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FILLMORE'S THEOREM AND SUMS OF NILPOTENT QUATERNION MATRICES*

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Abstract. Fillmore's Theorem states that an $n \times n$ nonscalar matrix A over a field is similar to a matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$ provided that $\sum_{i=1}^{n} \lambda_i = \operatorname{Tr} A$. It is shown that an analog of Fillmore's Theorem holds for matrices over a quaternion division algebra. As a consequence, a matrix that is nonscalar over the underlying field and has zero reduced trace is a sum of at most three nilpotent matrices. The upper bound three is optimal since it is shown that there are matrices that are sums of exactly three nilpotent matrices. A characterization of sums of two square-zero real quaternion matrices is also presented.

Key words. Fillmore's Theorem, Quaternion, Nilpotent matrix, Trace.

AMS subject classifications. 15A21, 15A30, 15B33, 16S50 16U99.

1. Introduction. A *nilpotent matrix* over a ring is a matrix A with $A^k = 0_n$ for some positive integer k. In this case, A is also called *k-zero*; if k is the least such integer, then A is called *strictly k-zero*. In particular, 2-zero matrices are also known as *square-zero* matrices. Sums of nilpotent matrices have been of interest to many researchers. Fillmore [3] established that given a nonscalar $n \times n$ matrix A over a field, then A is similar to a matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$ provided that $\sum_{i=1}^{n} \lambda_i = \operatorname{Tr} A$. Consequently, a matrix over a field is a sum of nilpotent matrices if and only if the matrix is *traceless* (i.e., has zero trace) [8, Corollary 3]. Different aspects of studying sums of nilpotent matrices have been considered in the literature (see [1, 4, 8, 9, 10, 12]). For example, every traceless complex matrix is a sum of four square-zero matrices, while a complex matrix is a sum of two square-zero matrices if and only if it is similar to its negative multiple [12]; for fields with at least three elements, every traceless matrix is a sum of five strictly k-zero matrices [8]. For division rings, a matrix is a sum of nilpotent matrices if and only if its trace is a sum of *commutators* (ring elements of the form $[a, b] \coloneqq ab - ba$) [6]. In [2, Theorem 2], it was shown that a matrix A over a commutative ring is a sum of three nilpotent matrices provided that $\operatorname{Tr} A$ is nilpotent. When the ring is not necessarily commutative, the authors used the decomposition in [6, Proposition 1] to prove that the number of nilpotent summands is a function of the size and decomposition properties of $\operatorname{Tr} A$ [2, Corollary 7].

The current study focuses on matrices over a generalization of the real quaternions. Following [11, Definitions 2.2.1 and 6.2.1], an algebra $\mathcal{Q}$ over a field $\mathbb{F}$ is a *quaternion algebra* if it has an $\mathbb{F}$-basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \subset \mathcal{Q}$ satisfying

$$\begin{cases} \mathbf{i}^2 = a,\ \mathbf{j}^2 = b,\ \text{and } \mathbf{k} = \mathbf{ij} = -\mathbf{ji},\ \text{for some } a, b \in \mathbb{F} \setminus \{0\}, & \text{if } \operatorname{char}(\mathbb{F}) \neq 2 \\ \mathbf{i}^2 + \mathbf{i} = a,\ \mathbf{j}^2 = b,\ \text{and } \mathbf{k} = \mathbf{ij} = \mathbf{ji} + \mathbf{j},\ \text{for some } a \in \mathbb{F},\ b \in \mathbb{F} \setminus \{0\}, & \text{if } \operatorname{char}(\mathbb{F}) = 2. \end{cases}$$

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Denote by $\mathbb{H}_{\mathbb{F}}^{a,b}$ the quaternion algebra constructed above; in particular, $\mathbb{H} := \mathbb{H}_{\mathbb{R}}^{-1,-1}$ is the classical *real quaternion division algebra* discovered by Hamilton in 1843. Equivalently, a quaternion algebra $\mathcal{Q}$ is a central simple algebra over $\mathbb{F}$ of dimension $\dim_{\mathbb{F}}(\mathcal{Q}) = 4$ (see [11, Corollary 7.1.2]). A *quaternion division algebra* is a quaternion algebra $\mathcal{Q}$ that is also a division algebra (see [11, Theorems 5.4.4 and 6.4.11] for necessary and sufficient conditions when a quaternion algebra is a division algebra).

In this paper, we prove an analog of Fillmore's Theorem for quaternion division algebras (Theorem 4.2). Consequently, every nonscalar matrix over $\mathbb{F}$ with zero reduced trace is a sum of at most three nilpotent matrices (Corollary 5.1). This upper bound improves the one established in [2, Corollary 7]. Moreover, three nilpotent matrices is optimal, i.e., there are matrices that are sums of exactly three nilpotent matrices (Remarks 5.1 and 5.2). We also characterize real quaternion matrices that are sums of two square-zero matrices (Theorem 5.2 and Corollary 5.3).

2. Matrices over a division ring. Let $\mathcal{D}$ be a division ring with identity 1. Necessarily, $\mathcal{D}$ is *simple* as a ring (i.e., its only two-sided ideals are $\{0\}$ and $\mathcal{D}$). The *center* of $\mathcal{D}$ is defined as $Z(\mathcal{D}) := \{\beta \in \mathcal{D} : \alpha\beta = \beta\alpha, \text{ for any } \alpha \in \mathcal{D}\}$. If $\mathcal{D} = \mathcal{A}$ is an $\mathbb{F}$-algebra with $Z(\mathcal{A}) = \mathbb{F}$, then $\mathcal{A}$ is said to be *central*. For example, $\mathbb{H}_{\mathbb{F}}^{a,b}$ is central. For any $\alpha \in \mathcal{D}$, define $\mathrm{Orb}(\alpha) := \{\beta \in \mathcal{D} : \beta = \gamma^{-1}\alpha\gamma \text{ for some } \gamma \in \mathcal{D} \setminus \{0\}\}$.

Following [7], a (right) *$\mathcal{D}$-vector space* V is a unitary (right) $\mathcal{D}$-module. Recall that V has a basis, and more generally, every linearly independent subset of V is contained in a basis of V [7, Ch. IV, Theorem 2.5]. The *dimension* $\dim_{\mathcal{D}} V$ of V over $\mathcal{D}$ is the cardinality of any basis of V [7, Ch. IV, Theorem 2.7]. A *subspace* of V is a $\mathcal{D}$-submodule of V. If U, W are subspaces of V, then we have the familiar rules on dimension such as (i) $\dim_{\mathcal{D}} W \le \dim_{\mathcal{D}} V$; (ii) if $\dim_{\mathcal{D}} W = \dim_{\mathcal{D}} V$ and $\dim_{\mathcal{D}} V$ is finite, then $W = V$; and (iii) if $\dim_{\mathcal{D}} U$ and $\dim_{\mathcal{D}} W$ are finite, then $\dim_{\mathcal{D}} U + \dim_{\mathcal{D}} W = \dim_{\mathcal{D}}(U \cap W) + \dim_{\mathcal{D}}(U + W)$ (see [7, Ch. IV, Theorem 2.13 and Corollary 2.15]). A *linear transformation* is a $\mathcal{D}$-module homomorphism $\mathfrak{f} : V \to V'$ where V, V' are $\mathcal{D}$-vector spaces; the Rank-Nullity Theorem [7, Ch. IV, Corollary 2.14] guarantees $\dim_{\mathcal{D}} V = \dim_{\mathcal{D}}(\mathrm{Ker}\mathfrak{f}) + \dim_{\mathcal{D}}(\mathrm{Im}\mathfrak{f})$ where $\mathrm{Ker}\mathfrak{f} = \{\mathbf{v} \in V : \mathfrak{f}(\mathbf{v}) = 0\}$ ($\dim_{\mathcal{D}}(\mathrm{Ker}\mathfrak{f})$ is called the *nullity* of $\mathfrak{f}$) and $\mathrm{Im}\mathfrak{f} = \{\mathfrak{f}(\mathbf{v}) : \mathbf{v} \in V\}$ ($\dim_{\mathcal{D}}(\mathrm{Im}\mathfrak{f})$ is called the *rank* of $\mathfrak{f}$). An *endomorphism* is a linear transformation between the same $\mathcal{D}$-vector space, while an *automorphism* is a bijective endomorphism.

For $m, n \in \mathbb{N}$, $M_{m,n}(\mathcal{D})$ denotes the set of $m \times n$ matrices over $\mathcal{D}$. For convenience, let $M_n(\mathcal{D}) := M_{n,n}(\mathcal{D})$ and $\mathcal{D}^m := M_{m,1}(\mathcal{D})$. Given $X = [x_{ij}] \in M_{m,n}(\mathcal{D})$, $X^\top = [x_{ji}] \in M_{n,m}(\mathcal{D})$ denotes the *transpose* of X. The $m \times n$ *zero matrix* is written as $0_{m,n}$; write $0_n := 0_{n,n}$ and 0 if the size is clear from the context. The $n \times n$ *identity matrix* is denoted by I_n. If $\mathbf{e}_j$ is the j^{th} column of I_n, then $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the *standard basis* of $\mathcal{D}^n$. Let $A \in M_n(\mathcal{D})$. The *trace* of A is defined as $\mathrm{Tr}(A) = \mathbf{e}_1^\top A\mathbf{e}_1 + \cdots + \mathbf{e}_n^\top A\mathbf{e}_n$, i.e., the sum of the *diagonal entries* of A. The matrix A is said to be a *diagonal matrix*, written $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, if $\mathbf{e}_i^\top A\mathbf{e}_j = 0$ for all $i \neq j$. A diagonal matrix A with $\mathbf{e}_1^\top A\mathbf{e}_1 = \cdots = \mathbf{e}_n^\top A\mathbf{e}_n$ is called a *scalar matrix*; in particular, if $\mathcal{D}$ is an $\mathbb{F}$-algebra, a *scalar matrix over* $\mathbb{F}$ is a matrix of the form $\mathrm{diag}(r, \ldots, r) = I_n r$ where $r \in \mathbb{F}$. A matrix $P \in M_n(\mathcal{D})$ is *invertible* or *nonsingular* if there exists a $Q \in M_n(\mathcal{D})$ such that $PQ = QP = I_n$. Such Q is unique if it exists, and it is denoted by P^{-1}. Two matrices $A, B \in M_n(\mathcal{D})$ are said to be *similar* if there exists invertible $P \in M_n(\mathcal{D})$ such that $B = P^{-1}AP$. If $A_1, \ldots, A_k$ are square matrices over $\mathcal{D}$ (possibly of different sizes), then their *direct sum* is defined as

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}.$$

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For a given $\lambda \in \mathcal{D}$, the $k \times k$ *Jordan block* over $\mathcal{D}$ corresponding to λ, denoted as $J_k(\lambda)$, is defined as $J_1(\lambda) = [\lambda]$ for $k = 1$ and

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix},$$

for $k > 1$. A *Jordan matrix* is a direct sum of Jordan blocks over $\mathcal{D}$. Up to permutation of the Jordan blocks, any Jordan matrix $J \in M_n(\mathcal{D})$ has the form

$$J = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_r),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathcal{D}$ are distinct and for $i = 1, \ldots, r$,

$$J(\lambda_i) = J_{k_{i,1}}(\lambda_i) \oplus J_{k_{i,2}}(\lambda_i) \oplus \cdots \oplus J_{k_{i,t}}(\lambda_i) \in M_{m_i}(\mathcal{D}),$$

such that $\sum_{i=1}^{r} m_i = n$ and $\sum_{j=1}^{t} k_{i,j} = m_i$ where $k_{i,1} \geq k_{i,2} \geq \cdots \geq k_{i,t} \geq 1$.

An element $\lambda \in \mathcal{D}$ is a *(right) eigenvalue* of $A \in M_n(\mathcal{D})$ if there exists nonzero $\mathbf{v} \in \mathcal{D}^n$ such $A\mathbf{v} = \mathbf{v}\lambda$. Such $\mathbf{v}$ is called a *(right) eigenvector* of A corresponding to λ. The collection of eigenvalues of A, denoted $\sigma(A)$, is the *(right) spectrum* of A. If $\mathcal{D} = \mathbb{C}$ and $\lambda \in \sigma(A)$ where $\operatorname{Im} \lambda := \frac{\lambda - \overline{\lambda}}{2\mathbf{i}} \geq 0$, then λ is called a *standard eigenvalue* of A. Observe that similar matrices in $M_n(\mathcal{D})$ have the same spectrum. However, the number of eigenvalues may be infinite. To see this, note that if $A\mathbf{v} = \mathbf{v}\lambda$ for some nonzero $\mathbf{v} \in \mathcal{D}^n$, then for any nonzero $\gamma \in D$, $A(\mathbf{v}\gamma) = (A\mathbf{v})\gamma = (\mathbf{v}\lambda)\gamma = (\mathbf{v}\gamma)(\gamma^{-1}\lambda\gamma)$. This implies that $\operatorname{Orb}(\lambda) \subseteq \sigma(A)$ and $\operatorname{Orb}(\lambda)$ is infinite if $\lambda \in \mathcal{D} \setminus \mathbb{F}$.

Proposition 2.1. *Let $A, B \in M_n(\mathcal{D})$ be similar. The following hold.*

1. *A is nilpotent if and only if B is nilpotent.*
2. *A is a sum of (m strictly) k-zero matrices if and only if B is a sum of (m strictly) k-zero matrices.*
3. *Let $\alpha \in Z(\mathcal{D}) \setminus \{0\}$. Then A is nilpotent if and only if αA is nilpotent. Similarly, A is a sum of (m strictly) k-zero matrices if and only if αA is a sum of (m strictly) k-zero matrices.*

Proof. Let $\alpha \in Z(\mathcal{D}) \setminus \{0\}$ and suppose $B = P^{-1}AP$ for some invertible $P \in M_n(\mathcal{D})$. For any $k \in \mathbb{N}$,

$$B^k = (P^{-1}AP)^k = P^{-1}A^kP \text{ and } A^k = (PBP^{-1})^k = PB^kP^{-1}.$$

Hence, A is a (strictly) k-zero matrix if and only if B is a (strictly) k-zero matrix proving (1). Consequently, (2) holds. For (3), observe that $(\alpha A)^k = \alpha^k A^k$ for any $k \in \mathbb{N}$ since $\alpha \in Z(\mathcal{D})$. Moreover, $\alpha^k \neq 0$ for any $k \in \mathbb{N}$. Hence, A is nilpotent if and only if αA is nilpotent. □

Let $A \in M_2(\mathcal{D})$ be nilpotent and let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By Proposition 2.1, $P^{-1}AP$ is also nilpotent. Using [5, Lemma 2.1], either $P^{-1}AP = \begin{bmatrix} \alpha\beta & \alpha \\ -\beta\alpha\beta & -\beta\alpha \end{bmatrix}$ or $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$ for some $\alpha, \beta, \gamma \in \mathcal{D}$. The next result is immediate.

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PROPOSITION 2.2. *Let $A \in M_2(\mathcal{D})$. Then A is nilpotent if and only if* $A = \begin{bmatrix} -\beta\alpha & -\beta\alpha\beta \\ \alpha & \alpha\beta \end{bmatrix}$ *or* $A = \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix}$ *for some $\alpha, \beta, \gamma \in \mathcal{D}$.*

Similar results were presented in [5, Lemma 2.1], but the above formulation is more suitable for the purposes of the current study. Throughout the paper, we refer to $N_1(\alpha, \beta) := \begin{bmatrix} -\beta\alpha & -\beta\alpha\beta \\ \alpha & \alpha\beta \end{bmatrix}$ as a *type 1 nilpotent matrix* and $N_2(\gamma) := \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix}$ as a *type 2 nilpotent matrix*.

3. Matrices over a quaternion algebra. Following [11], a *standard involution* on an $\mathbb{F}$-algebra $\mathcal{A}$ is an $\mathbb{F}$-linear map $\overline{} : \mathcal{A} \to \mathcal{A}$ satisfying (i) $\overline{1} = 1$; (ii) $\overline{\overline{\alpha}} = \alpha$ for all $\alpha \in \mathcal{A}$; (iii) $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$ for all $\alpha, \beta \in \mathcal{A}$; and (iv) $\alpha\overline{\alpha} \in \mathbb{F}$ for all $\alpha \in \mathcal{A}$. Given a standard involution, $\alpha\overline{\alpha} + \alpha + \overline{\alpha} + 1 = (\alpha + 1)\overline{(\alpha + 1)} \in \mathbb{F}$ for all $\alpha \in \mathcal{A}$; hence $\alpha + \overline{\alpha} \in \mathbb{F}$ and $\alpha\overline{\alpha} = \overline{\alpha}\alpha$ for all $\alpha \in \mathcal{A}$.

DEFINITION 3.1. Let $\mathcal{A}$ be an $\mathbb{F}$-algebra with a standard involution $\overline{} : \mathcal{A} \to \mathcal{A}$. The *reduced trace* is the map $\mathbf{T} : \mathcal{A} \to \mathbb{F}$ defined by

$$\mathbf{T}(\alpha) = \alpha + \overline{\alpha} \text{ for all } \alpha \in \mathcal{A}.$$

The *reduced norm* is the map $\mathbf{N} : \mathcal{A} \to \mathbb{F}$ defined by

$$\mathbf{N}(\alpha) = \alpha\overline{\alpha} \text{ for all } \alpha \in \mathcal{A}.$$

For all $\alpha, \beta \in \mathcal{A}$, observe that $\mathbf{T}(\alpha + \beta) = \mathbf{T}(\alpha) + \mathbf{T}(\beta)$, $\mathbf{T}(\alpha\beta) = \mathbf{T}(\beta\alpha)$ [11, Lemma 3.3.7], and $\mathbf{N}(\alpha\beta) = \mathbf{N}(\alpha)\mathbf{N}(\beta)$. In particular, $\mathbf{T}$ is an $\mathbb{F}$-linear transformation. Any $\alpha \in \mathcal{A}$ satisfies its *reduced characteristic polynomial* $p(x) = x^2 - \mathbf{T}(\alpha)x + \mathbf{N}(\alpha) \in \mathbb{F}[x]$. When $\alpha \in \mathcal{A} \setminus \mathbb{F}$, the reduced characteristic polynomial of α is its minimal polynomial.

Consider $\mathbb{H}_{\mathbb{F}}^{a,b}$. For each $\alpha = \alpha_0 + \alpha_1\mathbf{i} + \alpha_2\mathbf{j} + \alpha_3\mathbf{k} \in \mathbb{H}_{\mathbb{F}}^{a,b}$ where $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, the following defines the unique standard involution on $\mathbb{H}_{\mathbb{F}}^{a,b}$ (see [11, Examples 3.2.9 and 6.2.6] and [11, Corollary 3.4.4]):

$$\overline{\alpha} = \begin{cases} \alpha_0 - \alpha_1\mathbf{i} - \alpha_2\mathbf{j} - \alpha_3\mathbf{k} = 2\alpha_0 - \alpha, & \text{if } \operatorname{char}(\mathbb{F}) \neq 2 \\ \alpha_0 + \alpha_1 + \alpha_1\mathbf{i} + \alpha_2\mathbf{j} + \alpha_3\mathbf{k} = \alpha_1 + \alpha, & \text{if } \operatorname{char}(\mathbb{F}) = 2; \end{cases}$$

hence,

$$\mathbf{T}(\alpha) = \begin{cases} 2\alpha_0, & \text{if } \operatorname{char}(\mathbb{F}) \neq 2 \\ \alpha_1, & \text{if } \operatorname{char}(\mathbb{F}) = 2 \end{cases},$$

and

$$\mathbf{N}(\alpha) = \begin{cases} \alpha_0^2 - a\alpha_1^2 - b\alpha_2^2 + ab\alpha_3^2, & \text{if } \operatorname{char}(\mathbb{F}) \neq 2 \\ \alpha_0^2 + \alpha_0\alpha_1 + a\alpha_1^2 + b\alpha_2^2 + b\alpha_2\alpha_3 + ab\alpha_3^2, & \text{if } \operatorname{char}(\mathbb{F}) = 2. \end{cases}$$

Recall the following property of central simple algebras.

THEOREM 3.2. *(Skolem–Noether Theorem [11, Theorem 7.7.1]) Let $\mathcal{A}, \mathcal{B}$ be simple $\mathbb{F}$-algebras and $\mathcal{B}$ be central. Suppose $f, g : \mathcal{A} \to \mathcal{B}$ are two homomorphisms of $\mathbb{F}$-algebras. Then there exists an invertible element $\beta \in \mathcal{B}$ such that $f(\alpha) = \beta^{-1}g(\alpha)\beta$ for all $\alpha \in \mathcal{A}$.*

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PROPOSITION 3.3. *Let $\mathcal{Q}$ be a quaternion division algebra and let $\alpha, \beta \in \mathcal{Q} \setminus \mathbb{F}$. Then $\beta \in \mathrm{Orb}(\alpha)$ if and only if $\mathbf{T}(\alpha) = \mathbf{T}(\beta)$ and $\mathbf{N}(\alpha) = \mathbf{N}(\beta)$.*

Proof. The forward implication is clear. Now, take $\alpha, \beta \in \mathcal{Q} \setminus \mathbb{F}$ with $\mathbf{T}(\alpha) = \mathbf{T}(\beta)$ and $\mathbf{N}(\alpha) = \mathbf{N}(\beta)$. Then α and β have the same minimal polynomial over $\mathbb{F}$. Hence, there is an isomorphism $f : \mathbb{F}[\alpha] \to \mathbb{F}[\beta]$ of $\mathbb{F}$-algebras such that $f(\alpha) = \beta$. By the Skolem–Noether Theorem applied to $f : \mathbb{F}[\alpha] \to \mathcal{Q}$ and $\mathrm{id}_{\mathbb{F}[\alpha]} : \mathbb{F}[\alpha] \to \mathcal{Q}$, there exists $\gamma \in \mathcal{Q} \setminus \{0\}$ such that $\beta = f(\alpha) = \gamma^{-1}\mathrm{id}_{\mathbb{F}[\alpha]}(\alpha)\gamma = \gamma^{-1}\alpha\gamma$. ∎

Similar matrices in $M_n(\mathbb{F})$ share the same trace. This is not true for matrices over $\mathcal{Q}$. For example, suppose char$\mathbb{F} \neq 2$ and consider $A = \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ \mathbf{i} & \mathbf{i} \end{bmatrix} \in M_2(\mathbb{H}_{\mathbb{F}}^{a,b})$. If $P = \mathrm{diag}(\mathbf{j}, \mathbf{j})$, then $P^{-1}AP = -A$, but $\mathrm{Tr}(A) = 2\mathbf{i} \neq -2\mathbf{i} = \mathrm{Tr}(-A)$ since char$\mathbb{F} \neq 2$. Though the traces are different, we have $\mathbf{T}(\mathrm{Tr}(A)) = 0 = \mathbf{T}(\mathrm{Tr}(P^{-1}AP))$. This is not a coincidence as shown next.

PROPOSITION 3.4. *Let $\mathcal{Q}$ be a quaternion division algebra. If $A, B \in M_n(\mathcal{Q})$ are similar, then $\mathbf{T}(\mathrm{Tr}(A)) = \mathbf{T}(\mathrm{Tr}(B))$.*

Proof. Let $A, B \in M_n(\mathcal{Q})$ such that $B = P^{-1}AP$ for some invertible $P \in M_n(\mathcal{Q})$. Write $P^{-1} = [x_{ij}]$ and $AP = [y_{ij}]$. Then

$$\begin{aligned}
\mathbf{T}(\mathrm{Tr}(B)) - \mathbf{T}(\mathrm{Tr}(A)) &= \mathbf{T}(\mathrm{Tr}(P^{-1}(AP)) - \mathrm{Tr}((AP)P^{-1})) \\
&= \mathbf{T}\left(\textstyle\sum_{i=1}^n \sum_{k=1}^n x_{ik}y_{ki} - \sum_{i=1}^n \sum_{k=1}^n y_{ik}x_{ki}\right) \\
&= \textstyle\sum_{i=1}^n \sum_{k=1}^n (\mathbf{T}(x_{ik}y_{ki}) - \mathbf{T}(y_{ki}x_{ik})) \\
&= 0.
\end{aligned}$$
∎

The remainder of the section focuses on matrices over $\mathbb{H}$. Observe that every $q \in \mathbb{H}$ can be written as $q = q_1 + q_2\mathbf{j}$, where $q_1, q_2 \in \mathbb{C}$ are uniquely determined by q. This decomposition carries over to square real quaternion matrices: for any $A \in M_n(\mathbb{H})$, there exist $A_1, A_2 \in M_n(\mathbb{C})$ for which $A = A_1 + A_2\mathbf{j}$. This induces a ring embedding $\mathcal{X} : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ defined by $\mathcal{X}(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$. The matrix $\mathcal{X}(A)$ is known as the *complex adjoint form* of A.

By [14, Theorem 5.4], any $A \in M_n(\mathbb{H})$ has n standard eigenvalues, up to similarity. A thorough analysis of the eigenvalues and eigenvectors of a real quaternion matrix, using its complex adjoint form, yields the Jordan Canonical Form Theorem for matrices over $\mathbb{H}$.

THEOREM 3.5. *[13, Theorem 1] Every $A \in M_n(\mathbb{H})$ is similar to a Jordan matrix whose diagonal entries are the standard eigenvalues of A.*

The Jordan matrix in Theorem 3.5 is called the *standard Jordan canonical form* or *standard Jordan form* of A.

4. Fillmore's Theorem for quaternion division algebras. To prove Fillmore's Theorem for quaternion division algebras, we first establish a property of the reduced trace map. Observe that the kernel of $\mathbf{T}$ is a 3-dimensional $\mathbb{F}$-subspace of $\mathbb{H}_{\mathbb{F}}^{a,b}$. Elements of Ker$\mathbf{T}$ are called *pure quaternions*, i.e., elements $\alpha \in \mathbb{H}_{\mathbb{F}}^{a,b}$ such that $\mathbf{T}(\alpha) = \alpha + \overline{\alpha} = 0$.

LEMMA 4.1. *Let $\mathcal{Q}$ be a quaternion division algebra and let $\beta \in \mathcal{Q}$ with $\mathbf{T}(\beta) = 0$. There exists $\delta \in \mathcal{Q} \setminus \mathbb{F}$ such that $\beta + \delta \in \mathrm{Orb}(\delta)$.*

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Proof. The statement is clear if $\beta = 0$. Assume that $\beta \neq 0$. Since $\mathbf{T}(x+\beta) = \mathbf{T}(x)$ for all $x \in \mathcal{Q}$, it suffices to find $x \in \mathcal{Q}$ such that $\mathbf{N}(x) = \mathbf{N}(x+\beta)$ by Proposition 3.3. Observe that

$$\mathbf{N}(x) = \mathbf{N}(x+\beta) = (x+\beta)(\overline{x}+\overline{\beta}) = x\overline{x} + x\overline{\beta} + \beta\overline{x} + \beta\overline{\beta} = \mathbf{N}(x) + \mathbf{T}(x\overline{\beta}) + \mathbf{N}(\beta),$$

if and only if $\mathbf{T}(x\overline{\beta}) = -\mathbf{N}(\beta)$. Since $\beta \neq 0$, the solution set $\mathcal{S} = \{x \in \mathcal{Q} : \mathbf{T}(x\overline{\beta}) = -\mathbf{N}(\beta)\}$ is a 3-dimensional affine subspace of $\mathcal{Q}$. Hence, $\mathbb{F}$ and $-\beta+\mathbb{F}$ do not cover $\mathcal{S}$, and so take $\delta \in \mathcal{S} \setminus (\mathbb{F} \cup (-\beta+\mathbb{F}))$. ∎

Let $n \geq 2$ and $A \in M_n(\mathcal{Q})$ such that $A \neq I_n r$ for any $r \in \mathbb{F}$. We claim that there exists $\mathbf{v} \in \mathcal{Q}^n$ such that $A\mathbf{v}$ and $\mathbf{v}$ are linearly independent. Suppose otherwise, i.e., for any $\mathbf{v} \in \mathcal{Q}$, $A\mathbf{v} = \mathbf{v}\alpha_{\mathbf{v}}$ for some $\alpha_{\mathbf{v}} \in \mathcal{Q}$. By assumption, $A\mathbf{e}_i = \mathbf{e}_i\alpha_i$ for some $\alpha_i \in \mathcal{Q}$, for $i = 1, \ldots, n$. Hence, $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. For any $i \neq j$, $\mathbf{e}_i\alpha_i + \mathbf{e}_j\alpha_j = A(\mathbf{e}_i+\mathbf{e}_j) = (\mathbf{e}_i+\mathbf{e}_j)\alpha_{ij}$, for some $\alpha_{ij} \in \mathcal{Q}$. This implies $\mathbf{e}_i(\alpha_i - \alpha_{ij}) = \mathbf{e}_j(\alpha_{ij} - \alpha_j)$ for any $i \neq j$, forcing $\alpha_1 = \cdots = \alpha_n$. Let $\gamma \in \mathcal{Q}$. There exists $\alpha \in \mathcal{Q}$ such that $\mathbf{e}_1\alpha_1\gamma + \mathbf{e}_2\alpha_1 = A(\mathbf{e}_1\gamma + \mathbf{e}_2) = (\mathbf{e}_1\gamma + \mathbf{e}_2)\alpha$. This implies $\alpha_1\gamma = \gamma\alpha = \gamma\alpha_1$. Since γ is arbitrary, $\alpha_1 \in Z(\mathcal{Q}) = \mathbb{F}$, a contradiction.

THEOREM 4.2. *Let $n \geq 2$ and $A \in M_n(\mathcal{Q})$ such that $A \neq I_n r$ for any $r \in \mathbb{F}$.*

1. *Suppose $n = 2$. If $\alpha \in \mathcal{Q}$ is such that $\mathbf{T}(\alpha) = \mathbf{T}(\operatorname{Tr}(A))$, then A is similar to a matrix whose diagonal entries are $\alpha_1, \alpha_2 \in \mathcal{Q}$ with $\alpha_1 + \alpha_2 = \alpha$.*
2. *Suppose $n > 2$. If $\alpha, \alpha_3, \ldots, \alpha_n \in \mathcal{Q}$ are such that $\mathbf{T}(\alpha + \sum_{i=3}^n \alpha_i) = \mathbf{T}(\operatorname{Tr}(A))$, then A is similar to a matrix whose diagonal entries are $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n \in \mathcal{Q}$ with $\alpha_1 + \alpha_2 = \alpha$.*

Proof. There exists $\mathbf{v}_1 \in \mathcal{Q}^n$ such that $\{\mathbf{v}_1, A\mathbf{v}_1\}$ is linearly independent. If $n = 2$, consider the basis $\mathcal{V} = \{\mathbf{v}_1, A\mathbf{v}_1\}$ of $\mathcal{Q}^2$; if $n > 2$, extend $\{\mathbf{v}_1, A\mathbf{v}_1\}$ to a basis of $\mathcal{Q}^n$, say $\{\mathbf{v}_1, A\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and consider the basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ where $\mathbf{v}_2 = A\mathbf{v}_1 - \sum_{i=3}^n \mathbf{v}_i$. The matrix representation of A with respect to $\mathcal{V}$ is

$$\begin{bmatrix} 0 & * & \cdots & * \\ 1 & \alpha_2' & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 1 & * & \cdots & \alpha_n' \end{bmatrix},$$

for some $\alpha_2', \ldots, \alpha_n' \in \mathcal{Q}$. If $n = 2$, define $\beta = \alpha - \alpha_2'$; if $n > 2$, define $\beta = \alpha + \sum_{i=3}^n (\alpha_i - \alpha_i') - \alpha_2'$. By assumption and Proposition 3.4,

$$\mathbf{T}(\beta) = \begin{cases} \mathbf{T}(\alpha) - \mathbf{T}(\alpha_2') = \mathbf{T}(\alpha) - \mathbf{T}(\operatorname{Tr}(A)) = 0, & \text{if } n = 2 \\ \mathbf{T}(\alpha) + \sum_{i=3}^n \mathbf{T}(\alpha_i - \alpha_i') - \mathbf{T}(\alpha_2') = \mathbf{T}(\alpha) + \sum_{i=3}^n \mathbf{T}(\alpha_i) - \mathbf{T}(\operatorname{Tr}(A)) = 0, & \text{if } n > 2. \end{cases}$$

Lemma 4.1 implies $\beta + \delta = \gamma^{-1}\delta\gamma$ for some $\delta \in \mathcal{Q} \setminus \mathbb{F}$ and $\gamma \in \mathcal{Q} \setminus \{0\}$. Define $\alpha_2 = \delta$ and consider the basis $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ where

$$\mathbf{w}_1 = \mathbf{v}_1 \text{ and } \mathbf{w}_i = \mathbf{v}_i + \mathbf{v}_1(\alpha_i - \alpha_i') \text{ for } i = 2, \ldots, n.$$

Observe that

$$\mathbf{w}_1 \sum_{i=2}^n (\alpha_i' - \alpha_i) + \sum_{i=2}^n \mathbf{w}_i = \mathbf{v}_1 \sum_{i=2}^n (\alpha_i' - \alpha_i) + \sum_{i=2}^n (\mathbf{v}_i + \mathbf{v}_1(\alpha_i - \alpha_i')) = \sum_{i=2}^n \mathbf{v}_i = A\mathbf{v}_1 = A\mathbf{w}_1.$$

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For $i = 2, \ldots, n$, write $A\mathbf{v}_i = \mathbf{v}_i\alpha_i' + \sum_{j\neq i} \mathbf{v}_j\alpha_{ji}$ for some $\alpha_{ji} \in \mathcal{Q}$. Then

$$\begin{array}{rcl} A\mathbf{w}_i & = & A\mathbf{v}_i + A\mathbf{v}_1(\alpha_i - \alpha_i') \\ & = & \mathbf{v}_i\alpha_i' + \sum_{j\neq i} \mathbf{v}_j\alpha_{ji} + \sum_{j=2}^{n} \mathbf{v}_j(\alpha_i - \alpha_i') \\ & = & \mathbf{v}_1\alpha_{1i} + \mathbf{v}_i\alpha_i + \sum_{j\neq 1,i} \mathbf{v}_j\alpha_{ji}' \text{ (with } \alpha_{ji}' = \alpha_{ji} + \alpha_i - \alpha_i' \text{ for } j \neq 1, i) \\ & = & \mathbf{v}_1\alpha_{1i}' + (\mathbf{v}_i + \mathbf{v}_1(\alpha_i - \alpha_i'))\alpha_i + \sum_{j\neq 1,i}(\mathbf{v}_j + \mathbf{v}_1(\alpha_j - \alpha_j'))\alpha_{ji}', \end{array}$$

where $\alpha_{1i}' = \alpha_{1i} - (\alpha_i - \alpha_i')\alpha_i - \sum_{j\neq 1,i}(\alpha_j - \alpha_j')\alpha_{ji}'$. Hence, $A\mathbf{w}_i = \mathbf{w}_1\alpha_{1i}' + \mathbf{w}_i\alpha_i + \sum_{j\neq 1,i} \mathbf{w}_j\alpha_{ji}'$ for $i = 2, \ldots, n$. It follows that the matrix representation of A with respect to $\mathcal{W}$ is

$$A' = \begin{bmatrix} \sum_{i=2}^{n}(\alpha_i' - \alpha_i) & * & \cdots & * \\ 1 & \alpha_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 1 & * & \cdots & \alpha_n \end{bmatrix}.$$

If $n = 2$, let $P = \text{diag}(1, \gamma)$; if $n > 2$, let $P = \text{diag}(1, \gamma, 1, \ldots, 1)$. Then A is similar to $P^{-1}A'P$ and the diagonal entries of $P^{-1}A'P$ are $\sum_{i=2}^{n}(\alpha_i' - \alpha_i), \gamma^{-1}\alpha_2\gamma, \ldots, \alpha_n$. Consider the sum of the first two diagonal entries of $P^{-1}A'P$: if $n = 2$, then

$$(\alpha_2' - \alpha_2) + \gamma^{-1}\alpha_2\gamma = (\alpha_2' - \delta) + (\beta + \delta) = \alpha_2' + \beta = \alpha;$$

if $n > 2$, then

$$\begin{array}{rcl} \sum_{i=2}^{n}(\alpha_i' - \alpha_i) + \gamma^{-1}\alpha_2\gamma & = & \alpha_2' - \alpha_2 + \sum_{i=3}^{n}(\alpha_i' - \alpha_i) + (\beta + \delta) \\ & = & \alpha_2' - \delta + (\alpha - \alpha_2' - \beta) + (\beta + \delta) \\ & = & \alpha. \end{array}$$

□

5. Consequences. Theorem 4.2 implies an analog of [2, Theorem 2] for matrices over a quaternion division algebra.

COROLLARY 5.1. *Let $n \geq 2$ and $A \in M_n(\mathcal{Q})$. If either $A = I_n r$ for some $r \in \mathbb{F}$ with $\text{Tr}(A) = 0$ or $A \neq I_n r$ for any $r \in \mathbb{F}$ with $\mathbf{T}(\text{Tr}(A)) = 0$, then A is a sum of at most three nilpotent matrices.*

Proof. The claim holds for $A = I_n r$ for some $r \in \mathbb{F}$ with $\text{Tr}(A) = 0$ due to [2, Theorem 2]. Assume that $A \neq I_n r$ for any $r \in \mathbb{F}$ with $\mathbf{T}(\text{Tr}(A)) = 0$. If $n = 2$, let $\alpha = 0$; if $n > 2$, let $\alpha = \alpha_3 = \cdots = \alpha_n = 0$. By Theorem 4.2, A is similar to a matrix B whose diagonal entries are $\alpha_1, \ldots, \alpha_n$ and with $\alpha_1 + \alpha_2 = 0$. Observe that B can be decomposed as

$$B = \begin{cases} U + L + N_1(\alpha_1, -1), & \text{if } n = 2 \\ U + L + N_1(\alpha_1, -1) \oplus 0_{n-2}, & \text{if } n > 2 \end{cases},$$

where $U, L^\top$ are strictly upper triangular matrices. Since each term above is nilpotent, A is a sum of at most three nilpotent matrices by Proposition 2.1. □

In Theorem 4.2, the first two diagonal entries α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$ cannot be selected arbitrarily.

REMARK 5.1. *Let $\text{char}\mathbb{F} \neq 2$. Consider the non-nilpotent $A = \text{diag}(\mathbf{i}, 0) \in M_2(\mathbb{H}_{\mathbb{F}}^{a,b})$ whose $\mathbf{T}(\text{Tr}(A)) = 0$. Suppose A is similar to a matrix of the form $B = \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix}$ for some $\alpha, \beta \in \mathbb{H}_{\mathbb{F}}^{a,b}$. Observe*

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that $B = N_1(\alpha, 0) + N_2(\beta)$*, and so* $A = N_1 + N_2$ *where* N_1 *and* N_2 *are square-zero matrices by Proposition 2.1. Then* N_1 *and* N_2 *commute with* $(N_1 + N_2)^2 = A^2 = \mathrm{diag}(\mathbf{i}^2, 0) = \mathrm{diag}(a, 0)$*. This implies that* N_1 *and* N_2 *are diagonal nilpotent matrices, and so* $N_1 = N_2 = 0_2$ *leading to* $A = 0_2$*, which is a contradiction. This argument also shows that the non-nilpotent* A *is a sum of exactly three nilpotent matrices due to Corollary 5.1.*

Let $\mathcal{Q}$ be a quaternion division algebra. Suppose $\lambda \in \mathbb{F} \setminus \{0\}$. We claim that for any n-dimensional $\mathcal{Q}$-vector space V_n,

$$\text{if } \mathfrak{u} = \mathrm{id}_{V_n}\lambda - \mathfrak{a} \text{ is a sum of two nilpotent endomorphisms on } V_n \\ \text{where } \mathfrak{a} \text{ is a rank 1 non-nilpotent endomorphism on } V_n, \tag{5.1}$$

then $\mathfrak{a}$ has a nonzero eigenvalue in $\mathbb{F}$.

Observe that if $n = 1$, then $\mathfrak{u} = \mathrm{id}_{V_1}\lambda - \mathfrak{a}$ being a sum of two nilpotent endomorphisms means $\mathfrak{u} = 0$, which is equivalent to $\mathfrak{a} = \mathrm{id}_{V_1}\lambda$, as desired. Assume that the claim is true for all $\mathcal{Q}$-vector spaces V with $\dim_{\mathcal{Q}} V < n$ and endomorphisms on V satisfying (5.1). Let $\mathrm{id}_{V_n}\lambda - \mathfrak{a} = \mathfrak{u} = \mathfrak{n}_1 + \mathfrak{n}_2$ for some nilpotent endomorphisms $\mathfrak{n}_1, \mathfrak{n}_2 : V_n \to V_n$ and suppose $\mathfrak{a}$ is a rank 1 non-nilpotent endomorphism on V_n that has no eigenvalue in $\mathbb{F}$. If $\mathrm{Ker}\mathfrak{n}_1 \cap \mathrm{Ker}\mathfrak{n}_2 \neq \{0\}$, then $\lambda \in \sigma(\mathfrak{a}) \cap (\mathbb{F} \setminus \{0\})$. Suppose $\mathrm{Ker}\mathfrak{n}_1 \cap \mathrm{Ker}\mathfrak{n}_2 = \{0\}$. This implies that if $\mathbf{x} \in \mathrm{Ker}\mathfrak{n}_2 \setminus \{0\}$, then $\{\mathbf{x}, \mathfrak{n}_1(\mathbf{x})\}$ is linearly independent; otherwise, such an $\mathbf{x}$ is an eigenvector of $\mathfrak{n}_1$ corresponding to a nonzero eigenvalue but $\sigma(\mathfrak{n}_1) = \{0\}$. Since $\mathbf{x}\lambda - \mathfrak{a}(\mathbf{x}) = \mathfrak{u}(\mathbf{x}) = \mathfrak{n}_1(\mathbf{x})$ and $\mathfrak{a}$ has rank 1, $\mathrm{Im}\mathfrak{a} = (\mathbf{x}\lambda - \mathfrak{n}_1(\mathbf{x}))\mathcal{Q}$. Observe that $\mathrm{Im}\mathfrak{a}$ is a subspace of $W = \sum_{k=0}^{n-1} \mathfrak{n}_1^k(\mathbf{x})\mathcal{Q}$. Hence,

$$\mathfrak{n}_2(\mathfrak{n}_1^k(\mathbf{x})) = \mathfrak{u}(\mathfrak{n}_1^k(\mathbf{x})) - \mathfrak{n}_1^{k+1}(\mathbf{x}) = \mathfrak{n}_1^k(\mathbf{x})\lambda - \mathfrak{a}(\mathfrak{n}_1^k(\mathbf{x})) - \mathfrak{n}_1^{k+1}(\mathbf{x}) \in W,$$

and so W is an $\mathfrak{n}_2$-invariant subspace. Since W is an $\mathfrak{n}_1$-invariant subspace (note that $\mathfrak{n}_1^n = 0$), W is also $\mathfrak{a}$-invariant. If $\dim_{\mathcal{Q}} W < n$, observe that the restriction $\mathfrak{a}|_W$ is of rank 1 and non-nilpotent. Moreover, $\mathfrak{u}|_W = \mathrm{id}_W\lambda - \mathfrak{a}|_W$ is a sum of two nilpotent endomorphisms $\mathfrak{n}_1|_W + \mathfrak{n}_2|_W$. The induction hypothesis guarantees that $\mathfrak{a}|_W$ has a nonzero eigenvalue in $\mathbb{F}$; thus, $\mathfrak{a}$ has a nonzero eigenvalue in $\mathbb{F}$. If $\dim_{\mathcal{Q}} W = n$, then $W = V_n$ and $\mathcal{B} = \{\mathbf{x}, \mathfrak{n}_1(\mathbf{x}), \mathfrak{n}_1^2(\mathbf{x}), \ldots, \mathfrak{n}_1^{n-1}(\mathbf{x})\}$ is a basis for V_n. With respect to $\mathcal{B}$, the matrix representations of $\mathfrak{a}$, $\mathfrak{n}_1$, and $\mathfrak{n}_2$ are respectively

$$A = \begin{bmatrix} \lambda & -\lambda\mu & * \\ -1 & \mu & * \\ 0 & 0 & 0 \end{bmatrix}, \; N_1 = J_n(0)^\top, \text{ and } N_2 = \begin{bmatrix} 0 & * \\ 0 & N_2' \end{bmatrix},$$

where $\mu \in \mathcal{Q}$ and $N_2' \in M_{n-1}(\mathcal{Q})$. If $\mu \in \mathbb{F}$, then $\lambda + \mu \in \mathbb{F} \cap \sigma(\mathfrak{a})$ since $\mathfrak{a}$ has rank 1. Note that $\lambda + \mu \neq 0$ due to $\mathfrak{a}$ being non-nilpotent, thereby proving the claim in this case. Suppose $\mu \in \mathcal{Q} \setminus \mathbb{F}$. Let $\mathcal{B}' = \mathcal{B} \setminus \{\mathbf{x}\}$ and $W' = \mathrm{Span}\mathcal{B}'$. Then $\mathrm{id}_{W'}\lambda - \mathfrak{a}|_{W'} = \mathfrak{n}_1|_{W'} + \mathfrak{n}_2|_{W'}$. With respect to $\mathcal{B}'$, the matrix representations of $\mathfrak{a}|_{W'}$, $\mathfrak{n}_1|_{W'}$, $\mathfrak{n}_2|_{W'}$ are respectively the principal submatrices of A, N_1, N_2 obtained by deleting their first row and first column, namely,

$$A' = \begin{bmatrix} \mu & * \\ 0 & 0 \end{bmatrix}, \; N_1' = J_{n-1}(0)^\top, \text{ and } N_2'.$$

Observe that N_1' and N_2' are nilpotent while A' has rank 1 and is non-nilpotent since $0 \neq \mu \in \sigma(A')$. Since $I_{n-1}\lambda - A' = N_1' + N_2'$ satisfies the induction hypothesis, A' has a nonzero eigenvalue in $\mathbb{F}$. This is a contradiction since $\mu \in \mathcal{Q} \setminus \mathbb{F}$ and all nonzero eigenvalues of A' are in $\mathrm{Orb}(\mu)$.

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As noted in Remark 5.1, there are 2×2 quaternion matrices that are sums of exactly three nilpotent matrices. The preceding discussion is used to generalize this observation for the $n \times n$ case.

Remark 5.2. *Let $\lambda \in \mathbb{F} \setminus \{0\}$ and $\alpha \in \mathcal{Q} \setminus \mathbb{F}$ such that $\mathbf{T}(\alpha) = n\mathbf{T}(\lambda)$. Define $U = I_n\lambda - A$ where $A = diag(\alpha, 0, \ldots, 0)$. Then $\mathbf{T}(\mathrm{Tr}(U)) = n\mathbf{T}(\lambda) - \mathbf{T}(\alpha) = 0$. By Corollary 5.1, U is a sum of at most three nilpotent matrices. By the preceding discussion, U is not a sum of two nilpotent matrices since all the nonzero eigenvalues of A are in $\mathrm{Orb}(\alpha) \not\subseteq \mathbb{F}$.*

Remark 5.2 guarantees the existence of matrices that are sums of exactly three nilpotent matrices. This establishes the sharpness of the bound in Theorem 5.1.

Remark 5.3. *A matrix $A \in M_n(\mathbb{C})$ is a sum of two square-zero matrices if and only if A is similar to $-A$ (see [12]). This fails for matrices in $M_n(\mathbb{H})$ (with the obvious counterexample of $\mathbf{i}$ in the $n = 1$ case).*

Motivated by Remarks 5.1-5.3 and as an application of the main results, we give a characterization of $n \times n$ real quaternion matrices that are sums of two square-zero matrices.

Theorem 5.2. *Let $A \in M_n(\mathbb{H})$. Then A is a sum of two square-zero matrices if and only if the standard Jordan form of A is a direct sum of the following blocks:*

1. *$J_k(0)$ or*
2. *$J_k(\lambda) \oplus J_k(-\overline{\lambda})$, where $\lambda \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Im}\,\lambda \geq 0$.*

Proof. We first prove the backward implication. Since $(I_k\mathbf{j})^{-1} J_k(\overline{\lambda})(I_k\mathbf{j}) = J_k(\lambda)$ for any $\lambda \in \mathbb{C}$, it follows that A is similar to a Jordan matrix which is a direct sum of the following blocks:

1. $J_k(0)$ or
2. $J_k(\lambda) \oplus J_k(-\lambda)$, where $\lambda \in \mathbb{C} \setminus \{0\}$.

Hence, A is similar to a complex matrix of the form $S \oplus -S \oplus N$, where S and N are complex matrices for which S is invertible while N is nilpotent. Applying [12, Theorem 2.11] and Proposition 2.1 proves the backward implication.

Now, we establish the forward implication. Suppose A is a sum of two square-zero matrices in $M_n(\mathbb{H})$. Consider the endomorphism $\mathfrak{a} : \mathbb{H}^n \to \mathbb{H}^n$ defined by $\mathfrak{a}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{H}^n$. By Fitting's Lemma, there are $\mathfrak{a}$-invariant subspaces V_N and V_I of $\mathbb{H}^n$ such that $\mathbb{H}^n = V_N \oplus V_I$, where the restriction $\mathfrak{a}|_{V_N}$ is nilpotent and $\mathfrak{a}|_{V_I}$ is invertible. Furthermore, since $\mathbb{H}^n$ is finite dimensional, we can choose a large positive even integer k such that $\mathrm{Im}\mathfrak{a}^k = V_I$ and $\mathrm{Ker}\mathfrak{a}^k = V_N$. The decomposition of A into a sum of two square-zero matrices is equivalent to the decomposition of $\mathfrak{a}$ into a sum $\mathfrak{a} = \mathfrak{n}_1 + \mathfrak{n}_2$ of square-zero endomorphisms $\mathfrak{n}_1, \mathfrak{n}_2$. Since $\mathfrak{n}_1$ and $\mathfrak{n}_2$ commute with $(\mathfrak{n}_1 + \mathfrak{n}_2)^2 = \mathfrak{a}^2$, both commute with $\mathfrak{a}^k$. We claim that V_N and V_I are both $\mathfrak{n}_1$-invariant and $\mathfrak{n}_2$-invariant subspaces. Indeed, if $\mathbf{x} \in V_N = \mathrm{Ker}\mathfrak{a}^k$, then

$$\mathfrak{a}^k(\mathfrak{n}_1(\mathbf{x})) = \mathfrak{n}_1(\mathfrak{a}^k(\mathbf{x})) = 0,$$

and so $\mathfrak{n}_1(\mathbf{x}) \in \mathrm{Ker}\mathfrak{a}^k = V_N$. Likewise, if $\mathbf{z} \in V_I = \mathrm{Im}\mathfrak{a}^k$, then $\mathfrak{a}^k(\mathbf{y}) = \mathbf{z}$ for some $\mathbf{y} \in \mathbb{H}^n$. Thus,

$$\mathfrak{n}_1(\mathbf{z}) = \mathfrak{n}_1(\mathfrak{a}^k(\mathbf{y})) = \mathfrak{a}^k(\mathfrak{n}_1(\mathbf{y})) \in \mathrm{Im}\mathfrak{a}^k = V_I.$$

Analogous arguments show that both V_N and V_I are $\mathfrak{n}_2$-invariant subspaces. Thus, both $\mathfrak{a}|_{V_I} = \mathfrak{n}_1|_{V_I} + \mathfrak{n}_2|_{V_I}$ and $\mathfrak{a}|_{V_N} = \mathfrak{n}_1|_{V_N} + \mathfrak{n}_2|_{V_N}$ are sums of two square-zero endomorphisms. Therefore, it suffices to determine the standard Jordan form of all automorphisms on $\mathbb{H}^n$ that are sums of two square-zero endomorphisms.

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Let $U \in M_n(\mathbb{H})$ be invertible and consider the automorphism $\mathfrak{u} : \mathbb{H}^n \to \mathbb{H}^n$ defined by $\mathfrak{u}(\mathbf{x}) = U\mathbf{x}$ for all $\mathbf{x} \in \mathbb{H}^n$. Suppose $\mathfrak{u} = \mathfrak{n}_1 + \mathfrak{n}_2$ where $\mathfrak{n}_1, \mathfrak{n}_2$ are square-zero endomorphisms on $\mathbb{H}^n$. Notice that $\mathrm{Ker}\mathfrak{n}_1 \cap \mathrm{Ker}\mathfrak{n}_2 = \{0\}$ and $\mathrm{Im}\mathfrak{n}_i \subseteq \mathrm{Ker}\mathfrak{n}_i$ for $i = 1, 2$. Let $k_i = \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_i)$ for $i = 1, 2$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k_1}\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_{k_2}\}$ be a basis for $\mathrm{Ker}\mathfrak{n}_1$ and $\mathrm{Ker}\mathfrak{n}_2$, respectively. By the Rank-Nullity Theorem, we have

$$n = \dim_{\mathbb{H}}(\mathrm{Im}\mathfrak{n}_i) + \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_i) \leq 2\dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_i) = 2k_i,$$

for $i = 1, 2$, and so $n \leq k_1 + k_2$. But

$$n \geq \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_1 + \mathrm{Ker}\mathfrak{n}_2) = \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_1) + \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_2) - \dim_{\mathbb{H}}(\mathrm{Ker}\mathfrak{n}_1 \cap \mathrm{Ker}\mathfrak{n}_2) = k_1 + k_2.$$

Thus, $n = k_1 + k_2$ and $k_1 = k_2$; it follows that n is even. Moreover, $\mathrm{Im}\mathfrak{n}_i = \mathrm{Ker}\mathfrak{n}_i$ for $i = 1, 2$. Direct calculations reveal that $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_{k_2}, \mathfrak{n}_1(\mathbf{w}_1), \ldots, \mathfrak{n}_1(\mathbf{w}_{k_2})\}$ is a basis for $\mathbb{H}^n$. Notice that $\mathfrak{u}(\mathfrak{n}_1(\mathbf{w}_i)) = \mathfrak{n}_2(\mathfrak{n}_1(\mathbf{w}_i)) \in \mathrm{Im}\mathfrak{n}_2$ for $i = 1, \ldots, k_2$. Thus, the matrix representation of $\mathfrak{u}$ with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 0 & Q \\ I_{n/2} & 0 \end{bmatrix},$$

for some invertible $Q \in M_{n/2}(\mathbb{H})$. Select an invertible matrix $S \in M_{n/2}(\mathbb{H})$ such that $Q' = S^{-1}QS$ is a complex matrix (this is possible by Theorem 3.5). Thus, U is similar to the complex matrix $\begin{bmatrix} 0 & Q' \\ I_{n/2} & 0 \end{bmatrix}$ which is a sum of two complex square-zero matrices as seen from

$$\begin{bmatrix} 0 & Q' \\ I_{n/2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_{n/2} & 0 \end{bmatrix}.$$

Applying Proposition 2.1, [12, Theorem 2.11], and Theorem 3.5 yields the desired result. □

The characterization of sums of two *nilpotent* (not necessarily square-zero) matrices remains an open problem. However, the 2×2 case is immediate due to Theorem 5.2.

Corollary 5.3. *Let $A \in M_2(\mathbb{H})$ such that $\mathbf{T}(\mathrm{Tr}(A)) = 0$. Then A is a sum of at most two nilpotent matrices if and only if the standard Jordan form of A is one of the following:*

1. $J_2(0)$ *or*
2. $\mathrm{diag}(\lambda, -\overline{\lambda})$ *for some* $\lambda \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda) \geq 0$.

Declaration of competing interest. The authors declare that there is no competing interest.

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