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ALMOST DISJOINT FAMILIES: AN APPLICATION TO LINEAR ALGEBRA*

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Abstract. Suppose that κ is an infinite cardinal, V is a κ-dimensional vector space over a field F, and A is a family of subspaces of V which is maximal with respect to the property: whenever U and W are distinct members of A, then $U \cap W$ has dimension less than κ. What is the cardinality of A? This expository paper explains how questions about the possible cardinality of A for vector spaces of infinite dimension over countable fields are independent of the axioms of ordinary set theory (ZFC).

Key words. linear algebra, almost disjoint, Martin's axiom, combinatorial set theory, logic

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1. Introduction. The questions studied in this expository paper have the following form. Suppose that κ is an infinite cardinal, V is a κ-dimensional vector space over a field F, and A is a family of κ-dimensional subspaces of V which is maximal with respect to the property:

(AD) whenever U and W are distinct members of A, then $U \cap W$ has dimension less than κ.

What is the cardinality of A? For example, is the cardinality of A always 2^κ? Simple applications of combinatorial set theory give comprehensive answers to this problem. New phenomena appear in the case of uncountably infinite-dimensional vector spaces. It emerges that these natural linear algebraic questions cannot be settled using the axioms of ordinary set theory (which for the sake of definiteness we specify as Zermelo-Fraenkel set theory with the axiom of choice (ZFC)).

Let us say that a family A of κ-dimensional subspaces of V which has the property (AD) is an *almost disjoint packing.* The family A is *a maximal almost disjoint packing* if whenever B is an almost disjoint packing containing A, then $B = A$. This is equivalent to saying that whenever M is a κ-dimensional subspace of V, then for some U in A, $U \cap M$ has dimension κ. In geometric terms, a maximal almost disjoint packing of subspaces is a densest possible packing of big subspaces into V in which one allows a relatively small amount of interpenetration (or squashing) between distinct pairs of subspaces.

Recall that the Hebrew letters $\aleph_0$, $\aleph_1$ and $\aleph_2$ stand for the first, second and third infinite cardinals respectively, while $\aleph_{\omega_1}$ is the $\aleph_1$-th infinite cardinal.

The main results of the paper are:

THEOREM 1.1. *Suppose that V is a vector space of countably infinite dimension over a countable field F. Then,*

1. *The vector space V has a maximal almost disjoint packing of cardinality $2^{\aleph_0}$.*

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2. *The Continuum Hypothesis CH* ($2^{\aleph_0} = \aleph_1$) *or Martin's Axiom MA* [15] *imply that every maximal almost disjoint packing of subspaces of* V *has cardinality* $2^{\aleph_0}$.
3. *There is a model of ZFC +* $2^{\aleph_0} = \aleph_2$ *in which every maximal almost disjoint packing of subspaces of* V *has cardinality* $2^{\aleph_0}$.
4. *There is a model of ZFC +* $2^{\aleph_0} = \aleph_2$ *in which every vector space of countably infinite dimension over a countable field has a maximal almost disjoint packing of size* $\aleph_1$.

In particular, ordinary set theory (ZFC), and even ZFC + $2^{\aleph_0} > \aleph_1$, cannot resolve whether every almost disjoint packing of size $\aleph_1$ of subspaces of a countably infinite-dimensional vector space fails to be maximal.

THEOREM 1.2. *Suppose that* V *is a vector space of dimension* $\aleph_1$ *over a countable field* F. *Then,*

1. *The Generalised Continuum Hypothesis GCH implies that every maximal almost disjoint packing of subspaces of* V *has cardinality* $2^{\aleph_1}$.
2. *Assume that* $2^{\aleph_0} < 2^{\aleph_1}$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$. *Then* V *has a maximal almost disjoint packing of* $2^{\aleph_1}$ *subspaces.*
3. *There is a model of ZFC in which* $2^{\aleph_1} = \aleph_3$ *(or as large as desired), and every maximal almost disjoint packing of subspaces of* V *has size* $\aleph_2$.

There is a striking difference between Theorem 1.1 and Theorem 1.2. Theorem 1.1.1 says that if $\dim V$ is countably infinite, then there is *always* a maximal almost disjoint packing of subspaces of V of size $2^{\dim V}$. However, this can fail in the case where V has uncountable dimension over a countable field: Theorem 1.2.3 shows that it is consistent with ordinary set theory for every maximal almost disjoint packing of subspaces of V to have size $\aleph_2 < 2^{\dim V}$.

In the proofs of Theorems 1.1 and 1.2, the concepts of almost disjoint families and cardinal functions, whose provenance lies outside algebra, in the domains of set theory and set-theoretic topology, play a role. Since the 1970's and Shelah's resolution of the Whitehead problem [18], cardinal functions allied with set-theoretic methods have continued their advances into algebra, beyond infinite Abelian group theory, reaching recently into Boolean algebra [5], [16], [17] and even into linear algebra [3], [23], [26], [27]. To mention just one example from quadratic form theory, in [27], Shelah and Spinas prove that the existence of Gross spaces over finite or countable fields is independent of the axioms of ZFC. A *Gross space* is a vector space V over a field F equipped with a symmetric bilinear form Φ from $V \times V$ into F, and possessing the property that for every subspace U of V of infinite dimension, $\dim U^{\perp} < \dim V$, where $U^{\perp}$ is the orthogonal complement of U in V. The Shelah-Spinas theorems say that the existence of Gross spaces can be neither refuted nor proven using the intuitive principles of everyday mathematics. Another of their results shows that the dimension of a Gross space over F is bounded by the cardinal $|F|^{\aleph_0}$, and so by pcf theory, if $2^{\aleph_0} < \aleph_\omega$, then every Gross space over a field F of cardinality at most $\aleph_\omega$ has dimension less than $\aleph_{\omega_4}$. The propositions of the present article are very elementary in comparison with these achievements. I hope they illustrate some straightforward applications of contemporary set theory in linear algebra.

The paper contains four sections and an introduction. In section 2, the main definitions, including in particular the concept of *almost disjointness with respect to a cardinal function* (or family of cardinal functions), are introduced and related to concepts in the literature, and some background history is briefly given (mainly through references). In sections 3 and 4, these ideas are applied to resolve the vector space problems stated in the opening of this paper. There is an immediate transposition of these solutions to infinite Abelian p-groups, and in the final section some extensions to the torsion-free Abelian groups are noted. The algebraic and set-theoretic notations used throughout the paper are standard and follow the references [1], [6], [7], [9], [12], [14]. A recap of the main set-theoretic notation follows. Ordinals are denoted by the initial letters of the Greek alphabet: $\alpha,\ \beta,\ \gamma,\ \ldots,\ \iota,\ \ldots$; every infinite cardinal is a limit ordinal, and for these we use $\chi,\ \kappa,\ \lambda,\ \mu, \ldots$. Designated ordinals are ω (the first infinite ordinal) and ω_1 (the first uncountable ordinal). The cardinality of a set X is $|X|$, and 2^κ is the cardinality of the power set of κ. The continuum (the cardinality of the power set of the natural numbers) is $2^{\aleph_0}$. The set of μ-element subsets of λ is denoted $[\lambda]^\mu$; the set of finite sequences of elements of X is written ${}^{<\omega}X$ or $X^{<\omega}$.

2. Almost disjointness with respect to a cardinal function. A cardinal function is a map Φ into the class Card of cardinal numbers such that if A and B are isomorphic members of the domain $\mathrm{dom}(\Phi)$, then $\Phi(A) = \Phi(B)$. In module and infinite Abelian group theory, cardinal functions are often called (cardinal) invariants [6]. It saves some unnecessary notation if one allows partially defined maps. Simple examples of cardinal functions are the cardinality function, $|A|$, giving the cardinality of a set A, the dimension $\dim_F(V)$ a vector space V over a field F, and the various ranks of infinite Abelian groups [7]. Cardinal functions also abound naturally in set-theoretic topology and Boolean algebra, and have proved influential in shaping recent expositions of these areas [5], [11], [13], [16], [17], [27].

The second main idea central to this paper is that of an almost disjoint family of sets. A family A of subsets of a cardinal κ is *almost disjoint* if for every $a \in A$, $|a| = \kappa$, and if $a \neq b \in A$, then $|a \cap b| < \kappa$. Suppose that $\kappa \leq \mu \leq \lambda < \chi$ are infinite cardinals. We use the notation $A(\chi, \lambda, \mu, \kappa)$ to abbreviate the assertion that there exists a subset $P \subseteq [\lambda]^\mu$ of cardinality χ such that $[X \neq Y \in P \Rightarrow |X \cap Y| < \kappa]$.

Almost disjoint families of sets were studied by Tarski and Sierpinski in the late 1920's [25], [28], [29] and they established most of what one could prove using just the resources of ordinary set theory (ZFC). Under the Generalised Continuum Hypothesis (GCH), their work resolved many of the interesting questions about the sizes of almost disjoint families. The post-Cohen era of set theory in the 1970's witnessed renewed interest in almost disjointness. Martin and Solovay's discovery of Martin's Axiom (MA) [15] and the development of increasingly sophisticated iterated forcing techniques [24] provided a principle and methods that could handle the set-theoretic independence of assertions concerning the size and existence of maximal almost disjoint families of subsets of the natural numbers ω. Baumgartner [2] proved independence results for almost disjoint families of subsets of the first uncountable cardinal ω_1 and many other regular uncountable cardinals. The main results relating to almost disjoint families of sets can be summarised in the following theorems [2], [9], [10], [12].

THEOREM 2.1.

1. *Suppose X is a countably infinite set. There is an almost disjoint family A of subsets of X of cardinality $2^{\aleph_0}$. So $A(2^{\aleph_0}, \aleph_0, \aleph_0, \aleph_0)$ holds.*
2. *Assume the Continuum Hypothesis CH ($2^{\aleph_0} = \aleph_1$) or Martin's Axiom MA. Suppose X is a countably infinite set. Every maximal almost disjoint family of subsets of X has cardinality continuum $2^{\aleph_0}$.*
3. *There is a model of ZFC + $2^{\aleph_0} > \aleph_1$ in which every countably infinite set X has a maximal almost disjoint family of size $\aleph_1$.*

Part 1 of this theorem is of course very easy to prove: for each real number r, let $Q(r)$ be an infinite sequence of distinct rationals which converges to r; now $\{Q(r) : r \in \mathbb{R}\}$ is an almost disjoint family of subsets of the rationals which has cardinality $2^{\aleph_0}$. The other parts are slightly more difficult.

THEOREM 2.2.

1. *Assume $2^{<\kappa} = \kappa$. Then there is an almost disjoint family A of subsets of κ of size 2^κ.*
2. *If $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$, then $A(2^{\aleph_1}, \aleph_1, \aleph_1, \aleph_1)$ holds.*
3. *It is consistent with ZFC that $2^{\aleph_1}$ is large and $\neg A(\aleph_3, \aleph_1, \aleph_1, \aleph_1)$.*
4. *Suppose that $\aleph_1 < \lambda < 2^{\aleph_1}$. It is consistent with ZFC + CH + $2^{\aleph_1} > \aleph_2$ that every almost disjoint family of λ subsets of $\aleph_1$ fails to be maximal.*
5. *Suppose that $\aleph_1 < \lambda < 2^{\aleph_1}$. It is consistent with ZFC + CH + $2^{\aleph_1} > \aleph_2$ that there exists a maximal almost disjoint family of λ subsets of $\aleph_1$.*

The following theorem of Shelah [22] will also make a fleeting appearance.

THEOREM 2.3. *Suppose that $\lambda > \kappa^{\aleph_0}$ and $\kappa > \aleph_0$. Under pcf conditions, there exists a family A of λ^+ subsets of λ, each of cardinality κ, such that the intersection of any pair is finite.*

Although not central to the results of this paper, it is worth mentioning some simple examples of applications of pcf theory to infinite Abelian group theory. Suppose that A is an infinite Abelian group. One of the most important invariants associated with A is the Ulm-Kaplansky sequence and the Ulm factors $\langle A_\alpha : \alpha < \tau\rangle$ of A where τ is the Ulm length of A [7]. The following examples are immediate consequences of the cardinal arithmetic bounds coming from pcf theory [20]-[22] and well-known inequalities for $|A|$ in terms of sizes of its Ulm factors.

THEOREM 2.4. *Suppose that $2^{\aleph_0} < \aleph_\omega$. Let A be a reduced Abelian group.*

1. *If $|A_0| \le \aleph_\omega$, then $|A| < \aleph_{\omega_4}$.*
2. *If A is a p-group, B is a basic subgroup of A and $|B| \le \aleph_\omega$, then $|A| < \aleph_{\omega_4}$.*
3. *If there exists $n < \omega$, $\aleph_1 < |A_n| < \aleph_\omega$ and for all $m < n$, $|A_m| < \aleph_{\omega_4}$, and $\omega \le \tau$, then $|A| < \aleph_{\omega_4}$.*
4. *If F is the Frattini subgroup of A and $|A/F| \le \aleph_\omega$, then $|A| < \aleph_{\omega_4}$.*
5. *If A has a subgroup B of cardinality at most $\aleph_\omega$, and $|A/B|$ is divisible, then $|A| < \aleph_{\omega_4}$.*

Sophisticated uses of pcf theory in Abelian group theory can be found in [4].

A natural strategy for proving results about almost disjoint families of vector subspaces is to try to apply results about almost disjoint families of sets to the index set of a fixed basis of the vector space. The most convenient way of putting this plan into effect is to define a general concept of almost disjointness with respect to a cardinal function. Suppose that Φ is a cardinal function and $\bullet$ is a symmetric binary operation defined on dom(Φ). Let us write $\mathbf{\Phi}$ for the cardinal function pair $(\Phi, \bullet)$. If $\Phi(A) = \Phi(B)$ and $\Phi(A \bullet B) < \Phi(A)$, then A and B are *$\mathbf{\Phi}$-almost disjoint.* A collection A is *$\mathbf{\Phi}$-almost disjoint* if every pair of members of A is. A subfamily B of A is *maximal $\mathbf{\Phi}$-almost disjoint in A* if B is $\mathbf{\Phi}$-almost disjoint and whenever C is a subfamily of A containing B, then $B = C$. Wherever no confusion is likely to arise, $\mathbf{\Phi}$ and A are omitted.

The first observation about almost disjoint families is a trivial application of Zorn's Lemma.

PROPOSITION 2.5. *Suppose that $\mathbf{\Phi} = (\Phi, \bullet)$ is a cardinal function pair. Every $\mathbf{\Phi}$-almost disjoint non-empty subfamily B of A can be extended to a maximal $\mathbf{\Phi}$-almost disjoint family in A.*

Proof. Apply Zorn's Lemma to the collection of $(\Phi, \bullet)$-almost disjoint families containing the family A, partially ordered by inclusion. □

Returning to concrete examples for a moment, if one considers the cardinality function $|\ |$ and the binary operation of intersection $\cap$ defined on the power set of a set X, then a $(|\ |, \cap)$-almost disjoint family is just an almost disjoint family of subsets of X.

If one considers the class of vector spaces and takes $\mathbf{\Phi} = \big(\dim(\ ), \cap\big)$, then one obtains the linear algebraic analogues of almost disjoint sets, i.e. almost disjoint vector spaces in the sense we first defined: a non-empty family $A = \{V_i : i \in I\}$ of subspaces of a vector space V over a field F is *an almost disjoint packing* if (1) for every $i \in I$, $\dim_F V_i = \dim_F V$, and (2) for distinct i and $j \in I$, $\dim_F(V_i \cap V_j) < \dim_F V$. This is not so interesting for finite-dimensional vector spaces, since every almost disjoint non-empty packing has exactly one member. What is surprising however is that the analogous infinite-dimensional problems cannot be solved using just the resources of ordinary set theory, and that the answers are quite different according to whether the dimension of the vector space V is countably infinite or uncountably infinite.

The general question which will form the object of sections 3 and 4 is the following: for a given $\mathbf{\Phi}$ and family A, what are the cardinalities of maximal $\mathbf{\Phi}$-almost disjoint families B in A? Theorems 2.1 and 2.2 answer these questions in the case of almost disjoint families of sets. The goal is to prove similar results for other types of (maximal) $\mathbf{\Phi}$-almost disjoint families.

REMARK 2.6. *Suppose that* $\dim V = \lambda \geq \mathrm{cf}(\lambda) > \kappa \geq \aleph_0$. *Then there is a maximal almost disjoint packing of subspaces of V of cardinality κ.*

[Why? Well, if $\{x_\alpha : \alpha < \lambda\}$ is a basis of V, let $\lambda = \cup_{\alpha<\kappa} A_\alpha$ be a disjoint union of κ subsets of λ each of cardinality λ, and put $V_\alpha = \langle x_\xi : \xi \in A_\alpha \rangle$. Obviously $\{V_\alpha : \alpha < \kappa\}$ is almost disjoint. If U is any subspace of V of dimension λ, then for some $\alpha < \kappa$, $\dim(U \cap V_\alpha) = \lambda$, since $\kappa < \mathrm{cf}(\lambda)$.] For a regular cardinal λ, it is easy to improve this result (Proposition 4.1).

COROLLARY 2.7. *Every almost disjoint packing A of subspaces of a vector space V over a field F can be extended to a maximal almost disjoint packing.*

Suppose that A is a packing of subspaces of an infinite-dimensional vector space, and for every W in A, $\dim W = \dim V$. We say that A is *large invariant subspace* *pair free* (lisp-free) if whenever φ is a linear homomorphism between distinct members U and W of A, then $\mathrm{Fix}(\varphi) = U \cap \mathrm{range}(\varphi)$ has dimension less than $\dim V$. In other words, no linear operator between distinct members of A has an invariant subspace of "large" dimension.

Lisp-free families and almost disjoint packings are related in the following simple way.

PROPOSITION 2.8. *Suppose that U and W are subspaces of the vector space V of dimension λ over the field F. The following are equivalent.*

1. $\dim(U \cap W) < \lambda$;
2. *whenever φ is a homomorphism from U to W, then* $\dim(\mathrm{Fix}\ \varphi) < \lambda$, *where* $\mathrm{Fix}\ \varphi = \mathrm{range}(\varphi) \cap \mathrm{U}$.

Proof. The forward direction is trivial since $\mathrm{Fix}\ \varphi \subseteq U \cap W$. If, on the other hand, $\dim(U \cap W) = \lambda$, then there is a basis $\{x_i : i < \lambda\}$ of $U \cap W$, and one can expand this to a basis $\{x_i, u_j : i < \lambda,\ j < \alpha\}$ of U. Now define φ from $\{x_i, u_j : i < \lambda,\ j < \alpha\}$ to W by $\varphi(x_i) = x_i,\ \varphi(u_j) = 0$, and extend to a homomorphism from U into W by linearity. Since $\mathrm{Fix}\ \varphi \supseteq \{x_i : i < \lambda\},\ \dim(\mathrm{Fix}\ \varphi) \geq \lambda$. □

So by Proposition 2.8, a packing of infinite-dimensional vector spaces is (maximal) lisp-free if and only if it is (maximal) almost disjoint.

3. Vector spaces of countably infinite dimension. We begin with a very simple fact. Countable almost disjoint packings of subspaces of an infinite-dimensional vector space are never maximal:

PROPOSITION 3.1. *Suppose that V is a vector space of countably infinite dimension over a field F, and $A = \{V_n : n \in \omega\}$ is an almost disjoint packing. Then A is not maximal.*

Proof. Let $\{x_n : n \in \omega\}$ be a basis of V over F. Define $\{u_m : m \in \omega\}$ by induction as follows. Let u_0 be any non-zero element of V_0. Given $u_0, \ldots, u_m$, note that $\dim\left(V_{m+1} - (V_0 \oplus V_1 \oplus \ldots \oplus V_m)\right) = \omega$, since $\dim(V_{m+1}) = \omega$ and A is almost disjoint. So one can find $u_{m+1} \in V_{m+1} - (V_0 \oplus V_1 \oplus \ldots \oplus V_m),\ u_{m+1} = \sum q_i x_i$ where $x_i \notin \{x_n : x_n$ appears in the basis representation of $u_0, \ldots, u_m\}$. Put $U = \{u_m : m \in \omega\}$. By construction, $\dim U = \omega$, and $\dim(U \cap V_n) < \omega$. □

THEOREM 3.2. *Suppose that* $\dim V = \omega$. *Then there exists an almost disjoint packing of subspaces of V of cardinality $2^{\aleph_0}$.*

Proof. Let A be as in Theorem 2.1.1, and let $\{x_n : n \in \omega\}$ be a basis of V over F. For $a \in A$, let $V_a = \mathrm{span}\{x_n : n \in a\}$. Easily the packing $\{V_a : a \in A\}$ is as required, since $\dim(V_a \cap V_b) = |a \cap b| < \omega$. □

The following immediate corollary establishes Theorem 1.1.1.

COROLLARY 3.3. *Every countably infinite dimensional vector space has a maximal almost disjoint packing of subspaces of cardinality $2^{\aleph_0}$.*

Proof. Apply Corollary 2.7 to the almost disjoint packing of Theorem 3.2. □

COROLLARY 3.4. *The Continuum Hypothesis CH ($2^{\aleph_0} = \aleph_1$) implies that every maximal almost disjoint packing of subspaces of a countably infinite dimensional vector space has power $2^{\aleph_0}$.*

Proof. By Proposition 3.1. □

Now let us consider what happens if the Continuum Hypothesis fails. In other words, let us consider the question (*) if $2^{\aleph_0} > \aleph_1$, does every almost disjoint packing of size $\aleph_1$ of subspaces of a countably infinite-dimensional vector space fail to be maximal? The next results show that the answer to this question is independent of ZFC + $\neg$ CH.

THEOREM 3.5. *Assume Martin's Axiom MA. If* $\dim V = \omega$, *then every maximal almost disjoint packing of subspaces of* V *has cardinality* $2^{\aleph_0}$.

Proof. The Solovay partial order (for proving Theorem 2.1.2), slightly modified, works. Suppose that $A = \{U_i : i \in I\}$ is almost disjoint and $|I| = \kappa < 2^{\aleph_0}$. Let $\{x_n : n \in \omega\}$ be a basis of V. Let $\mathbf{P} = (P, \leq)$ be the following partial order. $P = \{(h, a) : h$ and a are finite subsets of I and ω respectively$\}$, and $(h, a) \leq (k, b)$ iff $h \subseteq k,\ a \subseteq b$, and (*) $\langle x_n : n \in (b - a)\rangle \cap \langle \oplus_{i \in h} U_i\rangle = \{0\}$. Note that $\mathbf{P}$ is a c.c.c. partial order, i.e., every antichain (family of pairwise incompatible elements) in $\mathbf{P}$ is at most countable.

For $i \in I$ and $n \in \omega$, the sets $D_i = \{(h, a) : i \in h\}$ and $E_n = \{(k, b) : |b| > n\}$ are dense in $\mathbf{P}$, so by MA there is a generic filter G intersecting all of them non-trivially.

Now $W = \langle x_n : n \in \cup\{a : (\exists h)\big((h, a) \in G\big)\}\rangle$ is a subspace of V having dimension ω and almost disjoint from each U_i. ◻

Theorem 3.5 completes the proof of Theorem 1.1.2, and Theorem 1.1.3 is the next corollary.

COROLLARY 3.6. *There is a model of ZFC +* $2^{\aleph_0} > \aleph_1$ *in which every maximal almost disjoint packing of subspaces of* V *has cardinality* $2^{\aleph_0}$.

Proof. By Theorem 3.5, any model of ZFC + MA + $(2^{\aleph_0} = \aleph_2)$ will work. ◻

Thus a positive answer to the question (*) is consistent with ordinary set theory. However, using a forcing argument, it is possible to show that a negative answer is also consistent, finishing the proof of Theorem 1.1.4.

THEOREM 3.7. *There is a model of ZFC +* $2^{\aleph_0} = \aleph_2$ *in which every vector space* V *of countably infinite dimension over a countable field* F *has a maximal almost disjoint packing of size* $\aleph_1$.

Proof. We give just a sketch of the forcing argument. Start from a model M of ZFC + CH, and, as in Theorem VII.2.3 in [12], force with the partial order for adding $\aleph_2$ Cohen reals. In the generic extension $M[G]$ there is a maximal almost disjoint packing of size $\aleph_1$, but $2^{\aleph_0} = \aleph_2$. ◻

These results complete the proof of Theorem 1.1. The main points to emerge are the following. Suppose that V is a countably infinite-dimensional vector space over a countable field F. Then,

1. there is always a maximal almost disjoint packing of subspaces of V of size $2^{\aleph_0}$;
2. Martin's Axiom implies that every maximal almost disjoint packing has size $2^{\aleph_0}$;
3. it is consistent that there is a maximal almost disjoint packing of subspaces of size $\aleph_1 < 2^{\aleph_0}$.

In particular, neither ordinary set theory (ZFC) nor ZFC + $2^{\aleph_0} = \aleph_2$ is a strong enough axiomatic system to determine the cardinality of a maximal almost disjoint packing of subspaces of V.

4. Vector spaces of uncountably infinite dimension. It is not surprising that, just as the Continuum Hypothesis determines completely the cardinality of maximal almost disjoint packings of subspaces of countably infinite dimensional vector spaces, so instances of the Generalised Continuum Hypothesis (GCH: $(\forall\kappa)(2^\kappa = \kappa^+)$) are equally powerful in resolving higher dimensional questions.

First it is useful to write down the appropriate version of Proposition 3.1.

PROPOSITION 4.1. *Suppose that* $\dim V = \kappa$ *where* κ *is a regular cardinal. If* A *is an almost disjoint packing of size* κ*, then* A *is not maximal.*

Proof. Like Proposition 3.1. Use the regularity of κ to prolong the construction. □

THEOREM 4.2. *Assume GCH. Suppose that* κ *is a regular infinite cardinal. Every* κ*-dimensional vector space* V *has an almost disjoint packing of* 2^κ *subspaces.*

Proof. By Proposition 4.1, like Theorem 3.2. □

COROLLARY 4.3. *GCH implies that every maximal almost disjoint packing of subspaces of a* κ*-dimensional vector space* V *has cardinality* 2^κ.

Proof. By Proposition 4.1. □

The reader interested only in the proof of Theorem 1.2 should now skip forward to Theorem 4.10. Corollaries 4.8 and 4.9 are minor strengthenings of Corollary 4.3.

DEFINITION 4.4. *Suppose that* $\{x_\alpha : \alpha < \lambda\}$ *is a basis of* V*. A subset* U *of* V *is* stationary in $\{x_\alpha : \alpha < \lambda\}$ *if the set* $S(U) = \{\alpha : x_\alpha \in U\}$ *is a stationary subset of* λ.

REMARK 4.5. *Suppose that* $\dim V = \lambda \geq \mathrm{cf}(\lambda) > \kappa \geq \aleph_0$*, and* $\{x_\alpha : \alpha < \lambda\}$ *is a basis of* V*. Then there is a maximal almost disjoint packing of* κ *subspaces of* V*, each of which is stationary in the basis.*

Why? By Solovay's theorem (see [12]), one can write $\lambda = \cup_{\alpha<\kappa} A_\alpha$ as a disjoint union of κ stationary subsets of λ; put $V_\alpha = \langle x_\xi : \xi \in A_\alpha\rangle$. Obviously $\{V_\alpha : \alpha < \kappa\}$ is almost disjoint. If U is any subspace of V of dimension λ, then for some $\alpha < \kappa$, $\dim(U \cap V_\alpha) = \lambda$, since $\kappa < \mathrm{cf}(\lambda)$.

The next result appeals to a well-known combinatorial principle $\diamondsuit_\kappa$ called diamond on κ; see for example any one of [9], [12], [14] for introductory explanations. This principle asserts the existence of a sequence $\langle S_\alpha : \alpha < \kappa\rangle$ such that:
$(\forall\alpha < \kappa)(S_\alpha \subseteq \alpha)$ and $(\forall X \subseteq \kappa)(\{\beta < \kappa : X \cap \beta = S_\beta\}$ is a stationary subset of $\kappa)$.

Diamond $\diamondsuit$ is just diamond on $\aleph_1$. The intuition behind diamond on $\aleph_1$ is that if one views the sets S_α as guesses for the initial segments of an arbitrary subset X of ω_1, then these guesses are right a large (stationary) number of times. Diamond on $\aleph_1$ implies CH, since every subset of ω must turn up as an S_α for some (in fact stationarily many) $\alpha < \omega_1$, so that the sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ contains a complete listing of all the subsets of ω, and hence $2^{\aleph_0} = \aleph_1$. For convenience, recall the following result.

LEMMA 4.6. *Assume* $\diamondsuit_\kappa$*. Then there exists an almost disjoint family of* 2^κ *stationary subsets of* κ.

Proof. For $X \subseteq \kappa$, let $A_X = \{\alpha : X \cap \alpha = S_\alpha\}$, where $\{S_\alpha : \alpha < \kappa\}$ is a diamond sequence. Now $\{A_X : X \subseteq \kappa\}$ works. □

THEOREM 4.7. *Assume* $\diamondsuit_\kappa$*. Then every* κ*-dimensional vector space* V *with basis* $B = \{x_\alpha : \alpha < \kappa\}$ *has an almost disjoint packing* $\mathcal{V}$ *of* 2^κ *subspaces such that every member* U *of* $\mathcal{V}$ *is stationary in the basis* B.

Proof. Just like Theorem 3.2 but using the family from Lemma 4.6. □

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COROLLARY 4.8. *GCH implies that for every regular cardinal $\kappa > \aleph_1$, every κ-dimensional vector space V with basis $B = \{x_\alpha : \alpha < \kappa\}$ has an almost disjoint packing $\mathcal{V}$ of 2^κ subspaces such that every member U of $\mathcal{V}$ is stationary in the basis B.*

Proof. It is well-known that GCH implies $\diamondsuit_\kappa$ [8], [19]. □

COROLLARY 4.9. *Assume $\diamondsuit$. Then every $\aleph_1$-dimensional vector space V with basis $B = \{x_\alpha : \alpha < \aleph_1\}$ has an almost disjoint packing $\mathcal{V}$ of $2^{\aleph_1}$ subspaces such that every member U of $\mathcal{V}$ is stationary in the basis B.*

Proof. Theorem 4.7 for $\kappa = \aleph_1$. □

THEOREM 4.10. *Assume that $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$. Then every $\aleph_1$-dimensional vector space V has an almost disjoint packing $\mathcal{V}$ of $2^{\aleph_1}$ subspaces.*

Proof. Like 3.2, using Theorem 2.2.2 on almost disjoint families of subsets of $\aleph_1$ [2], [10] . □

THEOREM 4.11. *There is a model of ZFC in which $2^{\aleph_1} = \aleph_3$ (or as large as desired), and whenever V is an $\aleph_1$-dimensional vector space over a countable field F, then every maximal almost disjoint packing of subspaces of V has size $\aleph_2$.*

Proof. By Theorem 2.2.3, there is a model M of ZFC + $2^{\aleph_1} = \aleph_3$, and in which every almost disjoint family of subsets of ω_1 has cardinality at most $\aleph_2$. Since $|{}^{<\omega}\omega_1| = \aleph_1$, one may assume that every almost disjoint family of subsets of ${}^{<\omega}\omega_1$ has cardinality at most $\aleph_2$. We shall use the pigeonhole principle twice to obtain a contradiction. Suppose that $\dim_F V = \aleph_1$ and $\mathcal{V} = \{V_\alpha : \alpha < \omega_3\}$ is an almost disjoint packing of subspaces of V. Let $\{x_\alpha : \alpha < \omega_1\}$ be a basis of V over F, and for $\alpha < \omega_3$, let $\{v_{\alpha,\iota} : \iota < \omega_1\}$ be a basis of V_α over F. Fix α for the moment. For every ι, there exist $q_{\alpha,\iota,j} \in F$, $n(\alpha,\iota) < \omega$, and $\xi(\iota,1) < \xi(\iota,2) < \ldots < \xi\big(\iota, n(\alpha,\iota)\big)$ such that $v_{\alpha,\iota} = \sum_{j=1}^{n(\alpha,\iota)} q_{\alpha,\iota,j} x_{\xi_{(\iota,j)}}$. Since $|\omega \times {}^{<\omega} F| < \aleph_1$, there exist $n(\alpha)$, $q_{\alpha,j}$ and I_α such that $|I_\alpha| = \aleph_1$, and for every $\iota \in I_\alpha$, $n(\alpha,\iota) = n(\alpha)$ and $q_{\alpha,\iota,j} = q_{\alpha,j}$ (for all $j < n(\alpha)\big)$.

Let $A_\alpha = \big\{\{\xi(\iota,1),\ldots,\xi\big(\iota,n(\alpha)\big)\} : \iota \in I_\alpha\big\}$. Note that A_α is uncountable, since for $\iota \in I_\alpha$, the $v_{\alpha,\iota}$ are distinct and uncountable.

Unfix α. There exist n, q_j, and $Y \subseteq \omega_3$ such that $|Y| = \aleph_3$ and for every $\alpha \in Y$, $n(\alpha) = n$ and $q_{\alpha,j} = q_j$ (for all $j < n$). Now consider $\mathcal{A} = \{A_\alpha : \alpha \in Y\}$. It is easy to check that $\mathcal{A}$ is an almost disjoint family of subsets of cardinality $\aleph_3$, since if $\iota \in I_\alpha \cap I_\beta$, then $v_{\alpha,\iota} = v_{\beta,\iota}$ and hence if $|A_\alpha \cap A_\beta| = \aleph_1$, then $|I_\alpha \cap I_\beta| = \aleph_1$, and so $\dim_F(V_\alpha \cap V_\beta) = \aleph_1$ - a contradiction. □

The difference from the case of the countably infinite dimensional vector spaces is striking: by Theorem 4.11, it is consistent with ZFC that *every* maximal almost disjoint packing of subspaces of an $\aleph_1$-dimensional vector space over a countable field has size less than $2^{\aleph_1}$. By Corollary 3.3, the parallel result is false for $\aleph_0$-dimensional vector spaces. However, like Corollary 3.6 and Theorem 3.7, the question whether every almost disjoint packing of λ subspaces (for $(\aleph_1 < \lambda < 2^{\aleph_1})\big)$ fails to be maximal is independent of ZFC + CH + $2^{\aleph_1} > \aleph_2$ (appeal to Theorem 2.2.4 and 2.2.5).

This completes the proof of Theorem 1.2.

5. Cardinal functions and rank-almost disjoint Abelian groups. Suppose that A is an infinite Abelian group. The rank $r(A)$ of A is the cardinality of a maximal independent subset containing only elements of infinite and prime power orders. The torsion-free rank $r_0(A)$ is the cardinality of a maximal independent subset of elements of infinite order; the p-rank $r_p(A)$ is the cardinality of a maximal independent subset of elements whose orders are a power of the prime p. These ranks are all invariants of the group A; see [7].

If A is a p-group, then its socle $S(A)$, the set of elements of square-free order, equals $A[p] = \{a \in A : pa = 0\}$, and is a vector space over the finite field $\mathbf{F_p}$ of p elements; moreover, $r(A) = r\big(S(A)\big) = \dim_{\mathbf{F_p}}\big(S(A)\big)$. It follows that the results of sections 3 and 4 apply directly to p-groups of infinite rank, where one considers the family of subgroups of A, $\Phi(G)$ is $r(G)$, and $G \bullet H$ is $\langle G \cap H\rangle^*$, the pure closure of $G \cap H$ in A. In this case, a non-empty family $G = \{G_i : i \in I\}$ of subgroups of A is *rank-almost disjoint* if for every $i \in I$, $r(G_i) = r(A)$, and for distinct i and $j \in I$, $r(G_i \cap G_j) < r(A)$.

THEOREM 5.1.

1. *Suppose that A is an Abelian p-group of rank $\kappa = \mathrm{cf}(\kappa) \geq \aleph_0$, and $G = \{G_\alpha : \alpha \in \kappa\}$ is rank-almost disjoint. Then G is not maximal.*
2. *Every Abelian p-group of countably infinite rank has a maximal rank-almost disjoint family of subgroups of cardinality $2^{\aleph_0}$.*
3. *Assume Martin's Axiom MA. If A is an Abelian p-group of countably infinite rank, then every maximal rank-almost disjoint family of subgroups of A has cardinality $2^{\aleph_0}$.*
4. *There is a model of ZFC + $2^{\aleph_0} > \aleph_1$ in which every maximal rank-almost disjoint family of subgroups of an Abelian p-group A of countably infinite rank has cardinality $2^{\aleph_0}$.*
5. *There is a model of ZFC + $2^{\aleph_0} > \aleph_1$ in which every Abelian p-group A of countably infinite rank has a maximal rank-almost disjoint family of size $\aleph_1$.*

THEOREM 5.2.

1. *Assume GCH. Suppose that κ is a regular infinite cardinal. Every Abelian p-group A of rank κ has a rank-almost disjoint family of 2^κ subgroups.*
2. *GCH implies that every maximal rank-almost disjoint family of subgroups of a Abelian p-group A of rank κ has cardinality 2^κ.*

THEOREM 5.3.

1. *Assume that $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$. Then every Abelian p-group A of rank $\aleph_1$ has a rank-almost disjoint family of $2^{\aleph_1}$ subgroups.*
2. *There is a model of ZFC in which $2^{\aleph_1} = \aleph_3$ (or as large as prescribed), and whenever A is an Abelian p-group of rank $\aleph_1$, then every maximal rank-almost disjoint family of subgroups of A has size $\aleph_2$.*

Indeed, a moment's reflection reveals that the proofs in sections 3 and 4 use relatively little information about vector spaces, and could be generalised to an abstract dependency relation: the infinitary combinatorics dominates the linear algebra. In the case of infinite torsion-free Abelian groups, one obtains results of the following kind as routine elaborations.

Let A be a torsion-free infinite Abelian group and consider the family of subgroups of A; take the cardinal function $\Phi(G) = \mathrm{rk}_0(\mathrm{G})$, the torsion-free rank of G, and let $G \bullet H = \langle G \cap H\rangle^*$, the pure closure of $G \cap H$ in A. In this case, a non-empty family

$G = \{G_i : i \in I\}$ of subgroups of A is *rank-almost disjoint* if for every $i \in I$, $\mathrm{rk_0(G_i)} = \mathrm{rk_0(A)}$, and for distinct i and $j \in I$, $\mathrm{rk_0(G_i \cap G_j)} < \mathrm{rk_0(A)}$.

THEOREM 5.4.

1. *Suppose that A is a torsion-free Abelian group of rank $\kappa = \mathrm{cf}(\kappa) \geq \aleph_0$, and $G = \{G_\alpha : \alpha \in \kappa\}$ is rank-almost disjoint. Then G is not maximal.*
2. *Every torsion-free Abelian group of countably infinite rank has a maximal rank-almost disjoint family of pure subgroups of cardinality $2^{\aleph_0}$.*
3. *Assume Martin's Axiom MA. If A is a torsion-free Abelian group of countably infinite rank, then every maximal rank-almost disjoint family of subgroups of A has cardinality $2^{\aleph_0}$.*
4. *There is a model of ZFC $+2^{\aleph_0} > \aleph_1$ in which every maximal rank-almost disjoint family of subgroups of a torsion-free Abelian group A of countably infinite rank has cardinality $2^{\aleph_0}$.*
5. *There is a model of ZFC $+2^{\aleph_0} > \aleph_1$ in which every torsion-free Abelian group A of countably infinite rank has a maximal rank-almost disjoint family of pure subgroups of size $\aleph_1$.*

THEOREM 5.5.

1. *Assume GCH. Suppose that κ is a regular infinite cardinal. Every torsion-free Abelian group A of rank κ has a rank-almost disjoint family of 2^κ pure subgroups.*
2. *GCH implies that every maximal rank-almost disjoint family of subgroups of a torsion-free Abelian group A of rank κ has cardinality 2^κ.*

THEOREM 5.6.

1. *Assume that $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$. Then every torsion-free Abelian group A of rank $\aleph_1$ has a rank-almost disjoint family of $2^{\aleph_1}$ pure subgroups.*
2. *There is a model of ZFC in which $2^{\aleph_1} = \aleph_3$ (or as large as prescribed), and whenever A is a torsion-free Abelian group of rank $\aleph_1$, then every maximal rank-almost disjoint family of subgroups of A has size $\aleph_2$.*

Appealing to pcf theory [22] (see Theorem 2.3 above) characterising the existence of strongly almost disjoint families of sets, i.e., such that every pair has finite intersection, one obtains some parallel facts, and, for $\aleph_1$-free groups, the pairwise intersections are free by Pontryagin's criterion [7]:

THEOREM 5.7. *Suppose that $\lambda > \kappa^{\aleph_0}$ and $\kappa > \aleph_0$. Under pcf conditions, every torsion-free ($\aleph_1$-free) Abelian group of rank λ has a rank-almost disjoint family A of λ^+ pure subgroups, each of cardinality κ such that the intersection of any pair is of finite rank (free).*

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