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SCHUR FUNCTIONS AND IMMANANTAL IDENTITIES*

RYO TABATA†

Abstract. Littlewood developed the theory of symmetric functions and immanants. It is known that some identities for immanants correspond to the ordinary products of Schur functions via the Littlewood–Richardson rule. We discuss the relations between immanants and plethysm, another type of products of Schur functions. We present immanantal identities corresponding to the most basic formula of plethysm. As an application, we show some inequalities for positive semidefinite Hermitian matrices.

Key words. Symmetric functions, Immanants, Plethysm, Littlewood–Richardson rule.

AMS subject classifications. 15A15, 05E05.

1. Introduction.

Let $A = (a_{ij})$ be an $n \times n$ matrix, and let χ_λ denote the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to each partition λ of n. The immanant associated with λ is a matrix function

$$d_\lambda(A) = \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

If $\lambda = (1^n)$, then χ_λ is the alternating character and $d_\lambda(A) = \det A$. If $\lambda = (n)$, then χ_λ is the trivial character and $d_\lambda(A) = \operatorname{per} A$.

In [2], Littlewood and Richardson defined Schur functions s_λ as the immanants of a special matrix:

$$d_\lambda \begin{pmatrix} p_1 & 1 & 0 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & 0 & \cdots & 0 \\ p_3 & p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ p_{n-1} & & & & p_1 & n-1 \\ p_n & & & \cdots & p_2 & p_1 \end{pmatrix} = n!\ s_\lambda,$$

where p_i is the i-th power sum symmetric function. Moreover, the connection between symmetric functions and immanants was discussed in the paper. Littlewood summarized the following results on relations between Schur functions (or S-functions denoted by $\{\lambda\}$) and immanants.

THEOREM 1.1 (Littlewood [7]). *I. Corresponding to any relation between S-functions of total weight n, we may replace the S-functions by the corresponding immanants of complementary coaxial minors of $[a_{st}]$ provided that every product is summed for all sets of complementary coaxial minors.*
II. Corresponding to any relation between S-functions we may replace each S-function by the corresponding

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†Department of General Education, National Institute of Technology, Ariake College, Omuta, Fukuoka 836-8585, Japan (tabata@ariake-nct.ac.jp).

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immanant of a coaxial minor of $[a_{st}]$, provided that we sum with respect to all the coaxial minors of the appropriate order.
III. The S-function $\{\lambda\}$ of weight p of the characteristic roots of a matrix $[a_{st}]$ is equal to the sum of the immanants corresponding to the partition (λ) of all p-rowed coaxial minors of $[a_{st}]$.

The first two of the results above imply that there are immanantal identities with respect to principal submatrices corresponding to Schur functions. Let us call them Littlewood–Richardson's correspondence between Schur functions and immanants. For example, consider the expansion of the (ordinary) product of Schur functions,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu .$$

It is known that the coefficient $c^\nu_{\lambda\mu}$ can be computed in a combinatorial way, so-called the Littlewood–Richardson rule. The corresponding identity for immanants is

$$\sum_{(N_1,N_2)} d_\lambda\left(A[N_1]\right) d_\mu\left(A[N_2]\right) = \sum_\nu c^\nu_{\lambda\mu} d_\nu(A),$$

where the sum runs over all ordered pairs (N_1, N_2) of subsets of $\{1, 2, \cdots, n\}$ such that $N_1 \cup N_2 = \{1, 2, \cdots, n\}$, $N_1 \cap N_2 = \phi$, $|N_1| = |\lambda|$, $|N_2| = |\mu|$, and $A[N_i]$ is the $|N_i| \times |N_i|$ principal submatrix obtained by choosing rows and columns of N_i.

The purpose of this paper is to discuss the relations between Schur functions and immanants from the view of plethysm. Plethysm, first introduced by Littlewood, is another type of products of Schur functions and has a natural interpretation as the representations of the general linear groups. We review Littlewood's original approach to plethysm based on invariant matrices. Especially, we observe the contribution of immanants to the definition of plethysm. It remains an open problem to find a combinatorial rule to expand plethysm, and there are explicit formulae only for special cases. Our main theorem is to present the immanantal identities corresponding to the most basic ones. We also mention applications to the inequality problem of immanants.

2. Invariant matrices and plethysm. If for any two $m \times m$ matrices A and B

$$T(AB) = T(A)T(B),$$

$T(A)$ is called an invariant matrix of A. Littlewood [5] constructed a specific invariant matrix corresponding to each partition λ of n as follows.

For f^λ standard tableaux $T_1, T_2, \cdots, T_{f^\lambda}$ of shape λ, define elements P_i, Q_j, and τ_{ij} of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ to be

$$P_i = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ preserves each row in } T_i}} \sigma, \qquad Q_i = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ preserves each column in } T_i}} (\operatorname{sgn}\sigma)\sigma,$$

$$\tau_{ij} = (\textit{the permutation which transforms } T_i \textit{ into } T_j),$$

and define $\phi^{(\lambda)}_{ij}(\sigma)$ to be the coefficient of σ in $P_i \tau_{ij} Q_j$.

Let μ and ν be

$$\mu = \left(1^{\mu_1} \cdots m^{\mu_m}\right), \nu = \left(1^{\nu_1} \cdots m^{\nu_m}\right),$$

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satisfying $\mu_1 + \cdots + \mu_m = n$ and $\nu_1 + \cdots + \nu_m = n$. Let $A[\mu|\nu]$ be the $n \times n$ submatrix of A obtained by (repeatedly) choosing rows and columns belonging to μ and ν. Denoting (k, ℓ)-element of $A[\mu|\nu]$ by $q_{k\ell}$, define the invariant matrix of A corresponding to λ to be

$$T_\lambda(A) = \left(\sum_{\sigma \in \mathfrak{S}_n} \phi_{ij}^{(\lambda)}(\sigma) \cdot \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} q_{1\sigma(1)} q_{2\sigma(2)} \cdots q_{n\sigma(n)} \right)_{i,j}. \tag{2.1}$$

Note that if $\lambda = (n)$, then $T_\lambda(A)$ is the n-th induced matrix of A, and if $\lambda = (1^n)$, then $T_\lambda(A)$ is the n-th compound matrix of A.

Clearly, the definition of invariant matrices can be regarded as a representation of the general linear groups. Littlewood noted the following important theorem, which nowadays has the interpretation as the character of a Schur functor:

THEOREM 2.1 (Littlewood [3, p. 371]). *Let A be an $m \times m$ matrix with its eigenvalues $\alpha_1, \cdots, \alpha_m$, and $T_\lambda(A)$ the invariant matrix of A corresponding to partition λ of n. Then, the trace of $T_\lambda(A)$ is equal to the Schur function s_λ of $\alpha_1, \cdots, \alpha_m$. Namely,*

$$\text{tr } T_\lambda(A) = s_\lambda(\alpha_1, \cdots, \alpha_m).$$

Proof. Recall that the rows (or columns) of the invariant matrix corresponding to λ is labeled by (μ, i) (or (ν, j)). Then,

$$\begin{aligned} \text{tr } T_\lambda(A) &= \sum_{(\mu,i)} \sum_{\sigma \in \mathfrak{S}_n} \phi_{ii}^{(\lambda)}(\sigma) \cdot \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} q_{1\sigma(1)} q_{2\sigma(2)} \cdots q_{n\sigma(n)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{\mu} \sum_{i=1}^{f_\lambda} \phi_{ii}^{(\lambda)}(\sigma) \cdot \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} q_{1\sigma(1)} q_{2\sigma(2)} \cdots q_{n\sigma(n)}. \end{aligned}$$

Since $\phi_{ii}^{(\lambda)}(\sigma)$ is the coefficient of σ in the Young symmetrizer of the tableau T_i with the shape λ, we can rewrite

$$\sum_{i=1}^{f_\lambda} \phi_{ii}^{(\lambda)}(\sigma) = \chi_\lambda(\sigma),$$

so that

$$\begin{aligned} \text{tr } T_\lambda(A) &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{\mu} \chi_\lambda(\sigma) \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} q_{1\sigma(1)} q_{2\sigma(2)} \cdots q_{n\sigma(n)} \\ &= \sum_{\mu} \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) q_{1\sigma(1)} q_{2\sigma(2)} \cdots q_{n\sigma(n)} \\ &= \sum_{\mu} \frac{1}{\mu_1!\mu_2! \cdots \mu_m!} d_\lambda(A[\mu|\mu]). \end{aligned}$$

By III of Theorem 1.1, the theorem is proved. ∎

The invariant matrix of the invariant matrix is an invariant matrix of the original matrix and can be expressed as a direct sum

$$T_\lambda\left(T_\mu(A)\right) = \bigoplus_{\nu} a_{\lambda\mu}^{\nu} T_\nu(A).$$

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In [4], Littlewood defined another product of Schur functions called plethysm to be

$$s_\lambda \circ s_\mu = \sum_\nu a^\nu_{\lambda\mu} s_\nu.$$

This product is naturally interpreted as the composition of the representations of the general linear groups, or the Schur functors. It is an open problem to find a combinatorial rule to compute the coefficient $a^\nu_{\lambda\mu}$ in general, and there are explicit formulae only for some special cases. Due to its complexity, some methods have been developed to approach the plethysm problem even in recent years (see, for example, [1]). The followings are the most basic results.

Theorem 2.2 (Littlewood [6], Thrall [8]).

$$s_{(2)} \circ s_{(n)} = \sum_{i=0, i:\text{even}}^{2\lfloor n/2 \rfloor} s_{(2n-i,i)}.$$

$$s_{(1,1)} \circ s_{(n)} = \sum_{i=1, i:\text{odd}}^{2\lfloor (n-1)/2 \rfloor+1} s_{(2n-i,i)}.$$

3. Main theorem. In this section, we present the immanantal identities corresponding to Theorem 2.2.

Theorem 3.1. *Let A be a $2n \times 2n$ matrix. Then:*

$$\sum_{(N_1,N_2)} \operatorname{per} \begin{pmatrix} \operatorname{per} A[N_1|N_1] & \operatorname{per} A[N_1|N_2] \\ \operatorname{per} A[N_2|N_1] & \operatorname{per} A[N_2|N_2] \end{pmatrix} = \sum_{i=0, i:\text{even}}^{2\lfloor n/2 \rfloor} d_{(2n-i,i)}(A); \tag{3.2a}$$

$$\sum_{(N_1,N_2)} \det \begin{pmatrix} \operatorname{per} A[N_1|N_1] & \operatorname{per} A[N_1|N_2] \\ \operatorname{per} A[N_2|N_1] & \operatorname{per} A[N_2|N_2] \end{pmatrix} = \sum_{i=1, i:\text{odd}}^{2\lfloor (n-1)/2 \rfloor+1} d_{(2n-i,i)}(A), \tag{3.2b}$$

where the sum runs over all non-ordered pairs $(N_1,\ N_2)$ of subsets of $\{1, 2, \cdots, 2n\}$ such that $N_1 \cup N_2 = \{1, 2, \cdots, 2n\}$, $N_1 \cap N_2 = \phi$, $|N_1| = |N_2| = n$, and $A[N_i|N_j]$ is the $n \times n$ submatrix obtained by choosing rows of N_i and columns of N_j.

Example 3.2. *For any 4×4 matrix $A = (a_{ij})_{1 \le i,j \le 4}$,*

$$\begin{aligned}
&\operatorname{per}\begin{pmatrix} \operatorname{per}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \\ \operatorname{per}\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \end{pmatrix} \\
&+\operatorname{per}\begin{pmatrix} \operatorname{per}\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{pmatrix} \\ \operatorname{per}\begin{pmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix} \end{pmatrix} \\
&+\operatorname{per}\begin{pmatrix} \operatorname{per}\begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{pmatrix} \\ \operatorname{per}\begin{pmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{pmatrix} & \operatorname{per}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \end{pmatrix}
\end{aligned}
=
\begin{aligned}
& d_{(4)}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \\
+\ & d_{(2,2)}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
\end{aligned}.$$

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Lemma 3.3. *Suppose that the cycle type of the permutation $\sigma \in \mathfrak{S}_{2n}$ is $(1^{\sigma_1}2^{\sigma_2}\cdots(2n)^{\sigma_{2n}})$.*

$$\sum_{i=0}^{n}(-1)^i\chi_{(2n-i,i)}(\sigma) = \begin{cases} 2^{\sigma_2+\sigma_4+\cdots+\sigma_{2n}} & (\text{if the cycle type of } \sigma \text{ is } (2^{\sigma_2}4^{\sigma_4}\cdots(2n)^{\sigma_{2n}}))\,, \\ 0 & (\text{if } \sigma \text{ has at least one odd cycle})\,. \end{cases}$$

Proof. Using a basic property of plethysm for the n-th complete symmetric function h_n,

$$\begin{aligned} h_n \circ p_2 &= h_n(x_1^2, x_2^2, \cdots) \\ &= s_{(n)}(x_1^2, x_2^2, \cdots) \\ &= \sum_{\sigma}\prod_{k=1}^{n}\frac{p_k(x_1^2,x_2^2,\cdots)^{\sigma_k}}{\sigma_k!k^{\sigma_k}} \\ &= \sum_{\sigma}\prod_{k=1}^{n}\frac{p_{2k}(x_1,x_2,\cdots)^{\sigma_k}}{\sigma_k!k^{\sigma_k}} \\ &= \sum_{\sigma}\frac{p_2^{\sigma_1}p_4^{\sigma_2}\cdots p_{2n}^{\sigma_n}}{\sigma_1!\sigma_2!\cdots\sigma_n!1^{\sigma_1}2^{\sigma_2}\cdots n^{\sigma_n}}, \end{aligned}$$

where the sum runs over all cycle types $\sigma = (1^{\sigma_1}2^{\sigma_2}\cdots n^{\sigma_n})$ of $\mathfrak{S}_n$, satisfying $\sigma_1 + 2\sigma_2 + \cdots + n\sigma_n = n$. Thus, we have

$$h_n \circ p_2 = \sum_{\sigma} 2^{\sigma_2+\sigma_4+\cdots+\sigma_{2n}} \cdot \frac{p_2^{\sigma_2}p_4^{\sigma_4}\cdots p_{2n}^{\sigma_{2n}}}{\sigma_2!\sigma_4!\cdots\sigma_{2n}!2^{\sigma_2}4^{\sigma_4}\cdots(2n)^{\sigma_{2n}}}.$$

On the other hand,

$$\begin{aligned} \sum_{i=0}^{n}(-1)^i s_{(2n-i,i)} &= \sum_{i=0}^{n}(-1)^i\left\{\sum_{\sigma}\chi_{(2n-i,i)}(\sigma)\prod_{k=1}^{2n}\frac{p_k^{\sigma_k}}{\sigma_k!k^{\sigma_k}}\right\} \\ &= \sum_{\sigma}\left(\sum_{i=0}^{n}(-1)^i\chi_{(2n-i,i)}(\sigma)\right)\frac{p_1^{\sigma_1}p_2^{\sigma_2}\cdots p_{2n}^{\sigma_{2n}}}{\sigma_1!\sigma_2!\cdots\sigma_{2n}!\ 1^{\sigma_1}2^{\sigma_2}\cdots(2n)^{\sigma_{2n}}}, \end{aligned}$$

where the sum runs over all cycle types $\sigma = (2^{\sigma_2}4^{\sigma_4}\cdots(2n)^{\sigma_{2n}})$ of $\mathfrak{S}_{2n}$, satisfying $2\sigma_2+4\sigma_4+\cdots+2n\sigma_{2n} = 2n$. Now, the lemma follows from the equation ([9, p. 140])

$$h_n \circ p_2 = \sum_{i=0}^{n}(-1)^i s_{(2n-i,i)}.$$

□

Proof of Theorem 3.1. The left-hand side of (3.2a) becomes

$$\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_1]\operatorname{per}A[N_2|N_2] + \sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_2]\operatorname{per}A[N_2|N_1].$$

The first term is yielded by the principal submatrices of A. This enables us to apply Littlewood–Richardson's correspondence. Noticing that it is applied with respect to ordered pairs (N_1, N_2),

$$\begin{aligned} \sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_1]\operatorname{per}A[N_2|N_2] &= \sum_{(N_1,N_2)} d_{(n)}\left(A[N_1|N_1]\right)d_{(n)}\left(A[N_2|N_2]\right) \\ &= \frac{1}{2}\left(d_{(2n)}(A) + d_{(2n-1,1)}(A) + d_{(2n-2,2)}(A) + \cdots + d_{(n,n)}(A)\right). \end{aligned}$$

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To analyze the second term, suppose $N_1 = \{j_1, j_2, \cdots, j_n\}$ and $N_2 = \{j_{n+1}, j_{n+2}, \cdots, j_{2n}\}$. Then,

$$A[N_1|N_2] = \begin{pmatrix} a_{j_1,j_{n+1}} & a_{j_1,j_{n+2}} & \cdots & a_{j_1,j_{2n}} \\ a_{j_2,j_{n+1}} & a_{j_2,j_{n+2}} & \cdots & a_{j_2,j_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_n,j_{n+1}} & a_{j_n,j_{n+2}} & \cdots & a_{j_n,j_{2n}} \end{pmatrix} \quad \text{and} \quad A[N_2|N_1] = \begin{pmatrix} a_{j_{n+1},j_1} & a_{j_{n+1},j_2} & \cdots & a_{j_{n+1},j_n} \\ a_{j_{n+2},j_1} & a_{j_{n+2},j_2} & \cdots & a_{j_{n+2},j_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_{2n},j_1} & a_{j_{2n},j_2} & \cdots & a_{j_{2n},j_n} \end{pmatrix}.$$

Any term in $\operatorname{per} A[N_1|N_2] \operatorname{per} A[N_2|N_1]$ can be written as

$$a_{j_1,j_{n+\sigma(1)}} \cdots a_{j_n,j_{n+\sigma(n)}} a_{j_{n+1},j_{\tau(1)}} \cdots a_{j_{2n},j_{\tau(n)}}, \tag{3.3}$$

for some $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_n$. The map $f_{\sigma\tau} : \{1, 2, \cdots, 2n\} \to \{1, 2, \cdots, 2n\}$ (simply denoted by f) defined by

$$f_{\sigma\tau}(j_k) = \begin{cases} j_{n+\sigma(k)} & (if \quad 1 \le k \le n), \\ j_{\tau(k-n)} & (if \quad n+1 \le k \le 2n), \end{cases}$$

is regarded as a permutation in $\mathfrak{S}_{2n}$, so we can rewrite the term (3.3) as

$$a_{1f(1)} a_{2f(2)} \cdots a_{2nf(2n)}.$$

It is clear that $f \in \mathfrak{S}_{2n}$ only consists of even cycles. If the length of the cycle of f including $1 \le j_k \le n$ is denoted by p, then f has a cycle

$$\left(j_k,\ f(j_k),\ f^2(j_k),\ \cdots,\ f^{p-1}(j_k)\right),$$

which can be

$$\left(j_k,\ j_{n+\sigma(k)},\ j_{\tau\sigma(k)},\ \cdots,\ j_{n+(\sigma\tau\sigma\cdots\tau\sigma)(k)}\right).$$

Also, note that some of elements in N_1 and N_2 appear alternately in this cycle. There exists q such that $p = 2q$ and $(\tau\sigma)^q(k) = k$, which implies the length of each cycle of $f \in \mathfrak{S}_{2n}$ is twice as the length of $\tau\sigma \in \mathfrak{S}_n$. Thus, the number of terms in $\operatorname{per} A[N_1|N_2] \operatorname{per} A[N_2|N_1]$ yielded by the permutations with the cycle type $\left(2^{i_2} 4^{i_4} \cdots (2n)^{i_{2n}}\right)$ is

$$\frac{(n!)^2}{1^{i_2} i_2! 2^{i_4} i_4! \cdots n^{i_{2n}} i_{2n}!},$$

which is equal to the number of elements of the conjugacy class of $\mathfrak{S}_n$ with the type $\left(1^{i_2} 2^{i_4} \cdots n^{i_{2n}}\right)$ multiplied by $n!$.

If we sum up for $\binom{2n}{n}/2!$ pairs (N_1, N_2), we have

$$\frac{(n!)^2}{1^{i_2} i_2! 2^{i_4} i_4! \cdots n^{i_{2n}} i_{2n}!} \times \frac{1}{2!} \binom{2n}{n} = \frac{(2n)!}{2 \cdot 1^{i_2} i_2! 2^{i_4} i_4! \cdots n^{i_{2n}} i_{2n}!},$$

terms yielded by the cycle type $\left(2^{i_2} 4^{i_4} \cdots (2n)^{i_{2n}}\right)$ in $\sum_{(N_1,N_2)} \operatorname{per} A[N_1|N_2] \operatorname{per} A[N_2|N_1]$.

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On the other hand, since the number of elements of the conjugacy class of $\mathfrak{S}_{2n}$ with the type $\left(2^{i_2}4^{i_4}\cdots(2n)^{i_{2n}}\right)$ is

$$\frac{(2n)!}{2^{i_2}i_2!4^{i_4}i_4!\cdots(2n)^{i_{2n}}i_{2n}!},$$

there are

$$\frac{(2n)!}{2\cdot 1^{i_2}i_2!2^{i_4}i_4!\cdots n^{i_{2n}}i_{2n}!}\Big/\left(\frac{(2n)!}{2^{i_2}i_2!4^{i_4}i_4!\cdots(2n)^{i_{2n}}i_{2n}!}\right)=2^{i_2+i_4+\cdots i_{2n}-1},$$

sets of the terms corresponding to the conjugacy class of the type $\left(2^{i_2}4^{i_4}\cdots(2n)^{i_{2n}}\right)$ in $\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_2]\operatorname{per}A[N_2|N_1]$. By Lemma 3.3,

$$\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_2]\operatorname{per}A[N_2|N_1]=\frac{1}{2}\sum_{i=0}^{n}(-1)^i d_{(2n-i,i)}(A).$$

Therefore, we conclude

$$\begin{aligned}
&\sum_{(N_1,N_2)}\operatorname{per}\begin{pmatrix}\operatorname{per}A[N_1|N_1] & \operatorname{per}A[N_1|N_2]\\ \operatorname{per}A[N_2|N_1] & \operatorname{per}A[N_2|N_2]\end{pmatrix}\\
&=\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_1]\operatorname{per}A[N_2|N_2]+\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_2]\operatorname{per}A[N_2|N_1]\\
&=\frac{1}{2}\sum_{i=0}^{n}d_{(2n-i,i)}(A)+\frac{1}{2}\sum_{i=0}^{n}(-1)^i d_{(2n-i,i)}(A)\\
&=\sum_{i=0,i:\text{even}}^{2\lfloor n/2\rfloor}d_{(2n-i,i)}(A).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sum_{(N_1,N_2)}\det\begin{pmatrix}\operatorname{per}A[N_1|N_1] & \operatorname{per}A[N_1|N_2]\\ \operatorname{per}A[N_2|N_1] & \operatorname{per}A[N_2|N_2]\end{pmatrix}\\
&=\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_1]\operatorname{per}A[N_2|N_2]-\sum_{(N_1,N_2)}\operatorname{per}A[N_1|N_2]\operatorname{per}A[N_2|N_1]\\
&=\frac{1}{2}\sum_{i=0}^{n}d_{(2n-i,i)}(A)-\frac{1}{2}\sum_{i=0}^{n}(-1)^i d_{(2n-i,i)}(A)\\
&=\sum_{i=1,i:\text{odd}}^{2\lfloor (n-1)/2\rfloor+1}d_{(2n-i,i)}(A).
\end{aligned}$$

□

We obtain the similar result to Theorem 3.1, corresponding to the expansions of $s_{(2)}\circ s_{(1^n)}$ and $s_{(1^2)}\circ s_{(1^n)}$.

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THEOREM 3.4. *Let A be a $2n \times 2n$ matrix. Then:*

$$\sum_{(N_1,N_2)} \operatorname{per}\begin{pmatrix} \det A[N_1|N_1] & \det A[N_1|N_2] \\ \det A[N_2|N_1] & \det A[N_2|N_2] \end{pmatrix} = \begin{cases} \displaystyle\sum_{i=0, i:\text{even}}^{n} d_{(2^i,1^{2n-2i})}(A) & (if\ n\ is\ even), \\ \displaystyle\sum_{i=1, i:\text{odd}}^{n} d_{(2^i,1^{2n-2i})}(A) & (if\ n\ is\ odd), \end{cases}$$

$$\sum_{(N_1,N_2)} \det\begin{pmatrix} \det A[N_1|N_1] & \det A[N_1|N_2] \\ \det A[N_2|N_1] & \det A[N_2|N_2] \end{pmatrix} = \begin{cases} \displaystyle\sum_{i=1, i:\text{odd}}^{n-1} d_{(2^i,1^{2n-2i})}(A) & (if\ n\ is\ even), \\ \displaystyle\sum_{i=0, i:\text{even}}^{n-1} d_{(2^i,1^{2n-2i})}(A) & (if\ n\ is\ odd), \end{cases}$$

where the sum runs over all non-ordered pairs $(N_1,\ N_2)$ of subsets of $\{1,2,\cdots,2n\}$ such that $N_1 \cup N_2 = \{1,2,\cdots,2n\}$, $N_1 \cap N_2 = \phi$, $|N_1| = |N_2| = n$, and $A[N_i|N_j]$ is the $n \times n$ submatrix obtained by choosing rows of N_i and columns of N_j.

These results will be proved by the following lemma, whose proof is similar to the proof of Lemma 3.3 with a simple argument of signs.

LEMMA 3.5. *Suppose that the cycle type of the permutation $\sigma \in \mathfrak{S}_{2n}$ is $(1^{\sigma_1}2^{\sigma_2}\cdots(2n)^{\sigma_{2n}})$. If n is even, then*

$$\sum_{i=0}^{n}(-1)^i \chi_{(2^i,1^{2n-2i})}(\sigma) = \begin{cases} (-1)^{\sigma_4+\sigma_8+\cdots+\sigma_{2n}} 2^{\sigma_2+\sigma_4+\cdots+\sigma_{2n}} \\ \qquad (if\ the\ cycle\ type\ of\ \sigma\ is\ (2^{\sigma_2}4^{\sigma_4}\cdots(2n)^{\sigma_{2n}})), \\ 0 \quad (if\ \sigma\ has\ at\ least\ one\ odd\ cycle). \end{cases}$$

If n is odd, then

$$\sum_{i=0}^{n}(-1)^i \chi_{(2^i,1^{2n-2i})}(\sigma) = \begin{cases} (-1)^{\sigma_4+\sigma_8+\cdots+\sigma_{2n-2}} 2^{\sigma_2+\sigma_4+\cdots+\sigma_{2n}+1} \\ \qquad (if\ the\ cycle\ type\ of\ \sigma\ is\ (2^{\sigma_2}4^{\sigma_4}\cdots(2n)^{\sigma_{2n}})), \\ 0 \quad (if\ \sigma\ has\ at\ least\ one\ odd\ cycle). \end{cases}$$

4. Remarks. In this section, we make some connections to positive semidefiniteness and inequality problems. Let us assume that A is a positive semidefinite Hermitian matrix. Modifying the coefficients $1/\mu_1!\cdots\mu_m!$ in (2.1) to $1/\sqrt{\mu_1!\cdots\mu_m!\nu_1!\cdots\nu_m!}$ (as seen in [10, p. 87]), the induced matrix preserves Hermitian-ness. The following result is deduced from Theorem 2.1.

THEOREM 4.1. *If A is positive semidefinite Hermitian, then so is the induced matrix of A.*

Let $A(i|j)$ denote the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i-th row and j-th column. One can notice that the $(n-1)$-th induced matrix of A has the following principal submatrix

$$\begin{pmatrix} \operatorname{per} A(1|1) & \operatorname{per} A(1|2) & \cdots & \operatorname{per} A(1|n) \\ \operatorname{per} A(2|1) & \operatorname{per} A(2|2) & \cdots & \operatorname{per} A(2|n) \\ \vdots & & \ddots & \vdots \\ \operatorname{per} A(n|1) & \operatorname{per} A(n|2) & \cdots & \operatorname{per} A(n|n) \end{pmatrix},$$

of which the transpose is the permanental adjugate of A, that is, $B = (\operatorname{per}(A(j|i)))$ (cf. [10, p.27]).

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THEOREM 4.2. *Let A be an $n \times n$ matrix, and B the permanental adjugate of A. If A is positive semidefinite Hermitian, then B is positive semidefinite Hermitian.*

Let A be an $mn \times mn$ matrix partitioned into $m \times m$ blocks with $n \times n$ block matrices. Namely,

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix},$$

where each A_{ij} is an $n \times n$ matrix. It is known [11] that if A is a positive semidefinite Hermitian matrix, then

$$\det \begin{pmatrix} \det A_{11} & \cdots & \det A_{1m} \\ \vdots & \ddots & \vdots \\ \det A_{m1} & \cdots & \det A_{mm} \end{pmatrix} \geq \det A.$$

Using this fact, we show the following inequality, which can also be obtained by Schur's inequality [12].

COROLLARY 4.3. *If n is even (resp. odd) and A is a $2n \times 2n$ positive semidefinite Hermitian matrix, then*

$$\frac{1}{\binom{2n-1}{n-1}} \sum_{i=0, i:\text{odd (resp. even)}}^{n-1} d_{(2^i, 1^{2n-2i})}(A) \geq \det A.$$

Proof. Applying Thompson's inequality with $m = 2$, we have

$$\det \begin{pmatrix} \det A[N_1|N_1] & \det A[N_1|N_2] \\ \det A[N_2|N_1] & \det A[N_2|N_2] \end{pmatrix} \geq \det A.$$

Summing up for

$$\frac{1}{2!}\binom{2n}{n} = \binom{2n-1}{n-1},$$

pairs (N_1, N_2), we have

$$\sum_{(N_1,N_2)} \det \begin{pmatrix} \det A[N_1|N_1] & \det A[N_1|N_2] \\ \det A[N_2|N_1] & \det A[N_2|N_2] \end{pmatrix} \geq \binom{2n-1}{n-1} \det A.$$

It is sufficient to apply Theorem 3.4 to the left-hand side. ∎

EXAMPLE 4.4. *If A is a 10×10 positive semidefinite Hermitian matrix, then*

$$\frac{1}{126}\left(d_{(1^{10})}(A) + d_{(2^2,1^6)}(A) + d_{(2^4,1^2)}(A)\right) \geq \det A.$$

Equivalently,

$$d_{(2^2,1^6)}(A) + d_{(2^4,1^2)}(A) - 125 \det A \geq 0.$$

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