutes with $AB - BA$. If (A, B) is such a pair, then for any complex numbers λ, μ, $(A + \lambda I_n, B + \mu I_n)$ is another one. Then we may assume that A and B are invertible. In the sequel, we put $C = AB - BA$. We introduce notation and definitions that will be used in the paper.

Notation. i) If U is a square matrix, then $\sigma(U)$ and χ_U denote the spectrum and the characteristic polynomial of U.
ii) Denote by I_n and 0_n the identity matrix and the zero matrix of dimension n.

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Definition 1.3. (See [7]) A pair (A, B) of complex $n \times n$ matrices is said to have *property L* if for a special ordering of the eigenvalues $(\lambda_i)_{i \leq n}, (\mu_i)_{i \leq n}$ of A, B, the eigenvalues of $xA + yB$ are $(x\lambda_i + y\mu_i)_{i \leq n}$ for all values of the complex numbers x, y.

Remark 1.4. (See [7]) If A, B are ST, then the pair (A, B) has property L but, except if $n = 2$, the converse is false.

Several known results are gathered in the following Proposition.

Proposition 1.5. *Let A, B be complex $n \times n$ matrices. We assume that C and A commute. Then C is nilpotent and the pair (B, C) has property L. Moreover, if A, B are invertible, then $A^{-1}B^{-1}C, B^{-1}A^{-1}C$ and $B^{-1}C$ are nilpotent.*

Proof. By Jacobson's Lemma, see [5, Lemma 2], C is nilpotent. According to [3], one has, for every $t \in \mathbb{R}$ and for any $A, B \in \mathcal{M}_n(\mathbb{C})$, $e^{tA}Be^{-tA} = B + tC + \frac{t^2}{2!}[A, C] + \frac{t^3}{3!}[A, [A, C]] + \cdots$. By an analytic continuation, this equality works also for complex numbers t. Here, we obtain for every $t \in \mathbb{C}$

$$e^{tA}Be^{-tA} = B + tC,$$

and therefore, $\sigma(B + tC) = \sigma(B)$. It follows that the pair (B, C) has property L. Now we assume that A, B are invertible. We have

$$A^{-1}CB^{-1} = CA^{-1}B^{-1} = ABA^{-1}B^{-1} - I_n.$$

By [9, Theorem 2], $ABA^{-1}B^{-1} - I_n$ is nilpotent. Since

$$\sigma(A^{-1}B^{-1}C) = \sigma(CA^{-1}B^{-1}) = \{0\} \ \textit{and} \ \sigma(B^{-1}A^{-1}C) = \sigma(A^{-1}CB^{-1}) = \{0\},$$

we conclude that $A^{-1}B^{-1}C$ and $B^{-1}A^{-1}C$ are also nilpotent. By [9, proof of Theorem 1], we obtain that CB^{-1} is nilpotent (or equivalently $B^{-1}C$ is nilpotent). ◻

2. Positive and negative results. We may wonder whether A and B are ST or, at least, the pair (A, B) has property L. We have a positive answer in the following case.

Definition 2.1. A complex matrix A is said to be *non-derogatory* if for every $\lambda \in \sigma(A)$, the number of Jordan blocks of A associated with λ is 1.

Proposition 2.2. *If A is a non-derogatory matrix and if $AC = CA$, then A and B are ST.*

Proof. Necessarily, C is a polynomial in A. According to [2, Theorem 1], A and B are ST. ◻

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REMARK 2.3. i) Note that the set of derogatory matrices is included in the set NS of non-separable matrices, that is they have at least one multiple eigenvalue. The set NS is an algebraic variety in $\mathcal{M}_n(\mathbb{C})$ of codimension 1. Therefore, it is a null set in the sense of Lebesgue measure (see [8] for an outline of the proof).
ii) If we fix the matrix A, then the equation $A(AB - BA) = (AB - BA)A$ is linear in the unknown B. More precisely $B \in \ker(\phi)$ where $\phi : X \to A^2X + XA^2 - 2AXA$. Hence,

$$\phi = A^2 \otimes I + I \otimes (A^T)^2 - 2A \otimes A^T = \psi^2,$$

where $\psi = A \otimes I_n - I_n \otimes A^T$. Thus, if $\sigma(A) = (\lambda_i)_i$, then $\sigma(\psi) = (\lambda_i - \lambda_j)_{i,j}$ and $\sigma(\phi) = ((\lambda_i - \lambda_j)^2)_{i,j}$. The quantity

$$i(A) \; = \; \frac{\dim(\ker(\psi^2)) - \dim(\ker(\psi))}{n^2}$$

indicates the existence of a matrix B such that $AB - BA$ and A commute and such that A, B are not ST.

Now we prove our main result.

PROPOSITION 2.4. i) *If $n = 2$ and $AC = CA$, then A and B are ST.*
ii) *If $n \geq 3$, then there exists a pair (A, B) such that*
- *$AC = CA$,*
- *(A, B) does not have property L,*
- *B and C are not ST.*

Proof. i) According to Proposition 2.2, we may assume that A is derogatory, that is, A is a scalar matrix, which gives the conclusion immediately.

ii) It is sufficient to find such a counterexample (A_0, B_0) when $n = 3$. Indeed, if $n > 3$, consider the pair $(A_0 \bigoplus 0_{n-3}, B_0 \bigoplus 0_{n-3})$.

Now suppose that $n = 3$ and that A_0 is the derogatory matrix $A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then ψ is nilpotent and we have the equalities

$$\dim(\ker(\psi)) = 5, \dim(\ker(\psi^2)) = 8 \;\; \textit{and} \;\; i(A_0) = \frac{1}{3}.$$

The associated matrices B are the matrices with a zero entry in position $(2,1)$. We choose

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

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• (A_0, B) does not have property L because $\sigma(A_0) = \{0\}$ and for every pair of complex numbers (t, x), $\chi_{tA_0+B}(x) = x^3 - t$.
• We observe that $\mathrm{Tr}(B^2C^2) = -1$. This implies that B and C are not ST. □

Remark 2.5. We can prove i) by reducing A to Jordan canonical form and examining the cases in which A is diagonalizable or not.

Proposition 2.6. *For every $n \geq 4$, there exists a derogatory matrix A_1 such that A_1 and each associated matrix B are ST.*

Proof. We take $n = 4$ and $A_1 = \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix}$. Note that A_1 is in Weyr canonical form (see [10]) and not in Jordan canonical form. One has

$$\dim(\ker(\psi)) = 8, \dim(\ker(\psi^2)) = 12 \ \textit{and} \ \ i(A_1) = \frac{1}{4} < i(A_0).$$

The associated matrices B are in the form $B = \begin{bmatrix} E & F \\ 0 & G \end{bmatrix}$ where E, F, G are arbitrary 2×2 complex matrices. Let U, V be 2×2 invertible complex matrices such that $U^{-1}EU$ and $V^{-1}GV$ are upper triangular. We remark that $P^{-1}A_1P$ and $P^{-1}BP$ are upper triangular where $P = \mathrm{diag}(U, V)$. □

3. How to determine whether two matrices are ST.

In general, how can one determine whether two $n \times n$ complex matrices are ST or not? McCoy's Theorem (see Section 2.4 of [4]) is an available tool, but it does not give a finite verification procedure.

The following theorem leads to an algorithm to check whether two matrices are ST.

Theorem 3.1. (Alp'in-Koreshkov, see [1, Theorem 6]) *Two $n \times n$ complex matrices A and B are ST if and only if for every $k \in [[1, n^2 - 1]]$, each matrix of the form $U_1 \cdots U_k(AB - BA)$ (where, for every i, U_i is A or B) has a zero trace.*

Remark 3.2. If the entries of A, B are in a subring K of $\mathbb{C}$, then all computations are performed in K.

Using Theorem 3.1, we must check that $2^{n^2} - 2$ matrices have a zero trace. If A, B are not ST, then the test stops when it finds a matrix with non-zero trace. If A, B are ST, then the test requires 2^{n^2} matricial multiplications in $\mathcal{M}_n(\mathbb{C})$. We can deduce the following.

Proposition 3.3. *The complexity of the computation induced by Theorem 3.1 is equivalent to $2^{n^2}n^3$ complex multiplications.*

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Experiments. We used a cluster provided with 16 GB of RAM.

For the following 4×4 matrices, that are ST,

$$A = \begin{bmatrix} 308831848 & 3514720569 & -2393248600 & -933664618 \\ 1653458482 & -2646203334 & 1951033145 & 1428485078 \\ 185766230 & -909262575 & 2221156990 & 78496990 \\ 1349546744 & -2237843658 & 4279424410 & 96552841 \end{bmatrix},$$

$$B = \begin{bmatrix} -277500618 & 34522275 & 180434913 & -933966414 \\ 2348943678 & 1523928630 & -700130673 & 1316048154 \\ -97303050 & -203818485 & 577843890 & 179268180 \\ 394577946 & 431913075 & -336185991 & 967683108 \end{bmatrix},$$

the duration of the test was less than one second and the used memory was about 90 MB.

In dimension five, there is a big storage at the end of the penultimate step. Precisely, at this stage, we store 2^{23} matrices of dimension 5. We considered a pair of numerical 5×5 matrices, that were ST and such that their entries were integers with absolute value at most 1000. Then the duration of the test was 2 minutes 26 seconds.

In dimension six, the maximal storage theoretically uses tens of terabytes of RAM and consequently this test only works to show eventually that two matrices are not ST.

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