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AN EXTENSION OF THE PERTURBATION ANALYSIS FOR THE DRAZIN INVERSE*

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Abstract. Let A denote a square complex matrix and let E be a perturbation matrix. The purpose of this paper is to investigate the perturbation of the Drazin inverse when $B = A + E$ satisfies the rank conditions $\operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} A^r B^s$, where r and s denote the indices of A and B, respectively. We will derive an explicit representation of B^D as a function of A and $B^k - A^j$, for certain positive integers j, k. We emphasize that the matrix $I + (A^D)^j(B^k - A^j)$ could be singular and the perturbation analysis will be carried out by using inner inverses. In addition, we exhibit inequalities bounding the errors $\|B^D - A^D\|/\|A^D\|$ and $\|BB^D - AA^D\|$. Examples will be given which show that these bounds recover others given in the literature and can be significant to those cases which can not be bounded using the previous known results. Alternatively, we shall formulate analogous perturbation results for the perturbed matrix B such that $\operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} B^s A^r$.

Key words. Drazin inverse, Inner inverses, Projectors, Perturbation, Upper bound.

AMS subject classifications. 15A09, 65F20, 65F35.

1. Introduction. Let A denote a square complex matrix and let E be a perturbation matrix. Several authors have considered the perturbation of the Drazin inverse and exhibited inequalities bounding the relative error $\|(A+E)^D - A^D\|/\|A^D\|$, under specific conditions [4, 6, 7, 11, 12, 14, 15, 16]. Other papers are concerned with the perturbation of the group inverse, which plays an important role in the theory of Markov finite chains [5, 10].

We recall that the Drazin inverse A^D is the unique matrix X which satisfies the relations:

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k \quad \text{for all } k \geq r,$$

and the index of A, $\operatorname{ind}(A)$, is the smallest non-negative integer k for which the third equation holds. Alternatively, the index of A may be defined to be the smallest non-negative integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$.

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Electronic Journal of Linear Algebra ISSN 1081-3810
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If A is nonsingular then $\operatorname{ind}(A) = 0$ and $A^D = A^{-1}$. If $\operatorname{ind}(A) = 1$, then A^D is called the group inverse of A, denoted by $A^{\#}$.

Let $\operatorname{ind}(A) = r$. We will denote by A^π the spectral projector whose range is $\mathcal{N}(A^r)$ and whose kernel is $\mathcal{R}(A^r)$, i.e., $A^\pi = I - AA^D$, where as for notation, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range space of A and the null space of A, respectively.

In [2], Campbell and Meyer established that if $A + E_j$ converges to A, then $(A+E_j)^D$ converges to A^D if and only if $\operatorname{rank}(A+E_j)^{r_j} = \operatorname{rank} A^r$ for all sufficiently large j, where $r_j = \operatorname{ind}(A + E_j)$.

In [4] the authors investigated perturbations such that $B = A + E$ satisfies the following conditions:

$$(\mathcal{C}_s) \quad \operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} A^r B^s A^r,$$

where $\operatorname{ind}(B) = s$. These rank conditions turn out to be equivalent to the geometrical conditions $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$ and $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$.

In our development, we consider perturbations such that $B = A + E$ satisfies the conditions:

$$(\mathcal{B}_{s,1}) \quad \operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} A^r B^s,$$

or, equivalently, $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$.

Alternatively, we shall consider the perturbed matrix B such that:

$$(\mathcal{B}_{s,2}) \quad \operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} B^s A^r,$$

or, equivalently, $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$.

Our objective is to derive an explicit expression for B^D as a function of A and the perturbation matrix $E_{j,k} = B^k - A^j$, for certain integers j, k. The fundamental fact on which perturbation analysis of previous papers was based is that the matrix $I + (A^D)^j E_{j,k}$ is nonsingular, for some integers j, k. This happens in particular when the perturbation is "small enough", i.e., when $||(A^D)^j E_{j,k}|| < 1$. We are interested in finding perturbation formulas and bounds for the Drazin inverse in a more general setting. If B satisfies either condition $(\mathcal{B}_{s,1})$ or condition $(\mathcal{B}_{s,2})$, in general, we can't guarantee that previous non-singularity condition holds. In theses cases, the $\{1\}$-inverse will play an important role in the perturbation formula for B^D. Continuity properties and perturbation analysis of $\{1\}$-inverses can be found in [8, 13].

We recall that a $\{1\}$-inverse (or inner inverse) of a given matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $X \in \mathbb{C}^{n\times m}$ such that $AXA = A$. We will denote by $A\{1\}$ the set of all $\{1\}$-inverses of A. The symbol A^- will be used to designate an arbitrary element of $A\{1\}$.

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ELA

A $\{1,2\}$-inverse (or inner and outer inverse) of A is a matrix X such that $AXA = A$ and $XAX = X$. We will denote by $A\{1,2\}$ the set of all $\{1,2\}$-inverses of A.

The results in this paper are summarized in the following. Section 2 establishes several characterizations for the perturbed matrix B satisfying condition $(\mathcal{B}_{s,1})$. We give a block matrix representation for the perturbed matrix, which will be the main tool in our further development. On the other hand, we show that we can find inner inverses of the matrices $(A^D)^j B^k$, for all integers j and $k \geq s$, with prescribed projectors AA^D and BB^D. Moreover, we give a formula for the oblique projector $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)}$. This section establishes also the reciprocal results for the perturbed matrix B satisfying condition $(\mathcal{B}_{s,2})$ and finally shows that under both conditions, the characterization result for the class $(\mathcal{C}_s)$ holds.

In Section 3, first it is shown that certain inner inverses of $I + E_{j,k}(A^D)^j$ provide the key for giving an explicit representation of B^D as a function of A and $E_{j,k}$. We derive upper bounds for the relative errors $\|B^D - A^D\|/\|A^D\|$ and $\|B^\pi - A^\pi\|$. Further, we establish the analogous results for perturbations satisfying condition $(\mathcal{B}_{s,2})$. Finally, we give numerical examples to illustrate our results. In case that $I + E_{j,k}(A^D)^j$ is nonsingular, we compare our bound with the upper bound given recently in [16].

Some basic results are given next. We refer to [1, 3] for properties on generalized inverses.

Lemma 1.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = r$.

(a) *If P is nonsingular, then* $(PAP^{-1})^D = PA^DP^{-1}$.
(b) $(A^t)^D = (A^D)^t$, *where A^t is the transpose conjugate of A.*
(c) *The matrix A has a unique decomposition*

$$A = C_A + N_A, \quad \operatorname{ind}(C_A) = 1, \;\; C_AN_A = N_AC_A = 0, \;\; N_A^r = 0. \tag{1.1}$$

Moreover, we have $A^k = C_A^k + N_A^k$ *for all integers* $k \geq 1$, $A^{\mathrm{D}} = C_A^{\#}$ *and* $A^\pi = C_A^\pi$.

(d) $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^D = \begin{pmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{pmatrix}$.

(e) *Let P and Q be idempotent matrices. Then* $\mathcal{N}(P-Q) = (\mathcal{R}(P)\cap\mathcal{R}(Q)) \oplus (\mathcal{N}(P)\cap\mathcal{N}(Q))$.

The following results on ranks of matrices are well-known [9].

Lemma 1.2. *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{n\times p}$, $D \in \mathbb{C}^{n\times k}$, $G \in \mathbb{C}^{p\times k}$. *Then*

(i) $\operatorname{rank}(AG) = \operatorname{rank}(G) - \dim(\mathcal{R}(G) \cap \mathcal{N}(A))$.
(ii) $\operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}\begin{pmatrix} 0 & (I_m - AA^-)B \\ C(I_p - A^-A) & D - CA^-B \end{pmatrix}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

Lemma 1.3. *Let* $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$ *with* R_{11} *nonsingular matrix and* R_{22} *square matrix. Then* R *is nonsingular if and only if* $Z = R_{22} - R_{21}R_{11}^{-1}R_{12}$ *is nonsingular. In this case*

$$R^{-1} = \begin{pmatrix} R_{11}^{-1} + R_{11}^{-1}R_{12}Z^{-1}R_{21}R_{11}^{-1} & -R_{11}^{-1}R_{12}Z^{-1} \\ -Z^{-1}R_{21}R_{11}^{-1} & Z^{-1} \end{pmatrix}.$$

2. Characterization results. From now on, let $\text{ind}(A) = r > 0$. Then there exists a nonsingular matrix P such that we can write A in the core-nilpotent block form

$$A = P \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} P^{-1}, \quad A_1 \in \mathbb{C}^{d\times d} \text{ nonsingular}, \quad d = \text{rank}\, A^r, \quad A_2^r = 0. \tag{2.1}$$

We have divided this section into three parts, according to the assumptions on the perturbed matrix B.

2.1. Condition $(\mathcal{B}_{s,1})$**.** First, we give the following result.

Theorem 2.1. *Let* $M \in \mathbb{C}^{n\times n}$ *in the form* $M = \begin{pmatrix} C(I-TS) & CT \\ SC(I-TS) & SCT \end{pmatrix}$, *where* $C \in \mathbb{C}^{d\times d}$ *is nonsingular. Then*

$$M^k = \begin{pmatrix} C^k(I-TS) & C^kT \\ SC^k(I-TS) & SC^kT \end{pmatrix}, \quad \forall k \geq 1, \tag{2.2}$$

$\text{ind}(M) = 1$, *the group inverse of* M *is*

$$M^{\#} = \begin{pmatrix} C^{-1}(I-TS) & C^{-1}T \\ SC^{-1}(I-TS) & SC^{-1}T \end{pmatrix}, \tag{2.3}$$

and

$$M^{\pi} = \begin{pmatrix} TS & -T \\ -S(I-TS) & I-ST \end{pmatrix}. \tag{2.4}$$

Proof. The expression for the powers of M can be easily proved by induction on k. We can write

$$M = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} C & CT \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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On account that C is nonsingular, we derive that $\operatorname{rank} M = \operatorname{rank} C = d$. Analogously, we see that $\operatorname{rank} M^2 = \operatorname{rank} C^2 = d$. Therefore $\operatorname{ind}(M) = 1$. By Lemma 1.1, properties (a) and (d), we get

$$M^{\#} = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} C^{-1} & C^{-1}T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} = \begin{pmatrix} C^{-1}(I-TS) & C^{-1}T \\ SC^{-1}(I-TS) & SC^{-1}T \end{pmatrix}.$$

Further, the expression (2.4) holds, since $M^\pi = I - M^{\#}M$. ☐

We can now state the characterization result.

THEOREM 2.2. *Let* $A, B \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(B) = s$. *If* $\operatorname{rank} B^s = \operatorname{rank} A^r = d$, *then the following conditions are equivalent:*

(a) $\operatorname{rank} A^r B^s = d$.
(b) $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$.
(c) *If* C_B *is the core part in the decomposition of* B *defined as in (1.1), then, relative to the form (2.1),*

$$C_B = P \begin{pmatrix} C(I-TS) & CT \\ SC(I-TS) & SCT \end{pmatrix} P^{-1}, \quad C \in \mathbb{C}^{d\times d} \text{ is nonsingular.} \tag{2.5}$$

(d) $\operatorname{rank}(I - A^\pi - B^\pi) = n - d + \operatorname{rank} B^s A^r$.

Proof. (a) ⇔ (b): By Lemma 1.2, (i),

$$\operatorname{rank} A^r B^s = \operatorname{rank} B^s - \dim(\mathcal{R}(B^s) \cap \mathcal{N}(A^r)).$$

Hence it follows that $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$ if and only if $\operatorname{rank} A^r B^s = \operatorname{rank} B^s$.

(a) ⇒ (c): Firstly, we note that $\operatorname{ind}(C_B) = 1$. Then there exists a nonsingular matrix Q such that

$$C_B = Q \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}, \quad B_1 \text{ nonsingular.}$$

Condition $\operatorname{rank} B^s = \operatorname{rank} A^r = d$ holds if and only if the size of B_1 is $d \times d$.

Now, let $R = P^{-1}Q = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$ where $R_{11} \in \mathbb{C}^{d\times d}$. Then

$$A^r B^s = P \begin{pmatrix} A_1^r R_{11} B_1^S & 0 \\ 0 & 0 \end{pmatrix} R^{-1} P^{-1}.$$

Since $\operatorname{rank} A^r B^s = d$ it follows that $\operatorname{rank} A_1^r R_{11} B_1^s = d$ and, thus, R_{11} is nonsingular. From Lemma 1.3, by denoting $Z = R_{22} - R_{21}R_{11}^{-1}R_{12}$, it follows that Z is nonsingular

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Volume 22, pp. 539-556, May 2011

ELA

and

$$P^{-1}C_BP = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11}^{-1} + R_{11}^{-1}R_{12}Z^{-1}R_{21}R_{11}^{-1} & -R_{11}^{-1}R_{12}Z^{-1} \\ -Z^{-1}R_{21}R_{11}^{-1} & Z^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} C(I-TS) & CT \\ SC(I-TS) & SCT \end{pmatrix},$$

where $C = R_{11}B_1R_{11}^{-1}$, $T = -R_{12}Z^{-1}$ and $S = R_{21}R_{11}^{-1}$. We note that C is a nonsingular matrix.

(c) ⇒ (d): Since $B^\pi = C_B^\pi$, using formula (2.4), we have

$$I - A^\pi - B^\pi = P \begin{pmatrix} I-TS & T \\ S(I-TS) & -(I-ST) \end{pmatrix} P^{-1}.$$

Taking into account that

$$\begin{pmatrix} I-TS & T \\ -S & I \end{pmatrix} \begin{pmatrix} I-TS & T \\ S(I-TS) & -(I-ST) \end{pmatrix} = \begin{pmatrix} I-TS & 0 \\ 0 & -I \end{pmatrix}$$

and, noting that the first matrix on left hand side is nonsingular, we conclude that

$$\operatorname{rank}(I - A^\pi - B^\pi) = \operatorname{rank}(I - TS) + n - d. \tag{2.6}$$

On the other hand,

$$B^sA^r = C_B^sA^r = P \begin{pmatrix} C^s(I-TS)A_1^r & 0 \\ SC^s(I-TS)A_1^r & 0 \end{pmatrix} P^{-1}.$$

Hence it follows that

$$\operatorname{rank} B^sA^r = \operatorname{rank}(I - TS). \tag{2.7}$$

From (2.6) and (2.7) we get $\operatorname{rank}(I - A^\pi - B^\pi) = \operatorname{rank} B^sA^r + n - d$.

(d) ⇒ (b): Using Lemma 1.2, (i), we have

$$\operatorname{rank}(I - B^\pi - A^\pi) - (n-d) = \operatorname{rank} B^sA^r = \operatorname{rank} A^r - \dim(\mathcal{R}(A^r) \cap \mathcal{N}(B^s)).$$

Hence $\operatorname{rank}(I - B^\pi - A^\pi) = n - \dim(\mathcal{R}(A^r) \cap \mathcal{N}(B^s))$ and, thus, $\dim \mathcal{N}(I - B^\pi - A^\pi) = \dim(\mathcal{R}(A^r) \cap \mathcal{N}(B^s))$. From Lemma 1.1, property (e), it follows that $\mathcal{N}(I - B^\pi - A^\pi) = (\mathcal{N}(A^r) \cap \mathcal{R}(B^s)) \oplus (\mathcal{R}(A^r) \cap \mathcal{N}(B^s))$, Therefore, $\mathcal{N}(A^r) \cap \mathcal{R}(B^s) = \{0\}$. □

In the remainder of this subsection, we make the assumptions: $\operatorname{ind}(B) = s$ and B satisfies condition $(\mathcal{B}_{s,1})$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

From Theorem 2.2, it follows that $\mathbb{C}^n = \mathcal{R}(B^s) \oplus \mathcal{N}(A^r)$ and, thus, we can consider the oblique projector $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)}$. Moreover, applying Theorem 2.2 and using (2.2), we have

$$B^k = C_B^k = P \begin{pmatrix} C^k(I-TS) & C^kT \\ SC^k(I-TS) & SC^kT \end{pmatrix} P^{-1}, \quad k \geq s. \tag{2.8}$$

The following result shows that there exists a $\{1,2\}$-inverse of $(A^D)^j B^k$ ($j \geq 1, k \geq s$) such that its prescribed projectors are AA^D and BB^D. Further, it provides an expression for $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)}$.

THEOREM 2.3. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(B) = s$. *Let us denote $L_{j,k} = (A^D)^j B^k$ for arbitrary $j \geq 1$ and $k \geq s$. If B satisfies condition $(\mathcal{B}_{s,1})$, then*

(i) *There exists a matrix $X_{j,k} \in L_{j,k}\{1,2\}$ satisfying*

$$L_{j,k}X_{j,k} = AA^D, \quad X_{j,k}L_{j,k} = BB^D. \tag{2.9}$$

(ii) $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)} = B^k L_{j,k}^-(A^D)^j$, *(the right hand side is independent of the choice of the $\{1\}$-inverse $L_{j,k}^-$ and the subindices j, k). Moreover*

$$\Sigma := A^\pi P_{\mathcal{R}(B^s),\mathcal{N}(A^r)} = P \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} P^{-1}, \tag{2.10}$$

with respect to the form (2.8).

Proof. (i) We have that B^k has the expression (2.8) and, therefore,

$$L_{j,k} = P \begin{pmatrix} A_1^{-j}C^k(I-TS) & A_1^{-j}C^kT \\ 0 & 0 \end{pmatrix} P^{-1}.$$

Now, we consider

$$X_{j,k} = P \begin{pmatrix} C^{-k}A_1^j & 0 \\ SC^{-k}A_1^j & 0 \end{pmatrix} P^{-1}. \tag{2.11}$$

Next, we prove that $X_{j,k}$ satisfies (2.9). We have

$$L_{j,k}X_{j,k} = P \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} P^{-1} = AA^D, \quad X_{j,k}L_{j,k} = P \begin{pmatrix} I-TS & T \\ S(I-TS) & ST \end{pmatrix} P^{-1} = BB^D,$$

where the last identity is followed by Theorem 2.1, (2.4). It remains to check that $X_{j,k}$ is a $\{1,2\}$-inverse of $L_{j,k}$, but the proof is straightforward.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

ELA

(ii) Since $X_{j,k}$ defined as (2.11) is a $\{1\}$-inverse of $L_{j,k}$, then $L_{j,k}\{1\} = \{X_{j,k} + H - X_{j,k}L_{j,k}HL_{j,k}X_{j,k} : H \in \mathbb{C}^{n\times n}\}$ [3, Theorem 6.3.3]. If we write

$$H = P\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} P^{-1},$$

then any $\{1\}$-inverse $L^-_{j,k}$ of $L_{j,k}$ has the form

$$L^-_{j,k} = P\begin{pmatrix} H_{11} + \Omega_{j,k} & H_{12} \\ H_{21} + S\Omega_{j,k} & H_{22} \end{pmatrix} P^{-1},$$

where $\Omega_{j,k} = C^{-k}A_1^j - ((I - TS)H_{11} + TH_{21})$ and the submatrices $H_{i,l}$, for $i, l = 1, 2$, are arbitrary.

Further, we get $B^kL^-_{j,k}(A^D)^j = P\begin{pmatrix} I & 0 \\ S & 0 \end{pmatrix} P^{-1}$. It is clear that this expression is an idempotent matrix of rank d, and it is independent of the choice of the $\{1\}$-inverse $L^-_{j,k}$ and the subindices j, k with $k \geq s$. We also note that $\mathcal{R}(B^kL^-_{j,k}(A^D)^j) \subseteq \mathcal{R}(B^s)$ for $k \geq s$. Since both subspaces have dimension d we conclude that they are equal. Analogously, $\mathcal{N}(A^r) \subseteq \mathcal{N}(B^kL^-_{j,k}(A^D)^j)$ and, since both subspaces have dimension $n - d$ we get that they are equal. Therefore, $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)} = B^kL^-_{j,k}(A^D)^j$. Finally (2.10) follows. □

2.2. Condition ($\mathcal{B}_{s,2}$). Next, we state the counterparts of preceding theorems. These results may be proved in much the same way as their analogous results.

COROLLARY 2.4. *Let* $\hat{M} = \begin{pmatrix} (I - \hat{T}\hat{S})\hat{C} & (I - \hat{T}\hat{S})\hat{C}\hat{T} \\ \hat{S}\hat{C} & \hat{S}\hat{C}\hat{T} \end{pmatrix}$, $\hat{C} \in \mathbb{C}^{d\times d}$ *is nonsingular. Then*

$$\hat{M}^k = \begin{pmatrix} (I - \hat{T}\hat{S})\hat{C}^k & (I - \hat{T}\hat{S})\hat{C}^k\hat{T} \\ \hat{S}\hat{C}^k & \hat{S}\hat{C}^k\hat{T} \end{pmatrix}, \quad \forall k \geq 1,$$

$\mathrm{ind}(\hat{M}) = 1$,

$$\hat{M}^\# = \begin{pmatrix} (I - \hat{T}\hat{S})\hat{C}^{-1} & (I - \hat{T}\hat{S})\hat{C}^{-1}\hat{T} \\ \hat{S}\hat{C}^{-1} & \hat{S}\hat{C}^{-1}\hat{T} \end{pmatrix}, \quad \hat{M}^\pi = \begin{pmatrix} \hat{T}\hat{S} & -(I - \hat{T}\hat{S})\hat{T} \\ -\hat{S} & I - \hat{S}\hat{T} \end{pmatrix}.$$

COROLLARY 2.5. *Let* $A, B \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(B) = s$. *If* $\mathrm{rank}\, B^s = \mathrm{rank}\, A^r = d$, *then the following conditions are equivalent:*

(a) $\mathrm{rank}\, B^sA^r = d$.
(b) $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

ELA

(c) *If $\hat{C}_B$ is the core part in the decomposition of B defined as in (1.1), then, relative to the form (2.1),*

$$\hat{C}_B = P\begin{pmatrix}(I-\hat{T}\hat{S})\hat{C} & (I-\hat{T}\hat{S})\hat{C}\hat{T}\\ \hat{S}\hat{C} & \hat{S}\hat{C}\hat{T}\end{pmatrix}P^{-1}, \quad \hat{C}\in\mathbb{C}^{d\times d} \text{ is nonsingular.}$$

(d) $\operatorname{rank}(I-A^\pi-B^\pi) = n-d+\operatorname{rank}A^rB^s$.

When B satisfies condition $(\mathcal{B}_{s,2})$, by Corollary 2.4 we have

$$B^k = \hat{C}_B^k = P\begin{pmatrix}(I-\hat{T}\hat{S})\hat{C}^k & (I-\hat{T}\hat{S})\hat{C}^k\hat{T}\\ \hat{S}\hat{C}^k & \hat{S}\hat{C}^k\hat{T}\end{pmatrix}P^{-1}, \quad \forall k\geq s. \tag{2.12}$$

Corollary 2.6. *Let $A, B\in\mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(A)=r$, $\operatorname{ind}(B)=s$ *and B satisfies condition $(\mathcal{B}_{s,2})$. Let us denote $\hat{L}_{j,k} = B^k(A^D)^j$. Then, for arbitrary $j\geq 1$ and $k\geq s$,*

(i) *There exists a matrix $\hat{X}_{j,k}\in\hat{L}_{j,k}\{1,2\}$ satisfying*

$$\hat{X}_{j,k}\hat{L}_{j,k} = AA^D, \quad \hat{L}_{j,k}\hat{X}_{j,k} = BB^D.$$

(ii) $P_{\mathcal{R}(A^r),\mathcal{N}(B^s)} = (A^D)^j\hat{L}_{j,k}^-B^k$ *(the right hand side is independent of the choice of the $\{1\}$-inverse $\hat{L}_{j,k}^-$ and the subindices j,k). Moreover,*

$$\hat{\Sigma} := P_{\mathcal{R}(A^r),\mathcal{N}(B^s)}A^\pi = P\begin{pmatrix}0 & \hat{T}\\ 0 & 0\end{pmatrix}P^{-1}.$$

2.3. Condition $(\mathcal{B}_{s,1})$ and $(\mathcal{B}_{s,2})$. From Theorem 2.2 and Corollary 2.5 we get the following characterization for a matrix B satisfying condition $(\mathcal{C}_s)$, which was established in [4].

Corollary 2.7. *Let $A, B\in\mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(A)=r$ *and* $\operatorname{ind}(B)=s$. *If* $\operatorname{rank}B^s=\operatorname{rank}A^r=d$, *then the following conditions are equivalent:*

(a) $\operatorname{rank}A^rB^s = \operatorname{rank}B^sA^r = d$.
(b) $\mathcal{R}(B^s)\cap\mathcal{N}(A^r)=\{0\}$ *and* $\mathcal{N}(B^s)\cap\mathcal{R}(A^r)=\{0\}$.
(c) *If $\widetilde{C}_B$ is the core part in the decomposition of B defined as in (1.1), then, relative to the form (2.1),*

$$\widetilde{C}_B = P\begin{pmatrix}\widetilde{B}_1 & \widetilde{B}_1\widetilde{T}\\ \widetilde{S}\widetilde{B}_1 & \widetilde{S}\widetilde{B}_1\widetilde{T}\end{pmatrix}P^{-1}, \quad \widetilde{B}_1\in\mathbb{C}^{d\times d} \text{ is nonsingular.} \tag{2.13}$$

(d) $I-A^\pi-B^\pi$ *is nonsingular.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

Proof. From Theorem 2.2 and Corollary 2.5 it follows that (a) and (b) are equivalent. Now, we prove that (a)⇔(c). We have that C_B has the block form (2.5) and, in view of (2.7), $\operatorname{rank}(I-TS) = \operatorname{rank} B^sA^r = d$. Therefore $I-TS$ is nonsingular. By denoting $\widetilde{B}_1 = C(I-TS)$, $\widetilde{S} = S$ and $\widetilde{T} = (I-TS)^{-1}T$ we get (2.13). The converse part is clear.

Next, we will prove that (a)⇔(d). By Theorem 2.2, equivalence between (a) and (d) given therein, it follows that $\operatorname{rank}(I - A^\pi - B^\pi) = n - d + \operatorname{rank} B^sA^r$ and, thus, using that $\operatorname{rank} B^sA^r = d$, we conclude that $I - A^\pi - B^\pi$ is nonsingular. Conversely, if $I - A^\pi - B^\pi$ is nonsingular, then condition (d) in Theorem 2.2 implies that $\operatorname{rank} B^sA^r = d$ and condition (d) in Corollary 2.5 implies $\operatorname{rank} A^rB^s = d$. Consequently (a) holds. ∎

3. Perturbation results. We split this section in three parts. The first subsection provides an explicit representation of B^D, under condition $(\mathcal{B}_{s,1})$, and upper bounds for the relative errors $\|B^D - A^D\|/\|A^D\|$ and $\|B^\pi - A^\pi\|$. The second subsection establishes the analogous results for perturbations satisfying condition $(\mathcal{B}_{s,2})$. Finally, the third subsection gives numerical examples.

3.1. Condition $(\mathcal{B}_{s,1})$. Throughout this subsection, we assume that B satisfies condition $(\mathcal{B}_{s,1})$, and so, the powers of B, $B^k = C_B^k$ for $k \geq s$, have the expression (2.8), and by (2.3),

$$B^D = C_B^\# = P \begin{pmatrix} C^{-1}(I-TS) & C^{-1}T \\ SC^{-1}(I-TS) & SC^{-1}T \end{pmatrix} P^{-1}. \tag{3.1}$$

Set $E_{j,k} = B^k - A^j$. The next theorem expresses a link between the oblique projector $P_{\mathcal{R}(B^s),\mathcal{N}(A^r)}$, as in Theorem 2.3, and the perturbations $I + E_{j,k}(A^D)^j$.

THEOREM 3.1. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(A) = r$, $\operatorname{ind}(B) = s$ *and B satisfies condition $(\mathcal{B}_{s,1})$. Let us denote $F_{j,k} = I + E_{j,k}(A^D)^j$. If $\mathcal{R}(I - F_{j,k}F_{j,k}^-) \subseteq \mathcal{R}(B^k)$ then, for arbitrary $j \geq 1$ and $k \geq s$,*

$$\Sigma = -A^\pi(I + E_{j,k}(A^D)^j)^- AA^D = A^\pi P_{\mathcal{R}(B^s),\mathcal{N}(A^r)}. \tag{3.2}$$

Proof. Using (2.8) we get

$$F_{j,k} = P \begin{pmatrix} C^k(I-TS)A_1^{-j} & 0 \\ SC^k(I-TS)A_1^{-j} & I \end{pmatrix} P^{-1}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 539-556, May 2011

A $\{1\}$-inverse of the above matrix is given by $X_{j,k} = P \begin{pmatrix} A_1^j(I-TS)^-C^{-k} & 0 \\ -S & I \end{pmatrix} P^{-1}$. We have

$$F_{j,k}X_{j,k} = P \begin{pmatrix} \Delta_k & 0 \\ S(\Delta_k - I) & I \end{pmatrix} P^{-1}, \quad X_{j,k}F_{j,k} = P \begin{pmatrix} \Psi_j & 0 \\ 0 & I \end{pmatrix} P^{-1},$$

where $\Delta_k = C^k(I-TS)(I-TS)^-C^{-k}$ and $\Psi_j = A_1^j(I-TS)^-(I-TS)A_1^{-j}$. The set of all $\{1\}$-inverses of $F_{j,k}$ is $F_{j,k}\{1\} = \{X_{j,k} + H - X_{j,k}F_{j,k}HF_{j,k}X_{j,k} : H \in \mathbb{C}^{n\times n}\}$. We write $H = P \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} P^{-1}$. Then for $F_{j,k}^- \in F_{j,k}\{1\}$ we have

$$F_{j,k}^- = P \begin{pmatrix} A_1^j(I-TS)^-C^{-k} + \Omega_{j,k} & (I-\Psi_j)H_{12} \\ -S + (H_{21} + H_{22}S)(I-\Delta_k) & I \end{pmatrix} P^{-1},$$

where $\Omega_{j,k} = H_{11} + \Psi_j(H_{12}S(I-\Delta_k) - H_{11}\Delta_k)$. Further,

$$I - F_{j,k}F_{j,k}^- = P \begin{pmatrix} \Phi_{j,k} & 0 \\ S\Phi_{j,k} - (H_{21} + H_{22}S)(I-\Delta_k) & 0 \end{pmatrix} P^{-1},$$

where $\Phi_{j,k} = \left(I - C^k(I-TS)A_1^{-j}(H_{11} + H_{12}S)\right)(I-\Delta_k)$. Using (3.1) we obtain

$$BB^D(I - F_{j,k}F_{j,k}^-) = P \begin{pmatrix} \Phi_{j,k} - T(H_{21} + H_{22}S)(I-\Delta_k) & 0 \\ S(\Phi_{j,k} - T(H_{21} + H_{22}S)(I-\Delta_k)) & 0 \end{pmatrix} P^{-1}.$$

Hence, $BB^D(I - F_{j,k}F_{j,k}^-) = I - F_{j,k}F_{j,k}^-$ iff $(H_{21} + H_{22}S)(I-\Delta_k) = 0$. Therefore, for all $F_{j,k}^-$ such that $\mathcal{R}(I - F_{j,k}F_{j,k}^-) \subseteq \mathcal{R}(B^k)$, we get (2.10) and, consequently, (3.2) holds. ◻

Remark 3.2. If B satisfies both condition $(B_{s,1})$ and $(B_{s,2})$, then $I + E_{j,k}(A^D)^j$ is invertible. In this case

$$\Sigma = -A^\pi(I + E_{j,k}(A^D)^j)^{-1}AA^D.$$

Theorem 3.3. *Let* $A, B \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(B) = s$. *If* B *satisfies condition* $(\mathcal{B}_{s,1})$, *then, for arbitrary* $j \geq 1$ *and* $k \geq s$,

$$\begin{aligned} B^D &= (I+\Sigma)(I+\Gamma_{j+1,k+1})^{-1}A^D(I+\Gamma_{j,k})(I-\Sigma), \\ B^\pi &= (I+\Sigma)(I+\Gamma_{j,k})^{-1}A^\pi(I+\Gamma_{j,k})(I-\Sigma), \end{aligned} \tag{3.3}$$

where Σ *is as in Theorem 3.1 and*

$$\Gamma_{j,k} = (A^D)^jE_{j,k}(I+\Sigma). \tag{3.4}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 539-556, May 2011

Proof. Using expressions (2.8) and (3.2) we get

$$I + \Gamma_{j,k} = P \begin{pmatrix} A_1^{-j} C^k & A_1^{-j} C^k T \\ 0 & I \end{pmatrix} P^{-1}, \quad j \geq 1, k \geq s.$$

Hence,

$$\begin{aligned}(I + \Gamma_{j+1,k+1})^{-1} &= P \begin{pmatrix} A_1^{-(j+1)} C^{k+1} & A_1^{-(j+1)} C^{k+1} T \\ 0 & I \end{pmatrix}^{-1} P^{-1} \\ &= P \begin{pmatrix} C^{-(k+1)} A_1^{(j+1)} & -T \\ 0 & I \end{pmatrix} P^{-1}.\end{aligned}$$

Further,

$$(I + \Gamma_{j+1,k+1})^{-1} A^D (I + \Gamma_{j,k}) = P \begin{pmatrix} C^{-1} & C^{-1}T \\ 0 & 0 \end{pmatrix} P^{-1}$$

and, with the notation $\Delta = (I + \Sigma)(I + \Gamma_{j+1,k+1})^{-1} A^D (I + \Gamma_{j,k})(I - \Sigma)$,

$$\begin{aligned}\Delta &= P \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} C^{-1} & C^{-1}T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} P^{-1} \\ &= P \begin{pmatrix} C^{-1}(I - TS) & C^{-1}T \\ SC^{-1}(I - TS) & SC^{-1}T \end{pmatrix} P^{-1}.\end{aligned}$$

Hence, in view of (3.1) we conclude $B^D = C_B^{\#} = \Delta$. In the same manner we can see that B^π has the expression given in (3.3). ◻

Remark 3.4. A perturbation formula for $X_{j,k} \in L_{j,k}\{1,2\}$ with prescribed projectors AA^D and BB^D, as in Theorem 2.3, is

$$X_{j,k} = B^k L_{j,k}^- (A^D)^j (I + \Gamma_{j,k})^{-1} AA^D = (I + \Sigma)(I + \Gamma_{j,k})^{-1} AA^D,$$

where Σ and $\Gamma_{j,k}$ as in Theorem 3.3.

In the remainder of this section we consider that $||\cdot||$ is any norm on $\mathbb{C}^{n\times n}$. The following theorems provide upper bounds for the perturbation of the Drazin inverse and the perturbation of the projector.

Theorem 3.5. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\mathrm{ind}(A) = r$, $\mathrm{ind}(B) = s$ *and B satisfies condition $(\mathcal{B}_{s,1})$. Set $\Omega_{j,k} = (A^D)^{j+1}(B^k - B^{k+1}(I + \Sigma)A^D)$ and let Σ and $\Gamma_{j,k}$ be defined as in Theorem 3.3. Then, for arbitrary $j \geq 1$ and $k \geq s$,*

$$\frac{||B^D - A^D||}{||A^D||} \leq \frac{||\Sigma A^D||}{||A^D||} + \frac{||I + \Sigma|| ||(I + \Gamma_{j+1,k+1})^{-1}|| ||\Omega_{j,k}||}{||A^D||}. \tag{3.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 539-556, May 2011

Furthermore, if $||\Gamma_{j+1,k+1}|| < 1$ *then*

$$\frac{||B^D - A^D||}{||A^D||} \leq \frac{||\Sigma A^D||}{||A^D||} + \frac{||I+\Sigma||||\Omega_{j,k}||}{(1-||\Gamma_{j+1,k+1}||)||A^D||}. \tag{3.6}$$

Proof. By Theorem 3.3, first relation in (3.3), it follows that

$$\begin{aligned} B^D - (I+\Sigma)A^D(I-\Sigma) &= (I+\Sigma)(I+\Gamma_{j+1,k+1})^{-1}(A^D(I+\Gamma_{j,k}) \\ &\quad - (I+\Gamma_{j+1,k+1})A^D)(I-\Sigma) \\ &= (I+\Sigma)(I+\Gamma_{j+1,k+1})^{-1}\Omega_{j,k}. \end{aligned}$$

Now, on account that $(I+\Sigma)A^D(I-\Sigma) = A^D + \Sigma A^D$, we have

$$B^D - A^D = \Sigma A^D + (I+\Sigma)(I+\Gamma_{j+1,k+1})^{-1}\Omega_{j,k}.$$

Hence, taking norms we get (3.5).

If $||\Gamma_{j+1,k+1}|| < 1$ and, in consequence, $||(I+\Gamma_{j+1,k+1}))^{-1}|| \leq 1/(1-||\Gamma_{j+1,k+1}||)$, then the upper bound (3.6) follows. □

THEOREM 3.6. *Let* $A, B \in \mathbb{C}^{n\times n}$ *be such that* $\text{ind}(A) = r$, $\text{ind}(B) = s$ *and* B *satisfies condition* $(\mathcal{B}_{s,1})$. *Let* Σ *and* $\Gamma_{j,k}$ *be defined as in Theorem 3.3. Then, for arbitrary* $j \geq 1$ *and* $k \geq s$,

$$||B^\pi - A^\pi|| \leq \|\Sigma\| + (1+\|\Sigma\|)^2\|(A^D)^jE_{j,k}A^\pi\|\|(I+\Gamma_{j,k})^{-1}\|. \tag{3.7}$$

Moreover, if $\|\Gamma_{j,k}\| < 1$, *then*

$$||B^\pi - A^\pi|| \leq \|\Sigma\| + \frac{(1+\|\Sigma\|)^2\|(A^D)^jE_{j,k}A^\pi\|}{1-\|\Gamma_{j,k}\|}. \tag{3.8}$$

Proof. By Theorem 3.3, second relation in (3.3), it follows that

$$\begin{aligned} B^\pi - (I+\Sigma)A^\pi(I-\Sigma) &= (I+\Sigma)(I+\Gamma_{j,k})^{-1}(A^\pi\Gamma_{j,k} - \Gamma_{j,k}A^\pi)(I-\Sigma) \\ &= -(I+\Sigma)(I+\Gamma_{j,k})^{-1}(A^D)^jE_{j,k}A^\pi(I-\Sigma). \end{aligned}$$

Now, on account of $(I+\Sigma)A^\pi(I-\Sigma) = A^\pi - \Sigma$, we have

$$B^\pi - A^\pi = -\Sigma - (I+\Sigma)(I+\Gamma_{j,k})^{-1}(A^D)^jE_{j,k}A^\pi(I-\Sigma).$$

Taking norms, (3.7) follows. The upper bound (3.8) is clear. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 539-556, May 2011

3.2. Condition ($\mathcal{B}_{s,2}$). In this subsection, we consider that B satisfies condition ($\mathcal{B}_{s,2}$) and, thus, the powers of $B^k = \hat{C}_B^k$ have the expression (2.12), which can be used to state the reciprocal results of the previous subsection. Their proofs are omitted but they can be derived easily taking conjugate transpose and applying property (b) in Lemma 1.1.

COROLLARY 3.7. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\mathrm{ind}(A) = r$, $\mathrm{ind}(B) = s$ *and B satisfies condition ($\mathcal{B}_{s,2}$). Let us denote $\hat{F}_{j,k} = I + (A^D)^j E_{j,k}$. If $\mathcal{N}(B^k) \subseteq \mathcal{N}(I - \hat{F}_{j,k}^- \hat{F}_{j,k})$ then, for arbitrary $j \geq 1$ and $k \geq s$,*

$$\hat{\Sigma} = -AA^D(I + (A^D)^j E_{j,k})^- A^\pi = P_{\mathcal{R}(A^r),\mathcal{N}(B^s)} A^\pi. \tag{3.9}$$

COROLLARY 3.8. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\mathrm{ind}(A) = r$, $\mathrm{ind}(B) = s$ *and B satisfies condition ($\mathcal{B}_{s,2}$). Then, for arbitrary $j \geq 1$ and $k \geq s$,*

$$B^D = (I - \hat{\Sigma})(I + \hat{\Gamma}_{j+1,k+1})^{-1} A^D (I + \hat{\Gamma}_{j,k})(I + \hat{\Sigma}),$$
$$B^\pi = (I - \hat{\Sigma})(I + \hat{\Gamma}_{j,k})^{-1} A^\pi (I + \hat{\Gamma}_{j,k})(I + \hat{\Sigma}),$$

where $\hat{\Sigma}$ is as in Corollary 3.7 and

$$\hat{\Gamma}_{j,k} = (I + \hat{\Sigma}) E_{j,k} (A^D)^j. \tag{3.10}$$

COROLLARY 3.9. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that* $\mathrm{ind}(A) = r$, $\mathrm{ind}(B) = s$ *and B satisfies condition ($\mathcal{B}_{s,2}$). Set $\hat{\Omega}_{j,k} = (B^k - A^D(I + \hat{\Sigma})B^{k+1})(A^D)^{j+1}$ and let $\hat{\Sigma}$ and $\hat{\Gamma}_{j,k}$ be defined as in (3.9) and (3.10), respectively. Then, for arbitrary $j \geq 1$ and $k \geq s$,*

$$\frac{||B^D - A^D||}{||A^D||} \leq \frac{||A^D\hat{\Sigma}||}{||A^D||} + \frac{||I + \hat{\Sigma}||||(I + \hat{\Gamma}_{j+1,k+1})^{-1}||||\hat{\Omega}_{j,k}||}{||A^D||}$$

and

$$||B^\pi - A^\pi|| \leq \|\hat{\Sigma}\| + (1 + \|\hat{\Sigma}\|)^2 \|A^\pi E_{j,k}(A^D)^j\| \|(I + \hat{\Gamma}_{j,k})^{-1}\|.$$

Further, if $||\hat{\Gamma}_{j+1,k+1}|| < 1$ then

$$\frac{||B^D - A^D||}{||A^D||} \leq \frac{||A^D\hat{\Sigma}||}{||A^D||} + \frac{||I + \hat{\Sigma}||||\hat{\Omega}_{j,k}||}{(1 - ||\hat{\Gamma}_{j+1,k+1}||)||A^D||}.$$

Moreover, if $||\hat{\Gamma}_{j,k}|| < 1$ then

$$||B^\pi - A^\pi|| \leq \|\hat{\Sigma}\| + \frac{(1 + \|\hat{\Sigma}\|)^2 \|A^\pi E_{j,k}(A^D)^j\|}{1 - \|\hat{\Gamma}_{j,k}\|}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

3.3. Numerical examples. The first and second examples do not satisfy restrictions stated in recent papers where the perturbation was studied [4, 6, 11, 14, 16]. The upper bounds in these examples extend the perturbation analysis to a wide class of matrices. The third example satisfies conditions in this paper and conditions in [16], for some cases, and we will make a comparison of the upper bounds associated.

EXAMPLE 3.10. Let $A \in \mathbb{R}^{4\times 4}$ be the diagonal matrix defined by $A = \text{diag}\ (10^{-4}, 1, 1, 0)$, for which $A^{\#} = \text{diag}\ (10^{4}, 1, 1, 0)$, and let

$$B = \begin{bmatrix} 0 & 0 & 0 & 10^{-5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10^{-5} \end{bmatrix}, \quad B^{\#} = \begin{bmatrix} 0 & 0 & 0 & 10^{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10^{5} \end{bmatrix}.$$

We have $\text{ind}(A) = \text{ind}(B) = 1$, $\text{rank}\, A = \text{rank}\, B = \text{rank}\, AB = 3$ and $\text{rank}\, BA = 2$. The upper bounds given in [16] are not valid for this case. The block matrices in the representation of $B = C_B$, given in (2.5), are $S = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^t$ and $C = \text{diag}\ (10^{-5}, 1, 1)$.

Let $E = B - A$. We note that $||E||_1 = 10^{-4}$, but since $I + EA^{\#}$ and $I + A^{\#}E$ are singular matrices we can't apply the upper bounds given in [4, 14]. Since $\|\Gamma_{2,2}\|_i < 1$, $i = 1, 2$, where $\Gamma_{j,k}$ is defined as in (3.4), we can apply upper bound in Theorem 3.5 and we obtain the results that appear in Table 3.1.

We recall that Relative error = (upper bound - exact value)/exact value.

TABLE 3.1
Upper bound for the perturbation of the group inverse in Example 3.10

	Exact value	Upper bound (3.6)	Relative error
$\|\|B^{\#} - A^{\#}\|\|_1/\|\|A^{\#}\|\|_1$	20	21	5%
$\|\|B^{\#} - A^{\#}\|\|_2/\|\|A^{\#}\|\|_2$	14.1598	17.3436	22%

EXAMPLE 3.11. Let $\epsilon \geq 0$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \epsilon/2 & 0 & \epsilon/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon/2 & 0 & \epsilon/2 \end{bmatrix}.$$

We have $\text{ind}(A) = 3$, $\text{ind}(B_\epsilon) = 2$, $\text{rank}\, A^3 = \text{rank}\, B_\epsilon^2 = \text{rank}\, A^3 B_\epsilon^2 = 2$ and $\text{rank}\, B_\epsilon^2 A^3 = 1$. We will apply upper bounds (3.6) and (3.8) with $j = 1$ and $k = 2$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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In this case, $\|\Gamma_{2,3}\|_1 < 1$ and $\|\Gamma_{1,2}\|_1 < 1$, where $\Gamma_{j,k}$ is defined as in (3.4). Table 3.2 shows the upper bound for the perturbation of the Drazin inverse, and Table 3.3 shows the upper bound for the perturbation of the projector.

Table 3.2
Upper bound of $||B_\epsilon^D - A^D||_1/||A^D||_1$ *in Example 3.11*

ϵ	Exact value	Upper bound (3.6)	Relative error
0.5	6	6.5	8%
0.1	110	110.1	0.09%

Table 3.3
Upper bound of $||B_\epsilon^\pi - A^\pi||_1$ *in Example 3.11*

ϵ	Exact value	Upper bound (3.8)	Relative error
0.1	11	12.2	11%
0.01	101	102.02	1%

Example 3.12. Let $\alpha = 0.1$, $\epsilon \geq 0$ and

$$B_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-3\epsilon+2\epsilon^2 & \alpha\epsilon^{1/2} \\ 0 & \epsilon^{1/2}(1-3\epsilon+2\epsilon^2) & \alpha\epsilon \end{bmatrix},$$

$$B_\epsilon^\# = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1-3\epsilon+2\epsilon^2)/d & \alpha\epsilon^{1/2}/d \\ 0 & \epsilon^{1/2}(1-3\epsilon+2\epsilon^2)/d & \alpha\epsilon/d \end{bmatrix},$$

where $d = \big(1-(3-\alpha)\epsilon+2\epsilon^2\big)^2$. Let us consider $A = B_0$. We have $\operatorname{ind}(A) = \operatorname{ind}(B_\epsilon) = 1$. Since $\operatorname{rank} A = \operatorname{rank} B_\epsilon = \operatorname{rank} AB_\epsilon = 2$ for all ϵ, then B_ϵ satisfies condition $(\mathcal{B}_{s,1})$. Moreover, for all $\epsilon \neq 1, \frac{1}{2}$ we also have $\operatorname{rank} B_\epsilon A = 2$, and, thus, in these cases B_ϵ satisfies condition $(\mathcal{C}_s)$. In Table 3.4 we compare the upper bound in Theorem 3.5 with the upper bound for matrices satisfying condition $(\mathcal{C}_s)$ given in [16, Theorem 4.1], taking $j = k = 1$ in both bounds. In Table 3.5, we analyze the behavior of the upper bound (3.6), taking $j = k = 1$, for the values $\epsilon = 1$ and $\epsilon = \frac{1}{2}$. We abbreviate Relative error to RE.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 539-556, May 2011

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TABLE 3.4
Upper bound of $||B_\epsilon^{\#} - A^{\#}||_1/||A^{\#}||_1$ *in case* $(\mathcal{C}_s)$ *in Example 3.12*

ϵ	Exact value	Bound (3.6)	RE	[16, Bound (4.4)]	RE
0.25	2.5156	2.5156	0%	4.5803	82%
0.1	0.7783	0.7783	0%	1.0787	39%
10^{-3}	3.4517×10^{-2}	3.4904×10^{-2}	1%	3.6012×10^{-2}	4%
10^{-7}	3.1651×10^{-4}	3.4786×10^{-4}	9,90%	3.4797×10^{-4}	9.94%

TABLE 3.5
Upper bound of $||B_\epsilon^{\#} - A^{\#}||_1/||A^{\#}||_1$ *in Example 3.12*

ϵ	Exact value	Bound (3.6)	RE
1	20	21	5%
1/2	48.2842	48.9914	1%

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REFERENCES

[1] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses. Theory and Applications*, 2nd edition. Springer-Verlag, New York, 2003.

[2] S.L. Campbell and C.D. Meyer Jr. Continuity properties of the Drazin pseudoinverse. *Linear Algebra Appl.*, 10:77–83, 1975.

[3] S.L. Campbell and C.D. Meyer Jr. *Generalized Inverses of Linear Transformations*. SIAM, Philadelphia, 2009.

[4] N. Castro-González, J. Robles and J.Y. Vélez-Cerrada. Characterizations of a class of matrices and perturbation of the Drazin inverse. *SIAM J. Matrix Anal. Appl.*, 30(2):882–897, 2008.

[5] G.E. Cho and C.D. Meyer. Comparison of perturbation bounds for the stationary distribution of a Markov chain. *Linear Algebra Appl.*, 335:137–150, 2001.

[6] J.J. Koliha. Error bound for a general perturbation of the Drazin inverse. *Appl. Math. Comput.*, 126:181–185, 2002.

[7] J.J. Koliha and V. Rakočević. Continuity of the Drazin inverse II. *Studia Math.*, 131:167–177, 1998.

[8] X. Liu, W. Wang and Y. Wei. Continuity properties of the {1}-inverse and perturbation bounds for the Drazin inverse. *Linear Algebra Appl.*, 429:1026–1037, 2008.

[9] G. Marsaglia and G.P.H. Styan. Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra*, 2:269–292, 1974.

[10] C.D. Meyer. The condition of a finite Markov chain and perturbation bounds for the limiting probabilities. *SIAM J. Algebraic Discrete Methods*, 1(3):273–283, 1980.

[11] V. Rakočević and Y. Wei. The perturbation theory for the Drazin inverse and its applications II. *J. Aust. Math. Soc.* 70:189–197, 2001.

[12] G.H. Rong. The error bound of the perturbation of the Drazin inverse. *Linear Algebra Appl.*, 47:159–168, 1982.

[13] M. Wei and S. Ling. On the perturbation bounds of g-inverses and oblique projections. *Linear*

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Algebra Appl., 433:1778–1792, 2010.

[14] Y. Wei and X. Li. An improvement on perturbation bounds for the Drazin inverse. *Numer. Linear Algebra Appl.*, 10:563-575, 2003.

[15] Y. Wei, X. Li and F. Bu. A perturbation bound of the Drazin inverse of a matrix by separation of simple invariant subspaces. *SIAM J. Matrix Anal. Appl.*, 27(1):72–81, 2005.

[16] Q. Xu, C. Song, and Y. Wei. The stable perturbation of the Drazin inverse of the square matrices. *SIAM J. Matrix Anal. Appl.*, 31(3):1507–1520, 2009.