Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 152-152, February 2025.

SEMIRINGS IN WHICH THE PERMANENT OF INVERTIBLE MATRICES IS MULTIPLICATIVE*

YAROSLAV SHITOV†

Abstract. We show that, if $1 + 2xy = 1$ holds for all elements x, y with additive inverses in a commutative semiring $\mathcal{S}$, then the function of permanent is multiplicative on the matrices with multiplicative inverses over $\mathcal{S}$.

Key words. Commutative semirings, Matrix inversion.

AMS subject classifications. 15A15, 16Y60.

Let $\mathcal{S}$ be a commutative semiring, and let $V(\mathcal{S})$ be the set of all elements with additive inverses in $\mathcal{S}$. We write $A \in \mathrm{GL}_n(\mathcal{S})$ if an $n \times n$ matrix A has a mutliplicative inverse over $\mathcal{S}$, and its *permanent* is

$$\operatorname{per} A = \sum_{\sigma \in S_n} \left(A_{1\sigma_1} \cdot \ldots \cdot A_{n\sigma_n} \right),$$

where S_n is the symmetric group on $\{1, \ldots, n\}$. In [1], Dolžan observed that, for any $n \geqslant 2$, the condition

$$(\forall A,\, B \in \mathrm{GL}_n(\mathcal{S}) \;\; \operatorname{per}(A \cdot B) = \operatorname{per} A \cdot \operatorname{per} B) \;\; \text{implies} \;\; (\forall x,\, y \in V(\mathcal{S}) \;\; 1 + 2xy = 1)\,, \tag{1}$$

and he conjectured the converse implication [1, page 449]. We recall that, for any $A \in \mathrm{GL}_n(\mathcal{S})$ and i, j, k in $\{1, \ldots, n\}$ with $j \neq k$, the elements $A_{ij} \cdot A_{ik}$ and $A_{ji} \cdot A_{ki}$ are both in $V(\mathcal{S})$, see Proposition 3.1 in [2].

LEMMA 1. *If $A, B \in \mathrm{GL}_n(\mathcal{S})$, then the equality* $\operatorname{per}(A \cdot B) = (\operatorname{per} A) \cdot (\operatorname{per} B) + 2 \cdot (x_1 \cdot y_1 + \ldots + x_k \cdot y_k)$ *is valid for some $x_1, y_1, \ldots, x_k, y_k \in V(\mathcal{S})$.*

Proof. If $i \neq \hat{\imath}$ and $j \neq \hat{\jmath}$, then $(A_{it} \cdot B_{tj}) \cdot (A_{\hat{\imath}t} \cdot B_{t\hat{\jmath}}) = (A_{it} \cdot B_{t\hat{\jmath}}) \cdot (A_{\hat{\imath}t} \cdot B_{tj})$ are products of pairs of additively invertible elements, and hence, in $\operatorname{per}(A \cdot B) = \sum_{\sigma \in S_n} \left(\sum_{i=1}^n A_{1i} \cdot B_{i\sigma_1}\right) \cdot \ldots \cdot \left(\sum_{i=1}^n A_{ni} \cdot B_{i\sigma_n}\right)$, all summands that are not involved in the corresponding expression

$$(\operatorname{per} A) \cdot (\operatorname{per} B) = \textstyle\sum_{\sigma \in S_n} \sum_{\tau \in S_n} \left(A_{1\tau_1} \cdot B_{\tau_1\sigma_1}\right) \cdot \ldots \cdot \left(A_{n\tau_n} \cdot B_{\tau_n\sigma_n}\right)$$

are split into pairs of equal products of the form $x \cdot y$ with x, $y \in V(\mathcal{S})$. ∎

In particular, if $A \in \mathrm{GL}_n(\mathcal{S})$, then $\operatorname{per}(A \cdot A^{-1}) = 1 = \operatorname{per}(A) \cdot \operatorname{per}(A^{-1})$ provided that the condition on the right of (1) is valid. Therefore, $\operatorname{per}(A)$ is a unit, and Lemma 1 implies the condition on the left of (1).

REFERENCES

[1] D. Dolžan. Multiplicativity of permanents over matrix semirings. *Electron. J. Linear Algebra*, 40:446–452, 2024.
[2] Y.-J. Tan. On invertible matrices over commutative semirings. *Linear Multilinear Algebra*, 61:710–724, 2013.

*Received by the editors on December 2, 2024. Accepted for publication on February 3, 2025. Handling Editor: Froilán Dopico. Corresponding Author: Yaroslav Shitov.

†Russia, 129346 Moscow, Izumrudnaya ulitsa, dom 65, kvartira 4 (yaroslav-shitov@yandex.ru).