Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 595-601, October 2025.

MORE INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES[*]

FENG ZHANG[†] AND HEFANG JING[‡]

Abstract. In this paper, we first present a necessary and sufficient condition for a class of block matrices to be positive semidefinite. Second, we demonstrate the significance of a known inequality (as presented in [5]) through a norm inequality. Finally, utilizing the polar decomposition, we provide a functional version of a singular value inequality.

Key words. Positive semidefinite matrix, Norm, Inequality, Singular value.

AMS subject classifications. 15A18, 15A60, 15A42.

1. Introduction.

Let M_n be the set of n square complex matrices and $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in M_n$. For $A \in M_n$, we use $\lambda_i(A)$ to denote the eigenvalues of A with $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$, the singular values of A are $s_i(A) = \lambda_i(|A|)$, $i = 1, \cdots, n$, where $|A| = (A^*A)^{\frac{1}{2}}$ and A^* is the conjugate transpose of A. Let $A \in M_n$ with $s_1(A) < 1$, we write $\frac{I+A}{I-A}$ to present $(I-A)^{-1}(I+A) = (I+A)(I-A)^{-1}$. Considering A, $B \in M_n$ with singular value vectors $s(A) = (s_1(A), s_2(A), \cdots, s_n(A))$ and $s(B) = (s_1(B), s_2(B), \cdots, s_n(B))$, $s(A) \prec_w s(B)$ holds if

$$\sum_{i=1}^{k} s_i(A) \leq \sum_{i=1}^{k} s_i(B),$$

for $1 \leq k \leq n$ and $s(A) \prec_{w\log} s(B)$ is used to mean that

$$\prod_{i=1}^{k} s_i(A) \leq \prod_{i=1}^{k} s_i(B).$$

For simplicity, $A \geq B$ $(A > B)$ shows that A, B are Hermitian matrices and $A - B$ is a positive semidefinite matrix (positive definite matrix). A matrix $X \in M_n$ is normal if $X^*X = XX^*$. Let $\|\cdot\|$ be a unitarily invariant norm, it is clear that $s(A) \prec_w s(B) \Longleftrightarrow \|A\| \leq \|B\|$.

Let $A \in M_n$, in [3], M. Sababheh, I. Gümüş, and H. Moradi utilized the properties of geometric mean to prove that

$$\begin{bmatrix} |A| & A^* \\ A & |A| \end{bmatrix} \geq 0,$$

---

[*]Received by the editors on December 15, 2024. Accepted for publication on October 15, 2025. Handling Editor: Fuad Kittaneh. Corresponding Author: Feng Zhang.

[†]School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China (fzhang1024@163.com).

[‡]School of Mathematics and Information Science, School of Civil Engineering, North Minzu University, Yinchuan, 750021, China (Jinghef@163.com). Supported by the National Natural Science Foundation of China (12461084) and the Natural Science Foundation of Ningxia, China (2023AAC02049).

if and only if A is normal. We note that $\begin{bmatrix} |A| & A^* \\ A & |A| \end{bmatrix}$ is a special case of $\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix}$ $(0 \leq \alpha \leq 1)$. So whether the sufficient and necessary condition for

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq 0,$$

is A is normal is a question we are interested in this paper. Due to the changes of diagonal elements, using the geometric mean method to solve it is difficult, we consider to give a new method and address this question partially.

Let $A_i \in M_n$ and let g be a nonnegative concave function on $[0, \infty)$, Zhang in [5] obtained

$$\left\| g\left(\left| \sum_{i=1}^{m} A_i \right| \right) \right\| \leq \left\| g\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \left(\sum_{i=1}^m A_i\right)^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \leq \sum_{i=1}^{m} \| g\left(|A_i|\right) \|, \tag{1.1}$$

for $m \geq 2$ as a generalization of the following inequality

$$\|g(|A+B|)\| \leq \left\| g\left(\frac{1}{2} \begin{pmatrix} |A| + |B| & A^* + B^* \\ A + B & |A^*| + |B^*| \end{pmatrix} \right) \right\| \leq \|g(|A|)\| + \|g(|B|)\|, \quad A, B \in M_n, \tag{1.2}$$

shown in [6]. Putting $A = A_1$ and $B = \sum_{i=2}^m A_i$ in inequality (1.2), we obtain an expression which is similar to the left side of inequality (1.1) as follows:

$$\left\| g\left(\left| \sum_{i=1}^{m} A_i \right| \right) \right\| \leq \left\| g\left(\frac{1}{2} \begin{pmatrix} |A_1| + \left|\sum_{i=2}^m A_i\right| & \left(\sum_{i=1}^m A_i\right)^* \\ \sum_{i=1}^m A_i & |A_1^*| + \left|\left(\sum_{i=2}^m A_i\right)^*\right| \end{pmatrix} \right) \right\|. \tag{1.3}$$

Indeed, inequality (1.3) is equivalent to the left side of inequality (1.2). If a fixed relation between

$$\left\| g\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \left(\sum_{i=1}^m A_i\right)^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\|, \tag{1.4}$$

and

$$\left\| g\left(\frac{1}{2} \begin{pmatrix} |A_1| + \left|\sum_{i=2}^m A_i\right| & \left(\sum_{i=1}^m A_i\right)^* \\ \sum_{i=1}^m A_i & |A_1^*| + \left|\left(\sum_{i=2}^m A_i\right)^*\right| \end{pmatrix} \right) \right\|, \tag{1.5}$$

is provided without any further assumptions, we can determine whether inequality (1.1) is a trivial generalization of inequality (1.2). Inspired by this thinking, we give a certain inequality and some numerical examples involving (1.4) and (1.5) under several conditions to suggest no fixed relation between (1.4) and (1.5).

In [7], Lewent gave a well-known inequality, i.e.,

$$\frac{1 + \sum_{i=1}^m \lambda_i x_i}{1 - \sum_{i=1}^m \lambda_i x_i} \leq \prod_{i=1}^{m} \left(\frac{1 + x_i}{1 - x_i} \right)^{\lambda_i}, \tag{1.6}$$

for $x_i \in [0, 1)$, $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Lin in [8] proved a determinantal type of (1.6):

$$\left| \det\left(\frac{1 + \sum_{i=1}^m \lambda_i A_i}{1 - \sum_{i=1}^m \lambda_i A_i} \right) \right| \leq \prod_{i=1}^{m} \det\left(\frac{I + |A_i|}{I - |A_i|} \right)^{\lambda_i}, \tag{1.7}$$

for $s_1(A_i) < 1$, $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Zhang [9] used the log-convexity of $\frac{1+t}{1-t}$ and spectral decomposition to generalize inequality (1.6) and inequality (1.7) to

$$s\left(\frac{1+\sum_{i=1}^m \lambda_i A_i}{1-\sum_{i=1}^m \lambda_i A_i}\right) \prec_{w \log} \prod_{i=1}^m s\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}, \tag{1.8}$$

under the same conditions in inequality (1.7), we suppose that $A_i = U_i\,|A_i|$ are the polar decompositions of A_i, $i = 1, \cdots, m$. Then inequality (1.8) can be restated as

$$s\left(\frac{1+\sum_{i=1}^m \lambda_i U_i\,|A_i|}{1-\sum_{i=1}^m \lambda_i U_i\,|A_i|}\right) \prec_{w \log} \prod_{i=1}^m s\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}.$$

At the last of this paper, we provide a nonnegative functional version of inequality (1.8).

2. Main results. In this part, we first list the following lemmas.

Lemma 2.1. *[1] Let $A \geq B \geq 0$. Then*

$$A^r \geq B^r,$$

for $0 \leq r \leq 1$.

Lemma 2.2. *[3] Let $A \in M_n$, Then A is normal if $|A^*| \leq |A|$.*

Theorem 2.3. *Let $A \in M_n$. Then*

$$\begin{bmatrix} |A|^{2a} & A^* \\ A & |A|^{2(1-a)} \end{bmatrix} \geq 0,$$

for $0 \leq a \leq \frac{1}{2}$ if and only if A is normal.

Proof. Let $A = V\,|A|$ be the polar decomposition of A. Then

$$\begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} |A|^a & 0 \\ |A|^{1-a} & 0 \end{bmatrix}\begin{bmatrix} |A|^a & 0 \\ |A|^{1-a} & 0 \end{bmatrix}^*\begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}^*$$
$$= \begin{bmatrix} |A|^{2a} & A^* \\ A & |A^*|^{2(1-a)} \end{bmatrix} \geq 0,$$

for $0 \leq a \leq \frac{1}{2}$. Since A is normal, we get

$$\begin{bmatrix} |A|^{2a} & A^* \\ A & |A|^{2(1-a)} \end{bmatrix} = \begin{bmatrix} |A|^{2a} & A^* \\ A & |A^*|^{2(1-a)} \end{bmatrix} \geq 0.$$

On the other hand. If

$$\begin{bmatrix} |A|^{2a} & A^* \\ A & |A|^{2(1-a)} \end{bmatrix} \geq 0.$$

Then

$$\begin{bmatrix} |A|^{2a} & |A| \\ |A| & V^*|A|^{2(1-a)}V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}^*\begin{bmatrix} |A|^{2a} & A^* \\ A & |A|^{2(1-a)} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \geq 0,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 595-601, October 2025.

where V is a unitary with $A = V|A|$. For $0 \le a \le \frac{1}{2}$,

$$V^*|A|^{2(1-a)}V - |A|^{2(1-a)} \ge 0, \tag{2.9}$$

is a consequence of

$$\begin{bmatrix} I & 0 \\ -|A|^{1-2a} & I \end{bmatrix} \begin{bmatrix} |A|^{2a} & |A| \\ |A| & V^*|A|^{2(1-a)}V \end{bmatrix} \begin{bmatrix} I & 0 \\ -|A|^{1-2a} & I \end{bmatrix}^* \ge 0.$$

Inequality (2.9) yield that

$$|A|^{2(1-a)} \ge V|A|^{2(1-a)}V^* = |A^*|^{2(1-a)}.$$

An application of Lemma 2.1 reveals that

$$|A^*| \le |A|.$$

Hence, A is normal due to Lemma 2.2. □

Lemma 2.4. *[4] Let* $\begin{bmatrix} A & X \\ X & B \end{bmatrix} \ge 0$ *with* $X \in M_n$. *Then*

$$X \le A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}. \tag{2.10}$$

Lemma 2.5. *[4] Let* $A \ge B \ge 0$ *and let* f *be a nonnegative function on* $[0,\infty)$. *Then*

$$\|f(B)\| \le \|f(A)\|.$$

Next, with the help of the above two lemmas, we give a relation between (1.4) and (1.5).

Theorem 2.6. *Let* A, $B_i \in M_n$ $(1 \le i \le m)$, $B_i = B_i^*$ *and* $\sum_{i=1}^m B_i \ge 0$. *Then*

$$\begin{aligned} &\left\| f\left(\frac{1}{2}\begin{pmatrix} |A| + |\sum_{i=1}^m B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + |\sum_{i=1}^m B_i^*| \end{pmatrix}\right)\right\| \\ \le &\left\| f\left(\frac{1}{2}\begin{pmatrix} |A| + \sum_{i=1}^m |B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + \sum_{i=1}^m |B_i^*| \end{pmatrix}\right)\right\|, \end{aligned}$$

for a nonnegative function f *on* $[0,\infty)$.

Proof. An application of the polar decomposition $B_i = V_i|B_i|$ shows that

$$\begin{pmatrix} |B_i| & B_i^* \\ B_i & |B_i^*| \end{pmatrix} \ge 0.$$

Therefore, due to B_i is Hermitian and

$$\begin{pmatrix} \sum_{i=1}^m |B_i| & \sum_{i=1}^m B_i^* \\ \sum_{i=1}^m B_i & \sum_{i=1}^m |B_i| \end{pmatrix},$$

are positive semidefinite, we obtain

$$\left|\sum_{i=1}^{m} B_i\right| \leq \sum_{i=1}^{m} |B_i|,$$

from Lemma 2.4 and $B_i = B_i^*$, $1 \leq i \leq m$. Hence,

$$\begin{aligned}&\begin{pmatrix} A + |\sum_{i=1}^m B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + |\sum_{i=1}^m B_i^*| \end{pmatrix} \\ \leq &\begin{pmatrix} A + \sum_{i=1}^m |B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + \sum_{i=1}^m |B_i^*| \end{pmatrix}.\end{aligned}$$

By Lemma 2.5, we get

$$\begin{aligned}&\left\| f\left(\frac{1}{2}\begin{pmatrix} |A| + |\sum_{i=1}^m B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + |\sum_{i=1}^m B_i^*| \end{pmatrix}\right)\right\| \\ \leq &\left\| f\left(\frac{1}{2}\begin{pmatrix} |A| + \sum_{i=1}^m |B_i| & A^* + \sum_{i=1}^m B_i^* \\ A + \sum_{i=1}^m B_i & |A^*| + \sum_{i=1}^m |B_i^*| \end{pmatrix}\right)\right\|.\end{aligned}$$

□

Remark 2.7. *In Theorem 2.6, we assume that*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix},$$

and $f(x) = x$, *we observe that* $B \neq B^*$, $C = C^*$ *and calculate that*

$$s_1 \begin{pmatrix} |A| + |B + C| & A^* + (B + C)^* \\ A + B + C & |A^*| + \left|(B + C)^*\right| \end{pmatrix} \approx 4.6852,$$

and

$$s_1 \begin{pmatrix} |A| + |B| + |C| & A^* + (B + C)^* \\ A + B + C & |A^*| + |B^*| + |C^*| \end{pmatrix} \approx 4.6165.$$

This indicates that Theorem (2.6) does not necessarily hold if B_i $(1 \leq i \leq m)$ *are not Hermitian matrices.*

Remark 2.8. *In Theorem 2.6, we set*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix},$$

and $f(x)=x$. *It is clear that* B, C *are Hermitian matrices, but* $B+C$ *is not a positive semidefinite matrix. A small calculation shows that*

$$s_1\begin{pmatrix} |A|+|B+C| & A^*+(B+C)^* \\ A+(B+C) & |A^*|+\left|(B+C)^*\right| \end{pmatrix}\approx 6.1770,$$

and

$$s_1\begin{pmatrix} |A|+|B|+|C| & A^*+(B+C)^* \\ A+(B+C) & |A^*|+|B^*|+|C^*| \end{pmatrix}\approx 6.1438.$$

From this, we see the condition that $\sum_{i=1}^m B_i$ *is a positive semidefinite matrix in Theorem 2.6 is necessary.*

Combining Theorem 2.6, Remarks 2.7 and 2.8, we see that inequality (1.3) is better than inequality (1.1) in certain cases. However, this does not mean that the proposal of inequality (1.1) is no meaningful.

LEMMA 2.9. *[2] Let* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\geq 0$ *with* $X\in M_n$. *Then*

$$s(X)\prec_{w\log} s\left(A^{\frac{1}{2}}\right)s\left(B^{\frac{1}{2}}\right).$$

LEMMA 2.10. *[8] Let* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\geq 0$ *with* $X\in M_n$. *Then, if* $I-A>0$, $I-B>0$, *so is*

$$\begin{bmatrix} \frac{I+A}{I-A} & \frac{I+X^*}{I-X^*} \\ \frac{I+X}{I-X} & \frac{I+B}{I-B} \end{bmatrix}.$$

LEMMA 2.11. *[9] Let* $A_i\geq 0$ *and* $s_1(A)<1$. *Then*

$$s\left(\frac{I+\sum_{i=1}^m\lambda_iA_i}{I-\sum_{i=1}^m\lambda_iA_i}\right)\prec_{w\log}\prod_{i=1}^m s^{\lambda_i}\left(\frac{I+A_i}{I-A_i}\right),$$

where $\lambda_i\geq 0$ *and* $\sum_{i=1}^m\lambda_i=1$.

THEOREM 2.12. *Let* $A_i\in M_n$ *with the polar decomposition* $A_i=U_i|A_i|$ *and* $s_1(A_i)<1$, $i=1,\cdots,m$. *Then*

$$s\left(\frac{I+\sum_{i=1}^m\lambda_iU_if(|A_i|)}{I-\sum_{i=1}^m\lambda_iU_if(|A_i|)}\right)\prec_{w\log}\prod_{i=1}^m s\left(\frac{I+f(|A_i|)}{I-f(|A_i|)}\right)^{\lambda_i},$$

for $f:[0,1)\to[0,1)$, $\lambda_i\geq 0$ *and* $\sum_{i=1}^m\lambda_i=1$.

Proof. Since $f:[0,1)\to[0,1)$ and $A_i=U_i|A_i|$, we have

$$\begin{aligned}&\begin{pmatrix} \sum_{i=1}^m\lambda_if(|A_i|) & \sum_{i=1}^m(\lambda_iU_if(|A_i|))^* \\ \sum_{i=1}^m\lambda_iU_if(|A_i|) & \sum_{i=1}^m\lambda_if(|A_i^*|) \end{pmatrix}\\ &=2\sum_{i=1}^m\lambda_iW_i\begin{pmatrix} f(|A_i|) & 0 \\ 0 & 0 \end{pmatrix}W_i^*\geq 0,\end{aligned}$$

where $W_i=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ U_i & -U_i \end{pmatrix}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 595-601, October 2025.

Then, by Lemma 2.10, we get

$$\begin{pmatrix} \frac{I+\sum_{i=1}^m \lambda_i f(|A_i|)}{I-\sum_{i=1}^m \lambda_i f(|A_i|)} & \frac{I+\sum_{i=1}^m (\lambda_i U_i f(|A_i|))^*}{I-\sum_{i=1}^m (\lambda_i U_i f(|A_i|))^*} \\ \frac{I+\sum_{i=1}^m \lambda_i U_i f(|A_i|)}{I-\sum_{i=1}^m \lambda_i U_i f(|A_i|)} & \frac{I+\sum_{i=1}^m \lambda_i f(|A_i^*|)}{I-\sum_{i=1}^m \lambda_i f\left(\left|A_i^*\right|\right)} \end{pmatrix},$$

is positive semidefinite. Thus,

$$\begin{aligned} & s\left(\frac{I+\sum_{i=1}^m \lambda_i U_i f\left(|A_i|\right)}{I-\sum_{i=1}^m \lambda_i U_i f\left(|A_i|\right)}\right) \\ \prec_{w \log} & s^{\frac{1}{2}}\left(\frac{I+\sum_{i=1}^m \lambda_i f\left(|A_i|\right)}{I-\sum_{i=1}^m \lambda_i f\left(|A_i|\right)}\right) s^{\frac{1}{2}}\left(\frac{I+\sum_{i=1}^m \lambda_i f\left(|A_i^*|\right)}{I-\sum_{i=1}^m \lambda_i f\left(|A_i^*|\right)}\right) \\ \prec_{w \log} & \prod_{i=1}^m s\left(\frac{I+f\left(|A_i|\right)}{I-f\left(|A_i|\right)}\right)^{\frac{\lambda_i}{2}} \prod_{i=1}^m s\left(\frac{I+f\left(|A_i^*|\right)}{I-f\left(|A_i^*|\right)}\right)^{\frac{\lambda_i}{2}} \\ \prec_{w \log} & \prod_{i=1}^m s\left(\frac{I+f\left(|A_i|\right)}{I-f\left(|A_i|\right)}\right)^{\lambda_i}, \end{aligned}$$

where the first inequality is formed by Lemma 2.9, the second inequality is a consequence of Lemma 2.11 and the last formula follows from

$$s_j\left(\frac{I+f\left(|A_i^*|\right)}{I-f\left(|A_i^*|\right)}\right)=s_j\left(\frac{UIU^*+Uf\left(|A_i|\right)U^*}{UIU^*-Uf\left(|A_i|\right)U^*}\right)=s_j\left(\frac{I+f\left(|A_i|\right)}{I-f\left(|A_i|\right)}\right),$$

for any i. ☐

REMARK 2.13. *Putting $f(x)=x$ in Theorem 2.12, we obtain Theorem 7 in [9].*

Due to the limitations of our research method, we cannot solve the problem that Theorem 2.3 holds for $\frac{1}{2} \le a \le 1$. At the end of this paper, we list an open question containing this case:

QUESTION 2.14. *Let $A \in M_n$. Then $\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} \ge 0$ if and only if A is normal under the conditions that $f(x)g(x)=x^2$ for $x \ge 0$, $f(x) \ge g(x)$ $(x \ge 1)$ and $g(x) \ge f(x)$ $(0 \le x \le 1)$.*

Acknowledgment. The authors are grateful to the referee for carefully reading the manuscript and providing constructive comments.

REFERENCES

[1] R. Bhatia. *Positive Definite Matrices.* Princeton University Press, Princeton, 2007.
[2] R. Bhatia. *Matrix Analysis.* Springer, New York, 1997.
[3] M. Sababheh, I. Gümüş, and H. Moradi. Further properties of PPT and (α, β)-normal matrices. *Oper. Matrices*, 18:257–271, 2024.
[4] M. Ghaemi, N. Gharakhanlu, T. Rassias, and R. Saadati. *Advances in Matrix Inequalities.* Springer, New York, 2021.
[5] F. Zhang. Concave function inequalities for sum of matrices. *Electron. J. Linear Algebra*, 40:663–666, 2024.
[6] Y. Zhang. Interpolating the Rotfel'd inequality for unitarily invariant norms. *Linear Algebra Appl.*, 574:60–66, 2019.
[7] L. Lewent. Über einige Ungleichungen. *Sitzungsber. Berl. Math. Ges.*, 7:95–100, 1908.
[8] M. Lin. A Lewent type determinantal inequality. *Taiwanese J. Math.*, 17:1303–1309, 2013.
[9] Y. Zhang. Singular value inequalities of Lewent type. *Taiwanese J. Math.*, 18:895–907, 2014.